VEKTORANALYS

Kursvecka 4

övningar

PROBLEM 1

Use "nablaräkning" to verify: $rot\left(\phi\overline{A}\right) = grad\,\phi \times \overline{A} + \phi\, rot\overline{A}$ ID3

SOLUTION

Add the dots

Now nabla can be considered as a vector.

$$rot\left(\phi\overline{A}\right) = \nabla \times \left(\phi\overline{A}\right) = \nabla \times \left(\dot{\phi}\overline{A}\right) + \nabla \times \left(\phi\dot{\overline{A}}\right) =$$

$$= \dot{\phi}\left(\nabla \times \overline{A}\right) + \phi\left(\nabla \times \dot{\overline{A}}\right) =$$

(rewrite the expression in order that only the field with the dot is after the nabla)

$$= -\dot{\phi}\left(\overline{A} \times \nabla\right) + \phi\left(\nabla \times \dot{\overline{A}}\right) =$$

$$= -\overline{A} \times \left(\nabla\dot{\phi}\right) + \phi\left(\nabla \times \dot{\overline{A}}\right) =$$

$$= \left(\nabla\phi\right) \times \overline{A} + \phi\left(\nabla \times \overline{A}\right)$$

PROBLEM 2

Use "nablaräkning" to verify: $div\, rot\overline{A} = 0$ ID8

SOLUTION

Now nabla can be considered as a vector.

$$div\, rot\overline{A} = \nabla \cdot \left(\nabla \times \dot{\overline{A}}\right) =$$

Because: $\overline{n} \cdot \left(\overline{n} \times \overline{a}\right) = \overline{a} \cdot \left(\underbrace{\overline{n} \times \overline{n}}_{=0}\right)$

$$= \dot{\overline{A}} \cdot \left(\nabla \times \nabla\right) = \left(\nabla \times \nabla\right) \cdot \dot{\overline{A}} = 0$$

PROBLEM 3

Use “nablaräkning” to verify: $\left(\bar{A}\times\nabla\right)\times\bar{A}=\frac{1}{2}\nabla A^2-\bar{A}\left(\nabla\cdot\bar{A}\right)$

SOLUTION

$$\left(\bar{A}\times\nabla\right)\times\bar{A}=\left(\bar{A}\times\nabla\right)\times\underset{\cdot}{\bar{A}}=$$

$$\left(\bar{a}\times\bar{n}\right)\times\bar{b}=\left(\bar{a}\cdot\bar{b}\right)\bar{n}-\left(\bar{n}\cdot\bar{b}\right)\bar{a}$$

$$=\nabla\left(\bar{A}\cdot\underset{\cdot}{\bar{A}}\right)-\bar{A}\left(\nabla\cdot\underset{\cdot}{\bar{A}}\right)=$$

$$\nabla A^2=\nabla\left(\bar{A}\cdot\bar{A}\right)=\nabla\left(\underset{\cdot}{\bar{A}}\cdot\bar{A}\right)+\nabla\left(\bar{A}\cdot\underset{\cdot}{\bar{A}}\right)=2\nabla\left(\bar{A}\cdot\underset{\cdot}{\bar{A}}\right)$$

$$\Rightarrow\quad\nabla\left(\bar{A}\cdot\underset{\cdot}{\bar{A}}\right)=\frac{1}{2}\nabla A^2$$

$$=\frac{1}{2}\nabla A^2-\bar{A}\left(\nabla\cdot\bar{A}\right)$$

PROBLEM 4

Use “nablaräkning” to simplify: $\bar{B}=\bar{A}\times\left(\nabla\times\bar{A}\right)-\left(\bar{A}\times\nabla\right)\times\bar{A}$

SOLUTION

$$\bar{B}=\bar{A}\times\left(\nabla\times\bar{A}\right)-\left(\bar{A}\times\nabla\right)\times\bar{A}=\bar{A}\times\left(\nabla\times\underset{\cdot}{\bar{A}}\right)-\left(\bar{A}\times\nabla\right)\times\underset{\cdot}{\bar{A}}=\bar{A}\times\left(\nabla\times\underset{\cdot}{\bar{A}}\right)-\underset{\cdot}{\bar{A}}\times\left(\bar{A}\times\nabla\right)=$$

using $\bar{a}\times\left(\bar{b}\times\bar{c}\right)=\bar{b}\left(\bar{a}\cdot\bar{c}\right)-\bar{c}\left(\bar{a}\cdot\bar{b}\right)$

$$=\nabla\left(\bar{A}\cdot\underset{\cdot}{\bar{A}}\right)-\underset{\cdot}{\bar{A}}\left(\bar{A}\cdot\nabla\right)-\nabla\left(\underset{\cdot}{\bar{A}}\cdot\bar{A}\right)+\bar{A}\left(\underset{\cdot}{\bar{A}}\cdot\nabla\right)=$$

$$=\nabla\left(\underset{\cdot}{\bar{A}}\cdot\bar{A}\right)-\left(\bar{A}\cdot\nabla\right)\underset{\cdot}{\bar{A}}-\nabla\left(\underset{\cdot}{\bar{A}}\cdot\bar{A}\right)+\bar{A}\left(\nabla\cdot\underset{\cdot}{\bar{A}}\right)=$$

$$=-\left(\bar{A}\cdot\nabla\right)\underset{\cdot}{\bar{A}}+\bar{A}\left(\nabla\cdot\underset{\cdot}{\bar{A}}\right)=\bar{A}\left(\nabla\cdot\bar{A}\right)-\left(\bar{A}\cdot\nabla\right)\bar{A}$$

PROBLEM 5

Calculate $\varepsilon_{ijk}\varepsilon_{ljk}$

SOLUTION

We know that $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

Therefore:

We re-arrange the suffixes to have an expression similar to

the same expression with m=j

$$\varepsilon_{ijk}\varepsilon_{ljk} = \varepsilon_{ijk}\varepsilon_{klj} = \delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl} = \delta_{il}3 - \delta_{il} = 2\delta_{il}$$

even permutations does NOT change the sign

Remember that $\delta_{ii} = 3$

Remember that $\delta_{km}p_m = p_k$

PROBLEM 6

Prove $\bar{a}\times\left(\bar{b}\times\bar{c}\right)=\left(\bar{a}\cdot\bar{c}\right)\bar{b}-\left(\bar{a}\cdot\bar{b}\right)\bar{c}$

using the suffix notation.

SOLUTION

We know that the *i-component* of the cross product can be written as: $\left(\bar{a}\times\bar{b}\right)_i=\varepsilon_{ijk}a_jb_k$
Therefore:

$$\left(\bar{b}\times\bar{c}\right)_k=\varepsilon_{klm}b_lc_m$$

$$\bar{a}\times\left(\bar{b}\times\bar{c}\right)_i=\varepsilon_{ijk}a_j\left(\bar{b}\times\bar{c}\right)_k=$$

$$\varepsilon_{ijk}\varepsilon_{klm}=\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}$$

$$=\varepsilon_{ijk}a_j\varepsilon_{klm}b_lc_m=$$

$$\delta_{ij}d_{kj}=d_{ki}$$

$$=\varepsilon_{ijk}\varepsilon_{klm}a_jb_lc_m=$$

$$=\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)a_jb_lc_m=\delta_{il}\delta_{jm}a_jb_lc_m-\delta_{im}\delta_{jl}a_jb_lc_m=$$

$$=a_mb_ic_m-a_lb_lc_i=\left(\bar{a}\cdot\bar{c}\right)b_i-\left(\bar{a}\cdot\bar{b}\right)c_i$$

PROBLEM 7

Use "indexräkning" to verify: $div\left(\overline{A}\times\overline{B}\right)=\left(rot\overline{A}\right)\cdot\overline{B}-\left(rot\overline{B}\right)\cdot\overline{A}$ **ID4**

SOLUTION

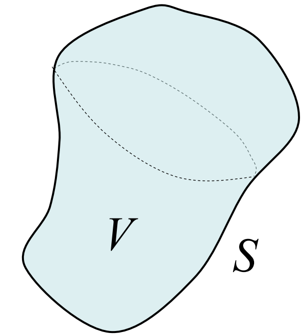
$$div\left(\overline{A}\times\overline{B}\right)=\left(\overline{A}\times\overline{B}\right)_{,i}=\left(\varepsilon_{ijk}A_jB_k\right)_{,i}=\varepsilon_{ijk}\left(A_jB_k\right)_{,i}=\varepsilon_{ijk}\left(A_{j,i}B_k+A_jB_{k.i}\right)=$$

$$=\varepsilon_{ijk}A_{j,i}B_k+\varepsilon_{ijk}A_jB_{k.i}=\varepsilon_{kij}A_{j,i}B_k-\varepsilon_{jik}B_{k.i}A_j=\left(rot\overline{A}\right)\cdot\overline{B}-\left(rot\overline{B}\right)\cdot\overline{A}$$

remember that: $\left(\nabla\times\overline{A}\right)_i=\varepsilon_{ijk}A_{k,j}$

PROBLEM 8

Show that: $$\iiint_V \overline{r} \times rot\overline{A} dV = 2\iiint_V \overline{A} dV$$



if on the boundary surface S of V the vector field is $\overline{A} = 0$

SOLUTION

Let's consider only the *i-th* component of the left hand side:

$$\hat{e}_i \cdot \iiint_V \overline{r} \times rot\overline{A}\, dV = \iiint_V \hat{e}_i \cdot \left(\overline{r} \times rot\overline{A}\right) dV = \iiint_V \left(rot\overline{A}\right) \cdot \left(\hat{e}_i \times \overline{r}\right) dV =$$

$$\overline{a} \cdot \left(\overline{b} \times \overline{c}\right) = \overline{c} \cdot \left(\overline{a} \times \overline{b}\right)$$

and we substitute

To continue, we must remember that: $\nabla \cdot \left(\overline{a} \times \overline{b}\right) = \overline{b} \cdot (\nabla \times \overline{a}) - \overline{a} \cdot \left(\nabla \times \overline{b}\right)$ (**ID4**)

therefore, $$\nabla \cdot \left(\overline{A} \times \left(\hat{e}_i \times \overline{r}\right)\right) = \left(\hat{e}_i \times \overline{r}\right) \cdot \left(\nabla \times \overline{A}\right) - \overline{A} \cdot \nabla \times \left(\hat{e}_i \times \overline{r}\right)$$

re-arranging the terms: $$\left(rot\overline{A}\right) \cdot \left(\hat{e}_i \times \overline{r}\right) = div\left(\overline{A} \times \left(\hat{e}_i \times \overline{r}\right)\right) + \overline{A} \cdot rot\left(\hat{e}_i \times \overline{r}\right)$$

$$= \iiint_V \left[div\left(\overline{A}\times(\hat{e}_i\times\overline{r})\right) + \overline{A}\cdot rot(\hat{e}_i\times\overline{r}) \right] dV =$$

$$= \iiint_V div\left(\overline{A}\times(\hat{e}_i\times\overline{r})\right) dV + \iiint_V \overline{A}\cdot rot(\hat{e}_i\times\overline{r})\, dV =$$

Generalized Gauss theorem

ID5

$$rot(\hat{e}_i\times\overline{r}) = (\overline{r}\cdot\nabla)\hat{e}_i - (\hat{e}_i\cdot\nabla)\overline{r} + \hat{e}_i(\nabla\cdot\overline{r}) - \overline{r}(\nabla\cdot\hat{e}_i)$$

$$= \quad 0 \quad - \quad \frac{\partial\overline{r}}{\partial x_i} \quad + \quad 3\hat{e}_i \quad - \quad 0 \quad = 2\hat{e}_i$$

$$= \underbrace{\iint_S \left(\overline{A}\times(\hat{e}_i\times\overline{r})\right)\cdot d\overline{S}}_{=0} + \iiint_V \overline{A}\cdot 2\hat{e}_i\, dV = 2\iiint_V A_i\, dV$$

Because on S, $\overline{A}=0$

So, we have: $$\hat{e}_i\cdot\iiint_V \overline{r}\times rot\overline{A}\, dV = 2\iiint_V A_i\, dV$$