VEKTORANALYS

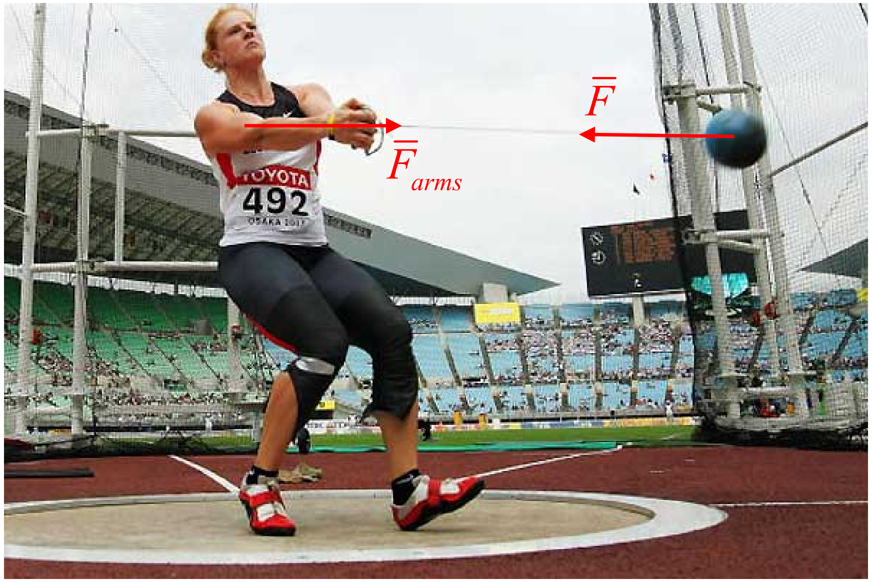
Kursvecka 5

COORDINATE TRANSORMATION

Kapitel 10
Sidor 99-121

TARGET PROBLEM

- An athlete is rotating a hammer
- Calculate the force on the arms.



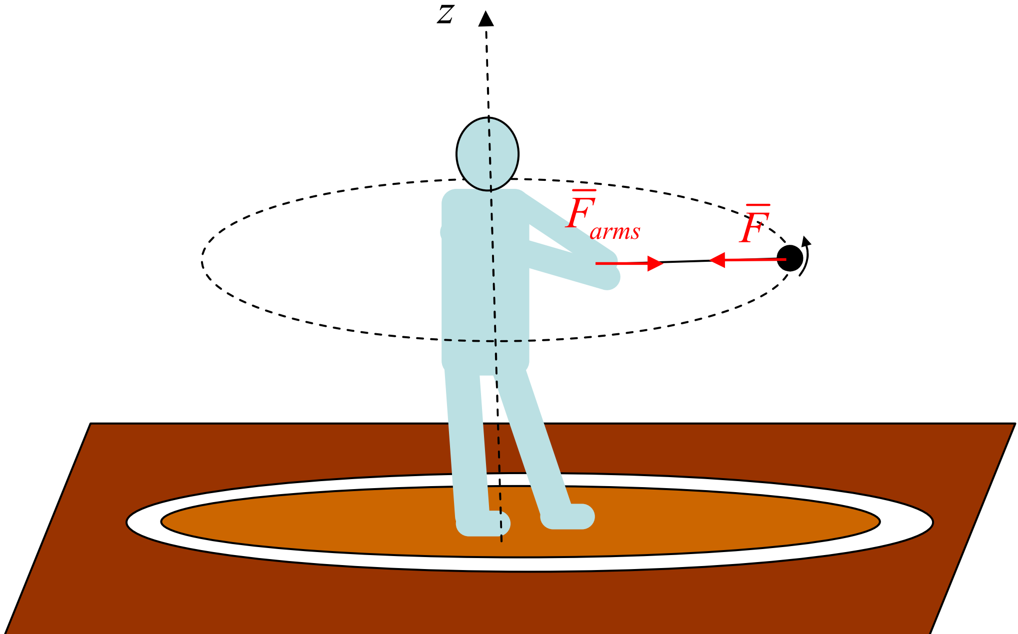
$$\overline{F}_{arms} = -\overline{F}$$

$$\overline{F} = m\overline{a}$$

$$\overline{a} = \frac{d\overline{v}}{dt} \equiv \dot{\overline{v}}$$

$$\overline{v} = \frac{d\overline{r}}{dt} \equiv \dot{\overline{r}}$$

$$\overline{r} = \rho_0 \hat{e}_\rho + z_0 \hat{e}_z \qquad \textit{(in cylindrical coord.)}$$



We need
- to introduce **curvilinear coordinates**
- to describe **cylindrical coordinates**
- to calculate **the derivative of** $\hat{e}_\rho$

CYLINDRICAL COORDINATES

Cylindrical coordinates are an example of curvilinear coordinates

cartesian coord.

$P\!:\ (x_0,\ y_0,\ z_0)$

cylindrical coord.

$P\!:\ (\rho,\ \varphi,\ z_0)$

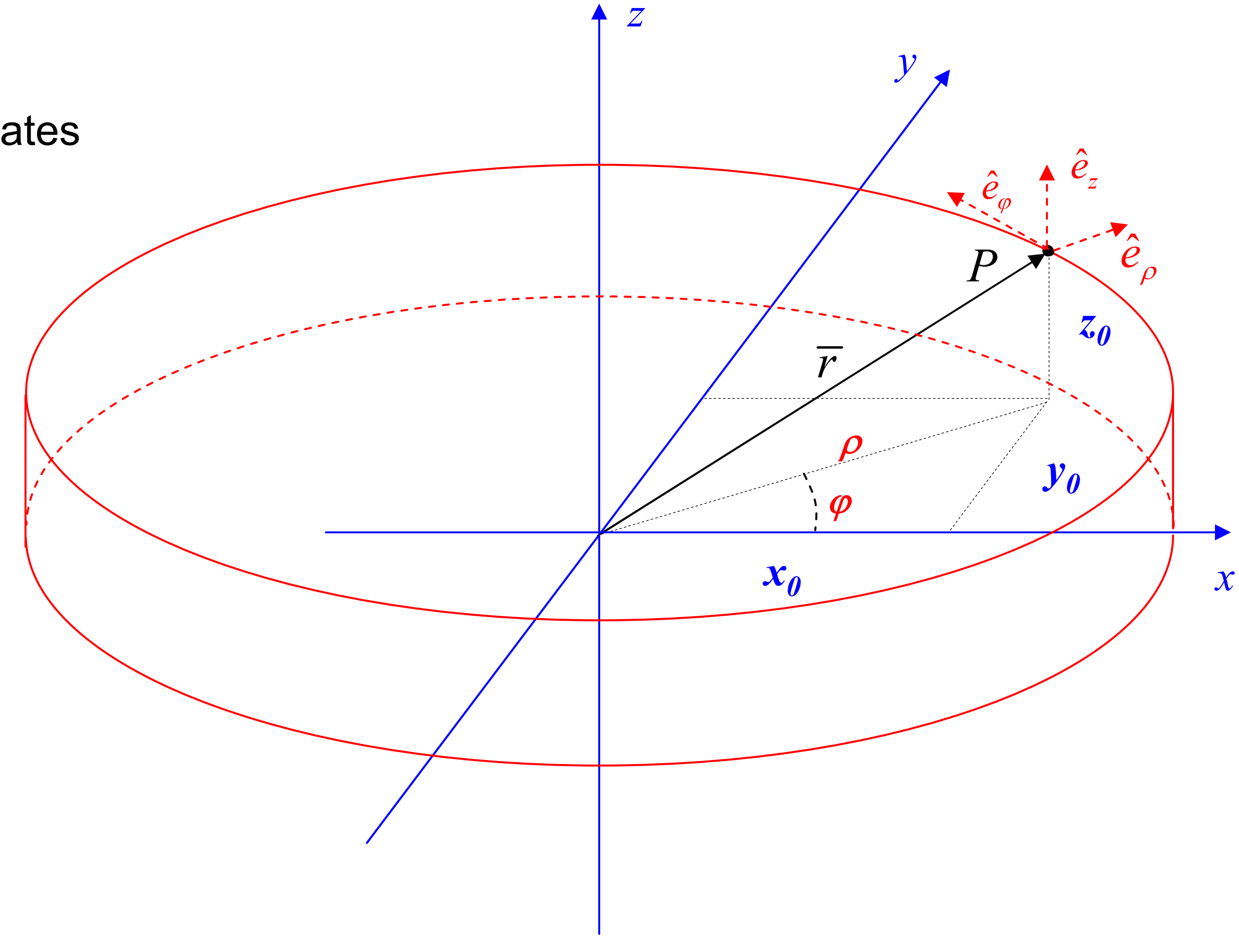
$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \tan\varphi = y / x \\ z = z \end{cases}$$

$$0 \le \rho \le \infty$$

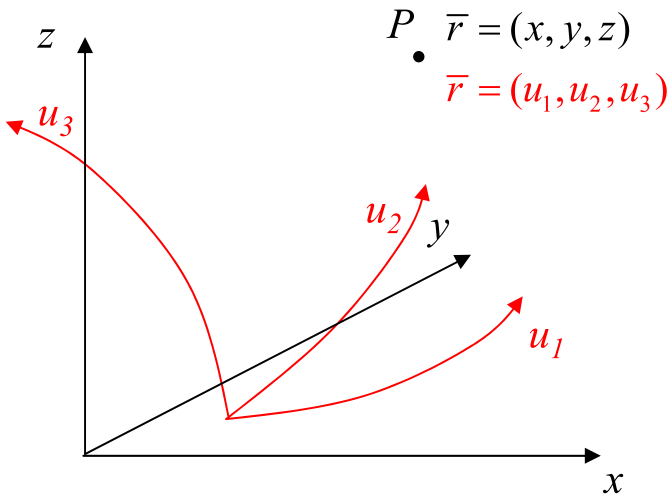
$$0 \le \varphi \le 2\pi$$

$$-\infty \le z \le +\infty$$

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$$



CURVILINEAR COORDINATES



Consider a cartesian coordinate system *(x, y, z)*
and then
another coordinate system (u_1, u_2, u_3),
which will be called **curvilinear**

- The 3 curves u_1, u_2, u_3 are the coordinate axes
- The surfaces defined by $u_i = const.$ are called **coordinate surfaces**
- The curves defined by the intersection of the coordinate surfaces are called **coordinate curves**

A point P has coordinates (x,y,z) in the cartesian and coordinates (u_1,u_2,u_3) in the curvilinear

We assume that there is a one-to-one relationship between x_i and u_i,
so that x_i can be expressed as a function of u_i *(and vice-versa)*:

$$\begin{cases} x = x(u_1, u_2, u_3) \\ y = y(u_1, u_2, u_3) \\ z = z(u_1, u_2, u_3) \end{cases}$$

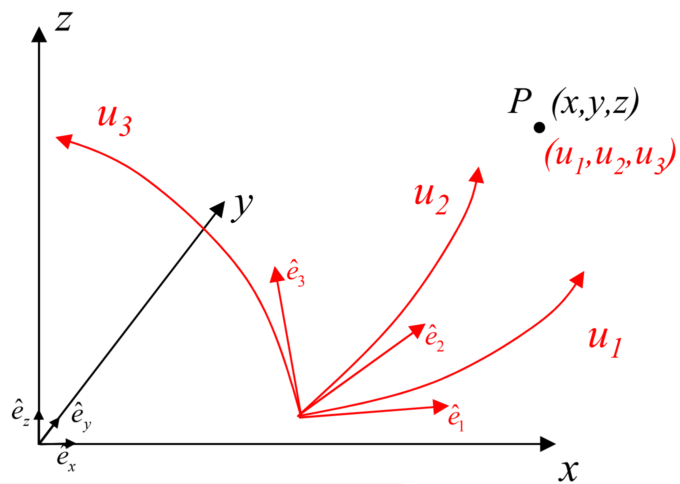
DEFINITION

A system of curvilinear coordinates is orthogonal
if the coordinate curves are perpendicular to each other
where they intersect

$$\overline{u}_i \cdot \overline{u}_j = \delta_{ik}$$

CURVILINEAR COORDINATES

	Point P	the basis is defined by the unit vectors:	$d\bar{r}$
CARTESIAN	(x, y, z)	$\hat{e}_x \quad \hat{e}_y \quad \hat{e}_z$	(dx, dy, dz)
CURVILINEAR	(u_1, u_2, u_3)	?	?



DEFINITION

An orthogonal curvilinear coordinate system has an orthonormal basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ in each point and

$$\hat{e}_i = \frac{1}{h_i}\frac{\partial \bar{r}}{\partial u_i} \quad with \quad scale\ factor \quad h_i = \left|\frac{\partial \bar{r}}{\partial u_i}\right|$$

Orthonormal:

- magnitude 1 $\quad |\hat{e}_i| = \left|\frac{1}{h_i}\frac{\partial \bar{r}}{\partial u_i}\right| = \frac{1}{|h_i|}\left|\frac{\partial \bar{r}}{\partial u_i}\right| = 1$
- orthogonal $\quad \hat{e}_i \cdot \hat{e}_j = \frac{1}{h_i}\frac{\partial \bar{r}}{\partial u_i} \cdot \frac{1}{h_j}\frac{\partial \bar{r}}{\partial u_j} = 0 \quad for\ i \neq j$

$$\Rightarrow \hat{e}_i \cdot \hat{e}_j = \underset{\uparrow \\ Kronecker\ delta}{\delta_{ij}}$$

$$d\bar{r} = \frac{\partial \bar{r}}{\partial u_1}du_1 + \frac{\partial \bar{r}}{\partial u_2}du_2 + \frac{\partial \bar{r}}{\partial u_3}du_3 = h_1\hat{e}_1 du_1 + h_2\hat{e}_2 du_2 + h_3\hat{e}_3 du_3$$

$$\boxed{d\bar{r} = \sum_i h_i du_i \hat{e}_i}$$

SURFACE ELEMENT AND VOLUME ELEMENT

In a Cartesian coordinate system:

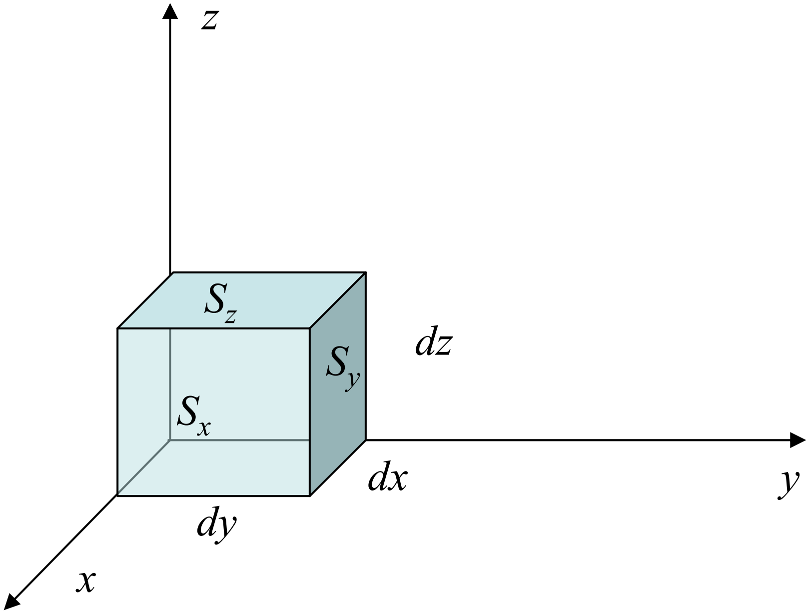
$$d\bar{r} = (dx, dy, dz) = dx\hat{e}_x + dy\hat{e}_y + dz\hat{e}_z$$

$$dS_x = dydz$$

$$dS_y = dxdz$$

$$dS_z = dxdy$$

$$dV = dxdydz$$



In a orthogonal curvilinear coordinate system:

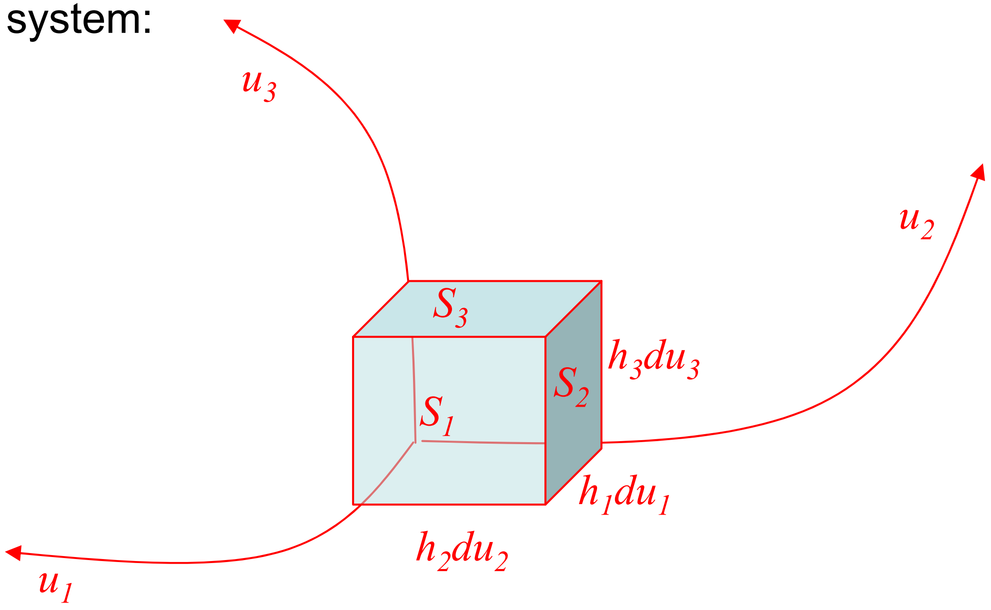
$$d\bar{r} = h_1du_1\hat{e}_1 + h_2du_2\hat{e}_2 + h_3du_3\hat{e}_3$$

$$dS_1 = h_2h_3du_2du_3$$

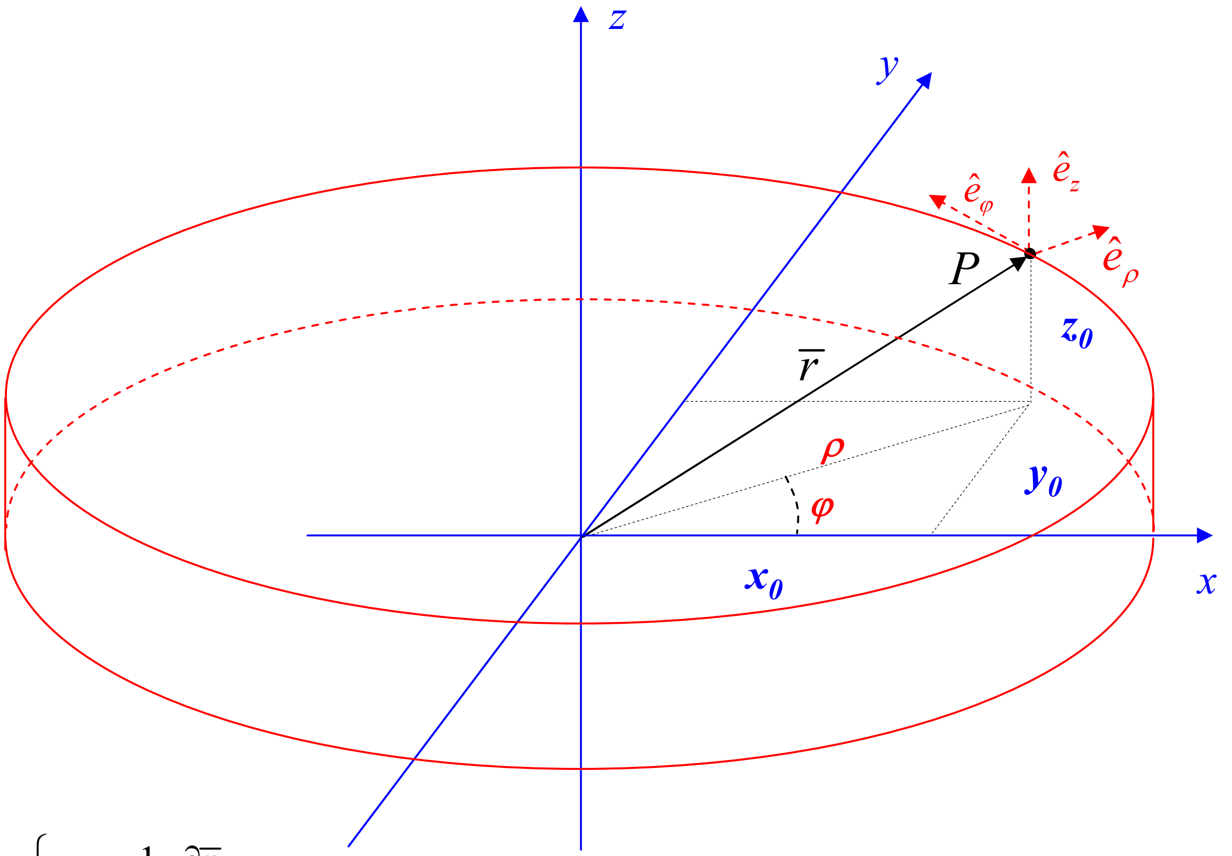
$$dS_2 = h_1h_3du_1du_3$$

$$dS_3 = h_1h_2du_1du_2$$

$$dV = h_1h_2h_3du_1du_2du_3$$



CYLINDRICAL COORDINATES



Orthonormal basis

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$$

$$\overline{r} = (\rho\cos\varphi, \rho\sin\varphi, z)$$

$$\hat{e}_i = \frac{1}{h_i}\frac{\partial\overline{r}}{\partial u_i} \quad with \quad h_i = \left|\frac{\partial\overline{r}}{\partial u_i}\right|$$

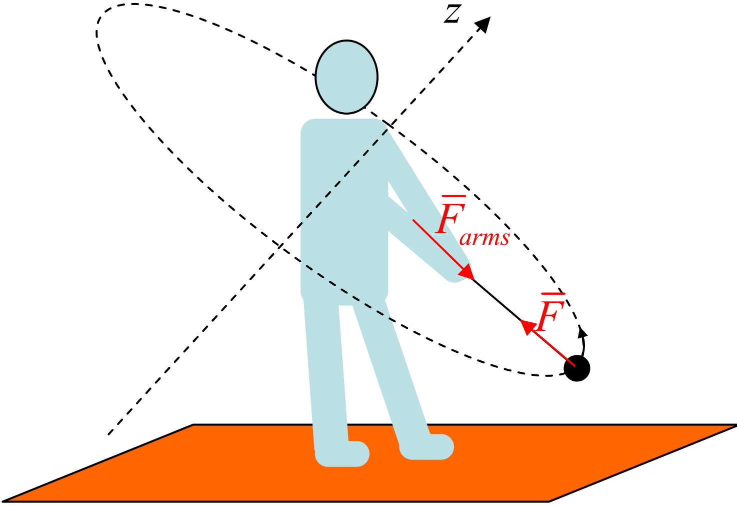
$$\begin{cases} \frac{\partial\overline{r}}{\partial\rho} = (\cos\varphi, \sin\varphi, 0) \\ \frac{\partial\overline{r}}{\partial\varphi} = (-\rho\sin\varphi, \rho\cos\varphi, 0) \\ \frac{\partial\overline{r}}{\partial z} = (0,0,1) \end{cases}$$

$$\begin{cases} h_\rho = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1 \\ h_\varphi = \sqrt{(-\rho\sin\varphi)^2 + (\rho\cos\varphi)^2} = \rho \\ h_z = 1 \end{cases}$$

$$\begin{cases} \hat{e}_\rho = \frac{1}{h_\rho}\frac{\partial\overline{r}}{\partial\rho} = (\cos\varphi, \sin\varphi, 0) = \cos\varphi\hat{e}_x + \sin\varphi\hat{e}_y \\ \hat{e}_\varphi = \frac{1}{h_\varphi}\frac{\partial\overline{r}}{\partial\varphi} = (-\sin\varphi, \cos\varphi, 0) = -\sin\varphi\hat{e}_x + \cos\varphi\hat{e}_y \\ \hat{e}_z = \frac{1}{h_z}\frac{\partial\overline{r}}{\partial z} = (0,0,1) = \hat{e}_z \end{cases}$$

$$\boxed{\begin{aligned} dS_\rho &= \rho d\varphi dz \\ dS_z &= \rho d\varphi d\rho \\ dV &= \rho d\rho d\varphi dz \end{aligned}}$$

TARGET PROBLEM



$$\overline{F}_{arms} = -\overline{F}$$

$$\overline{F} = m\overline{a}$$

$$\overline{a} = \frac{d\overline{v}}{dt} \equiv \dot{\overline{v}}$$

$$\overline{v} = \frac{d\overline{r}}{dt} \equiv \dot{\overline{r}}$$

$$\overline{r} = \rho_0 \hat{e}_\rho + z_0 \hat{e}_z$$ *(in cylindrical coord.)*

$$\overline{F}_{arms} = -m\overline{a} = -m\dot{\overline{v}} = -m\ddot{\overline{r}}$$

$$\ddot{\overline{r}} = \frac{d}{dt}\dot{\overline{r}} = \frac{d}{dt}\left(\overset{=0\ \text{(the length does not change)}}{\dot{\rho}_0}\hat{e}_\rho + \rho_0 \dot{\hat{e}}_\rho + \underset{=0\ \text{(the hammer rotates on a plane z=constant)}}{\dot{z}_0}\hat{e}_z + z_0 \dot{\hat{e}}_z\right) =$$

$$\boxed{\begin{aligned} \dot{\hat{e}}_\rho &= \frac{d}{dt}(\cos\varphi, \sin\varphi, 0) = (-\dot{\varphi}\sin\varphi, \dot{\varphi}\cos\varphi, 0) = \dot{\varphi}(-\sin\varphi, \cos\varphi, 0) = \dot{\varphi}\hat{e}_\varphi \\ \dot{\hat{e}}_z &= \frac{d}{dt}(0,0,1) = 0 \end{aligned}}$$

$$= \frac{d}{dt}\left(\rho_0 \dot{\varphi}\hat{e}_\varphi\right) = \left(\dot{\rho}_0\dot{\varphi}\hat{e}_\varphi + \rho_0\ddot{\varphi}\hat{e}_\varphi + \rho_0\dot{\varphi}\dot{\hat{e}}_\varphi\right) = \left(\dot{\rho}_0\dot{\varphi}\hat{e}_\varphi + \rho_0\ddot{\varphi}\hat{e}_\varphi - \rho_0\dot{\varphi}\dot{\varphi}\hat{e}_\rho\right) = \rho_0\left(\ddot{\varphi}\hat{e}_\varphi - \dot{\varphi}^2\hat{e}_\rho\right)$$

$$\boxed{\dot{\hat{e}}_\varphi = \frac{d}{dt}(-\sin\varphi, \cos\varphi, 0) = (-\dot{\varphi}\cos\varphi, -\dot{\varphi}\sin\varphi, 0) = -\dot{\varphi}(\cos\varphi, \sin\varphi, 0) = -\dot{\varphi}\hat{e}_\rho}$$

$$\overline{F}_{arms} = -m\ddot{\overline{r}} = m\rho_0\left(\dot{\varphi}^2\hat{e}_\rho - \ddot{\varphi}\hat{e}_\varphi\right)$$

WHICH STATEMENT IS WRONG?

1- $\hat{e}_\rho$ **does not depend on the position (yellow)**

2- $\hat{e}_z$ **is constant everywhere (red)**

3- Cylindrical coordinates are appropriate with a cylindrical symmetry (green)

4- Cylindrical coordinates are a curvilinear coordinate system (blue)

TARGET PROBLEM

A vector field that is conservative has a potential: $\overline{E} = \nabla\phi$ *(see theorems in week 2)*

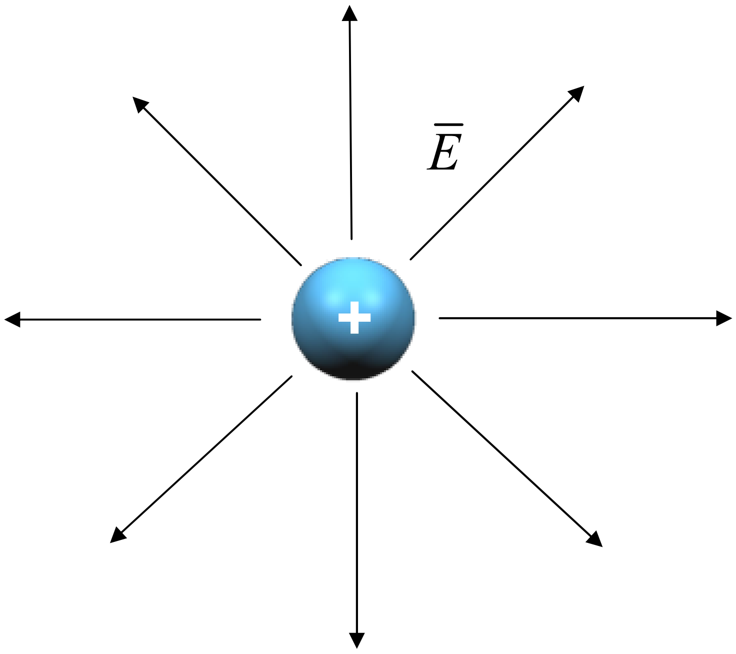
If $\overline{E}$ is the electric field, -ϕ is called electrostatic potential.

$$\left.\begin{aligned} \overline{E} &= \nabla\phi \\ \nabla\cdot\overline{E} &= 0 \end{aligned}\right\} \Longrightarrow \boxed{\nabla\cdot\left(\nabla\phi\right) = 0}$$

Laplace's equation

first Maxwell's equation with no charge

Calculate the electrostatic potential generated outside a spherical charge.



Due to the spherical symmetry, the solution will depend only on the radius: $\phi = \phi(r)$

with $r = \left|\overline{r}\right|$

We need to:
- introduce spherical coordinates
- calculate gradient and divergence in spherical coordinates
- solve the equation

SPHERICAL COORDINATES

spherical coordinates are an
example of curvilinear coordinates

cartesian coord.

$P: \ (x_0, \ y_0, \ z_0)$

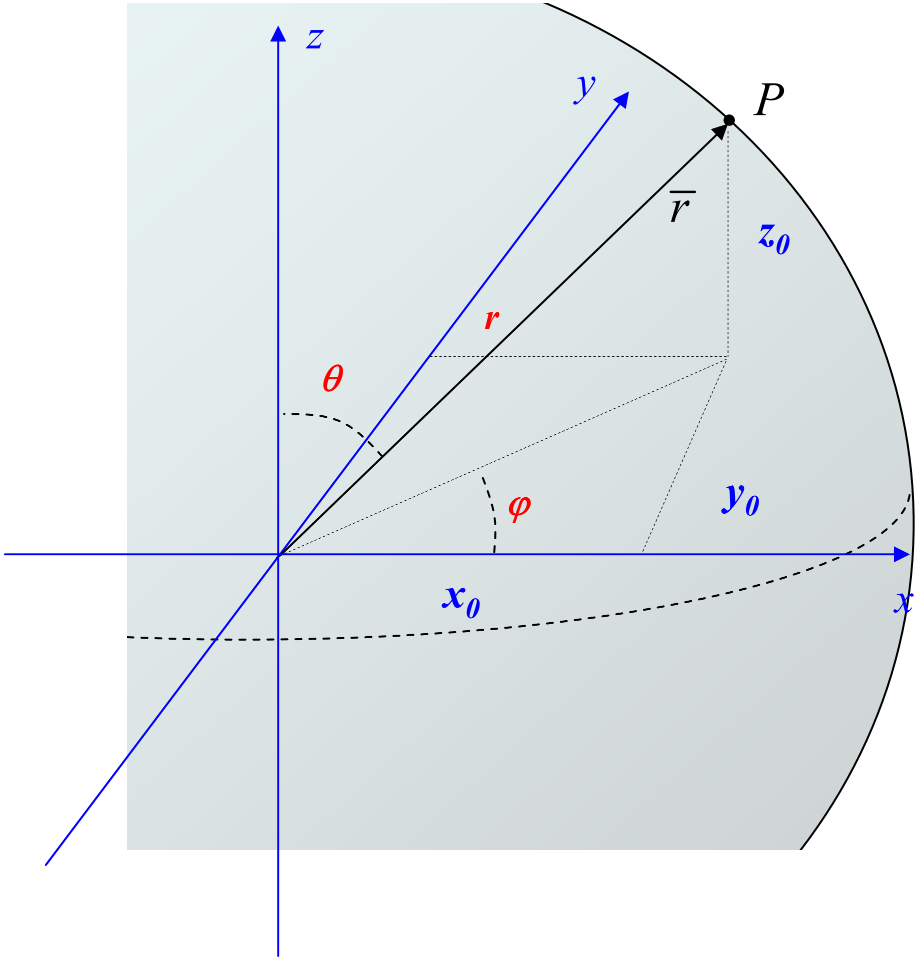
spherical coord.

$P: \ (r, \ \theta, \ \varphi)$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \tan\theta = \dfrac{\sqrt{x^2 + y^2}}{z} \\ \tan\varphi = y / x \end{cases}$$

$$0 \le r \le \infty$$
$$0 \le \theta \le \pi$$
$$0 \le \varphi \le 2\pi$$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$



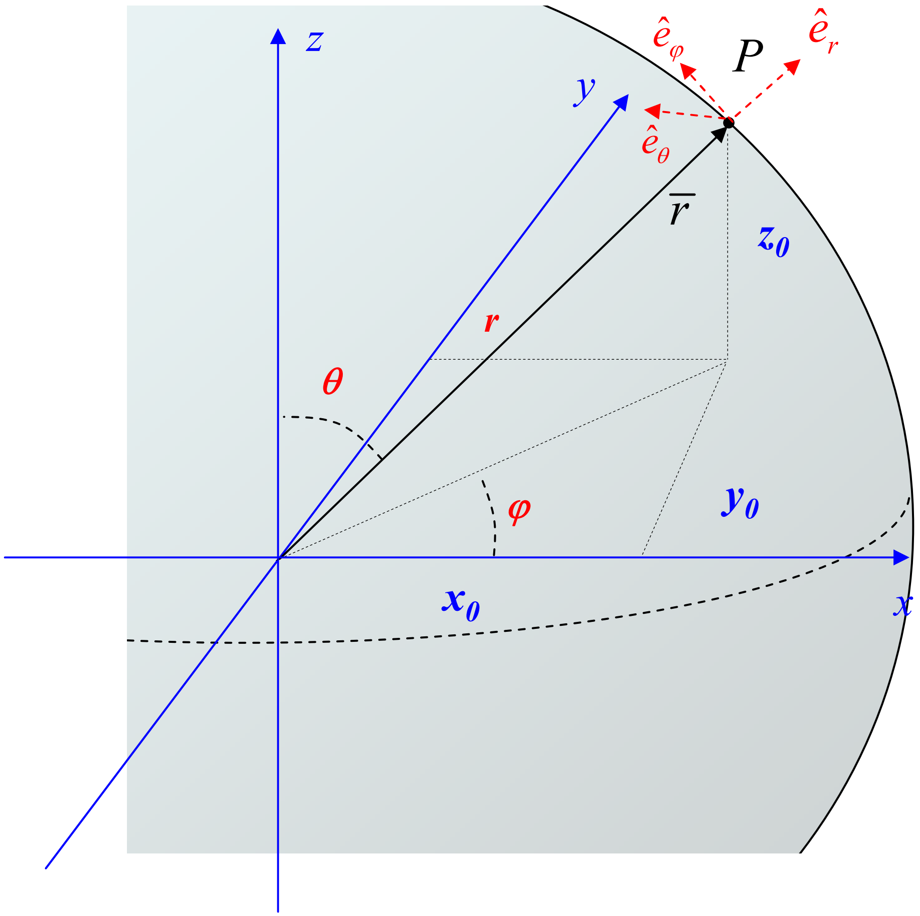
SPHERICAL COORDINATES

Orthonormal basis

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$$\overline{r} = \left(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta\right)$$

$$\hat{e}_i = \frac{1}{h_i}\frac{\partial\overline{r}}{\partial u_i} \quad with \quad h_i = \left|\frac{\partial\overline{r}}{\partial u_i}\right|$$



$$\begin{cases} \dfrac{\partial\overline{r}}{\partial r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\ \dfrac{\partial\overline{r}}{\partial\theta} = (r\cos\theta\cos\varphi, r\cos\theta\sin\varphi, -r\sin\theta) \\ \dfrac{\partial\overline{r}}{\partial\varphi} = (-r\sin\theta\sin\varphi, r\sin\theta\cos\varphi, 0) \end{cases}$$

$$\begin{cases} h_r = \sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta} = 1 \\ h_\theta = \sqrt{\left(r\cos\theta\cos\varphi\right)^2 + \left(r\cos\theta\sin\varphi\right)^2 + \left(r\sin\theta\right)^2} = r \\ h_\varphi = \sqrt{\left(r\sin\theta\sin\varphi\right)^2 + \left(r\sin\theta\cos\varphi\right)^2} = r\sin\theta \end{cases}$$

$$\Rightarrow \begin{cases} \hat{e}_r = \dfrac{1}{h_r}\dfrac{\partial\overline{r}}{\partial r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\ \hat{e}_\theta = \dfrac{1}{h_\theta}\dfrac{\partial\overline{r}}{\partial\theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) \\ \hat{e}_\varphi = \dfrac{1}{h_\varphi}\dfrac{\partial\overline{r}}{\partial\varphi} = (-\sin\varphi, \cos\varphi, 0) \end{cases}$$

$$\boxed{\begin{aligned} dS_r &= r^2\sin\theta d\theta d\varphi \\ dV &= r^2\sin\theta dr d\theta d\varphi \end{aligned}}$$

GRADIENT IN CURVILINEAR COORDINATES

In cartesian coordinates: $grad\,\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = \frac{\partial\phi}{\partial x}\hat{e}_x + \frac{\partial\phi}{\partial y}\hat{e}_y + \frac{\partial\phi}{\partial z}\hat{e}_z$

And in curvilinear coordinates?
We must express $grad\,\phi$ in terms of the curvilinear basis $\hat{e}_1, \hat{e}_2, \hat{e}_3$:

$$grad\,\phi = g_1\hat{e}_1 + g_2\hat{e}_2 + g_3\hat{e}_3$$

since $d\phi = grad\,\phi \cdot d\overline{r}$ and $d\overline{r} = h_1 du_1\hat{e}_1 + h_2 du_2\hat{e}_2 + h_3 du_3\hat{e}_3$

$$d\phi = \left(g_1\hat{e}_1 + g_2\hat{e}_2 + g_3\hat{e}_3\right)\cdot\left(h_1 du_1\hat{e}_1 + h_2 du_2\hat{e}_2 + h_3 du_3\hat{e}_3\right) = g_1 h_1 du_1 + g_2 h_2 du_2 + g_3 h_3 du_3$$

But also, writing ϕ as a function of u_i: $d\phi = \frac{\partial\phi}{\partial u_1}du_1 + \frac{\partial\phi}{\partial u_2}du_2 + \frac{\partial\phi}{\partial u_3}du_3$

Therefore: $g_1 = \frac{1}{h_1}\frac{\partial\phi}{\partial u_1}, \quad g_2 = \frac{1}{h_2}\frac{\partial\phi}{\partial u_2}, \quad g_3 = \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}$

$$\boxed{grad\,\phi = \sum_i \frac{1}{h_i}\frac{\partial\phi}{\partial u_i}\hat{e}_i}$$

GRADIENT IN CURVILINEAR COORDINATES

EXAMPLES

- Cylindrical coordinates $(\rho,\ \varphi,\ z)$:

$$grad\phi = \frac{1}{h_\rho}\frac{\partial\phi}{\partial\rho}\hat{e}_\rho + \frac{1}{h_\varphi}\frac{\partial\phi}{\partial\varphi}\hat{e}_\varphi + \frac{1}{h_z}\frac{\partial\phi}{\partial z}\hat{e}_z = \left(\frac{\partial\phi}{\partial\rho}, \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}, \frac{\partial\phi}{\partial z}\right)$$

- Spherical coordinates $(r,\ \theta,\ \varphi)$:

$$grad\phi = \frac{1}{h_r}\frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{h_\theta}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta + \frac{1}{h_\varphi}\frac{\partial\phi}{\partial\varphi}\hat{e}_\varphi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right)$$

DIVERGENCE IN CURVILINEAR COORD.

$$div\bar{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(A_1h_2h_3\right) + \frac{\partial}{\partial u_2}\left(A_2h_3h_1\right) + \frac{\partial}{\partial u_3}\left(A_3h_1h_2\right)\right]$$

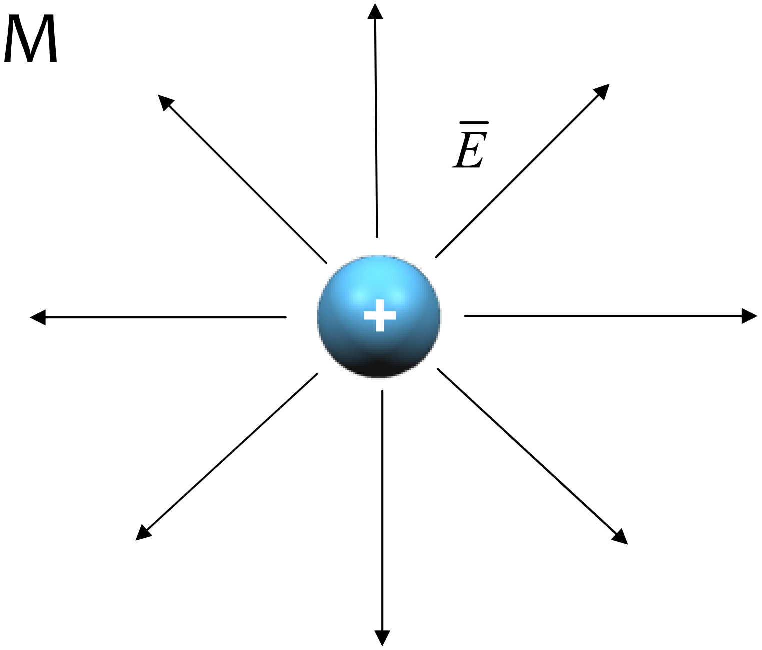
Proof: see theorem 10.4, page 109

CURL IN CURVILINEAR COORD.

$$rot\bar{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

Proof: see theorem 10.5, page 112

TARGET PROBLEM



$$\nabla \cdot \left(\nabla \phi \right) = 0$$

Which can be written as:

$$div\left(grad\phi \right) = 0$$

Due to spherical symmetry

$$\phi = \phi(r)$$

$$\text{with} \;\; r = \left| \overline{r} \right|$$

Due to spherical symmetry, the solution is easy in spherical coordinates

$$grad\phi = \left(\frac{\partial \phi}{\partial r}, \frac{1}{r}\frac{\partial \phi}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \varphi} \right)$$

$$div\overline{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(A_1 h_2 h_3 \right) + \frac{\partial}{\partial u_2}\left(A_2 h_3 h_1 \right) + \frac{\partial}{\partial u_3}\left(A_3 h_1 h_2 \right) \right] = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(A_r r^2 \sin\theta \right) + \frac{\partial}{\partial \theta}\left(A_\theta r \sin\theta \right) + \frac{\partial}{\partial \varphi}\left(r A_\varphi \right) \right]$$

$$div\left(grad\phi \right) = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(\frac{\partial \phi}{\partial r} r^2 \sin\theta \right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\underbrace{\frac{\partial \phi}{\partial \theta}}_{=0} r\sin\theta \right) + \frac{\partial}{\partial \varphi}\left(r\frac{1}{r\sin\theta}\underbrace{\frac{\partial \phi}{\partial \varphi}}_{=0} \right) \right] = \frac{2}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial r^2}$$

$$\phi = \phi(r) \;\; \Rightarrow \textit{No } \theta \textit{ and no } \varphi \textit{ dependence}$$

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} = 0 \quad \Rightarrow \quad \phi(r) = -\frac{c}{r} + d$$

WHICH STATEMENT IS WRONG?

1- The scale factor is necessary to calculate the gradient (yellow)

2- The scale factor is necessary to calculate the divergence (red)

3- Spherical coordinates are a curvilinear coordinate system (blue)

4- Spherical coordinates are appropriate with
a cylindrical symmetry (green)