VEKTORANALYS

Kursvecka 3

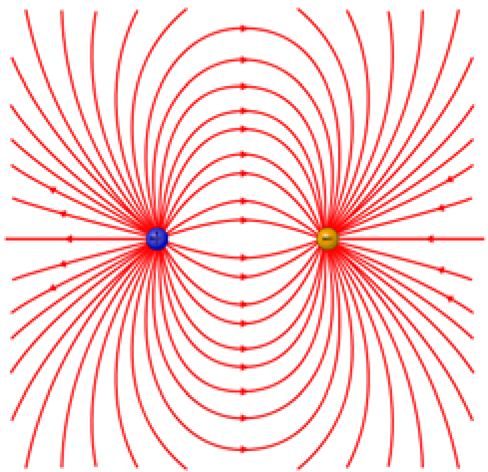
GAUSS' THEOREM
and
STOKES' THEOREM

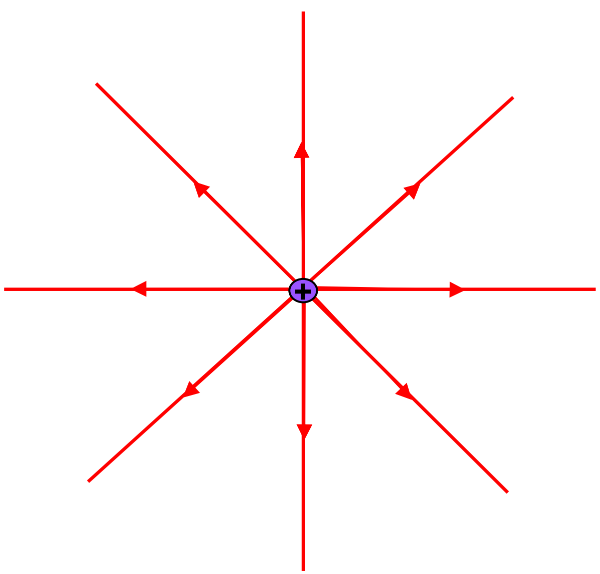
Kapitel 6-7
Sidor 51-82

TARGET PROBLEM

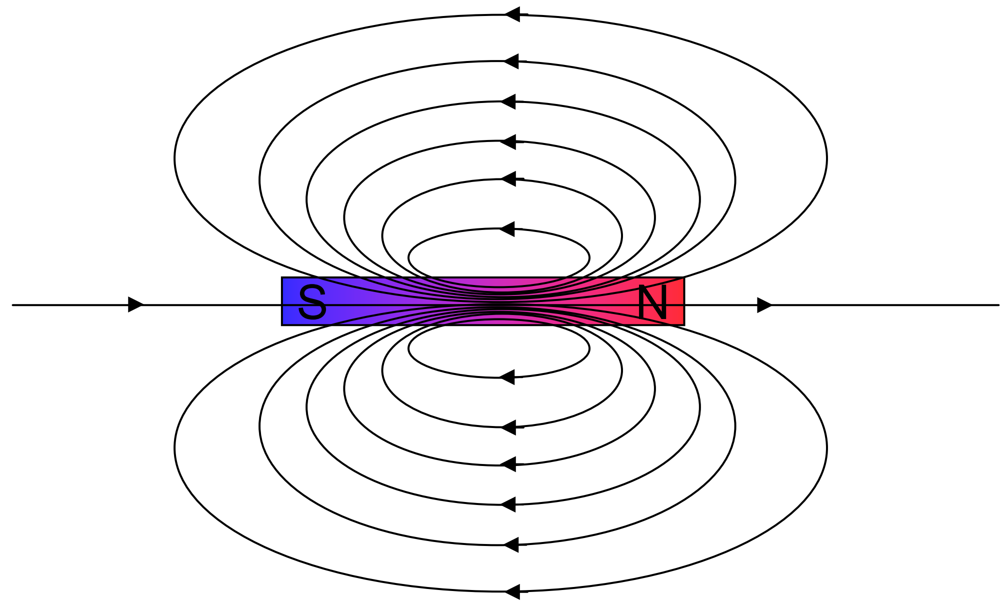
Do magnetic monopoles exist?

ELECTRIC FIELD





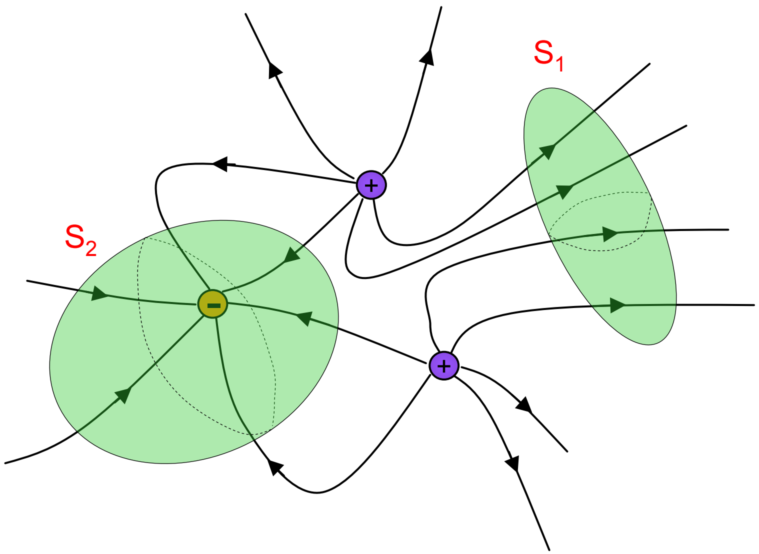
MAGNETIC FIELD



?

TARGET PROBLEM

Let's consider some ELECTRIC CHARGES and two closed surfaces, S_1 and S_2



S_1 does not contain any charge.
It has no sources and no sinks:
no field lines destroyed and
no field lines created inside S1

$$\iint_{S_1} \overline{E} \cdot d\overline{S} = 0$$

S_2 do contain a charge.
It has a sink. Field lines are destroyed inside S2

$$\iint_{S_2} \overline{E} \cdot d\overline{S} \neq 0$$

If $\iint_{S} \overline{B} \cdot d\overline{S} \neq 0$ then magnetic monopoles exist!!

To calculate this integral we need:

- to introduce the divergence of a vector field $\overline{A}$, $div\,\overline{A}$
- the Gauss' theorem $\iint_{S} \overline{A} \cdot d\overline{S} = \iiint_{V} div\overline{A}dV$
- one of the Maxwell equations: $div\,\overline{B} = 0$

THE DIVERGENCE

In cartesian coordinates, the divergence of a vector field $\overline{A}$ is:

DEFINITION

$$div\overline{A} \equiv \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad (1)$$

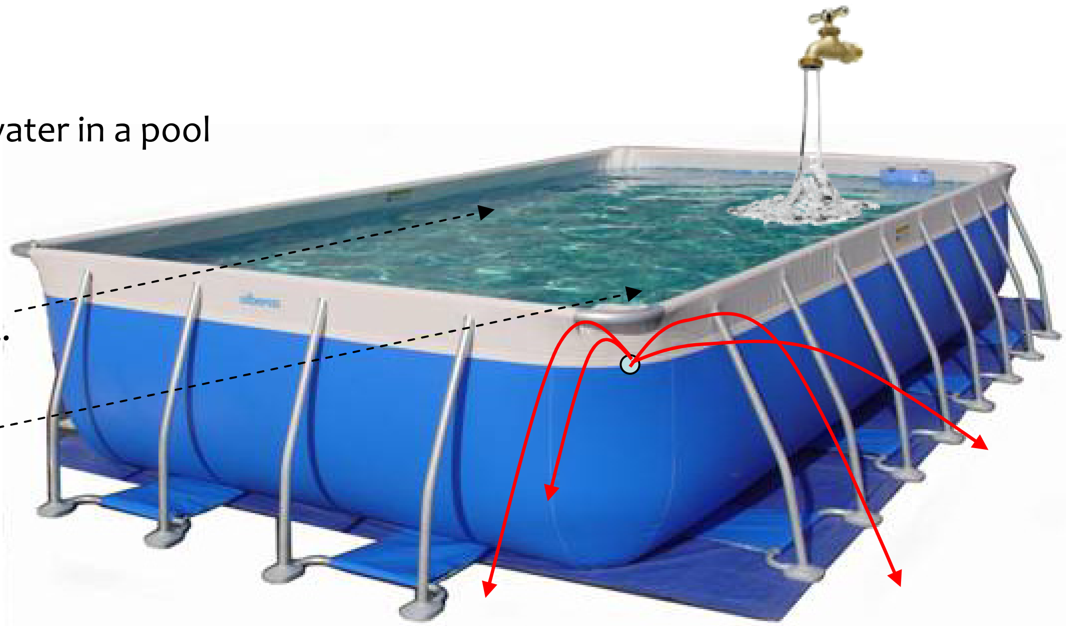
It is a measure of how much the field diverges (*or converges*) from (*to*) a point.

Let's suppose that $\overline{A}$ is the velocity field of the water in a pool

Let's make a hole in the pool…

Far from the hole, the velocity is almost constant.
The divergence is almost zero.

Close to the hole the velocity is changing a lot.
The divergence is high.



If there is no hole $\Rightarrow$ $div\,\overline{A} = 0$
(and no source…)

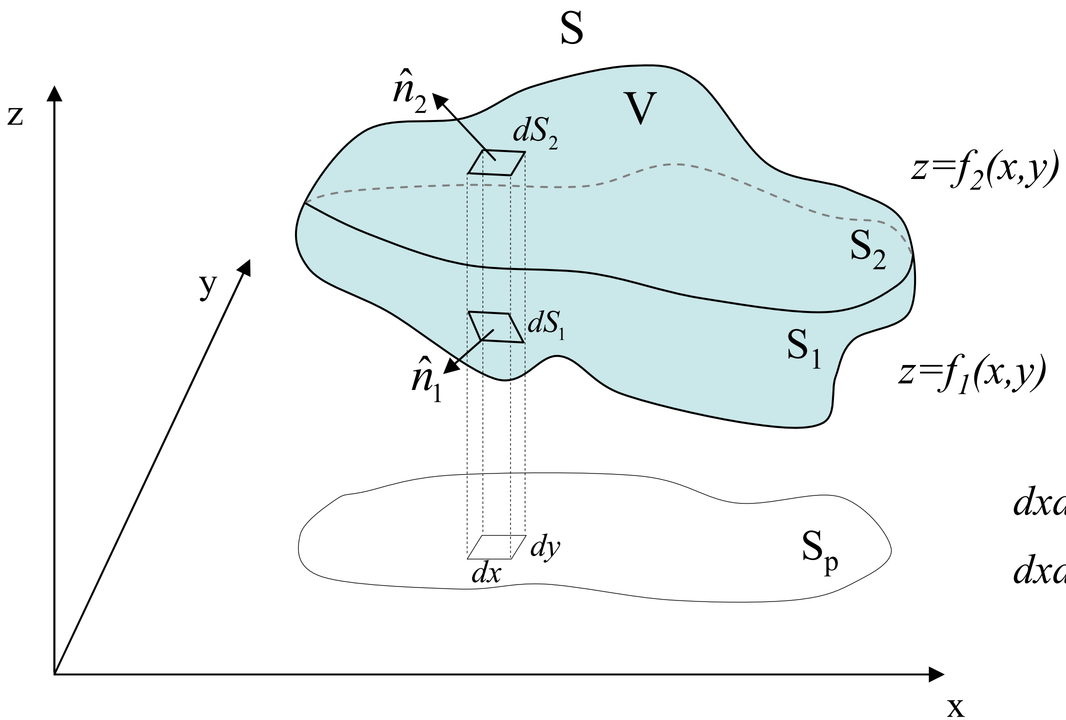
The divergence is also a measure of sources or sinks

(this concept will be more clear at the end of the lesson)

THE GAUSS' THEOREM

$$\iint_S \overline{A} \cdot d\overline{S} = \iiint_V div\overline{A}dV \qquad (2)$$

where **S is a closed surface** that forms the boundary of the volume V and $\overline{A}$ is a continuously differentiable vector field defined on V.



$$dxdy = dS_2\hat{n}_2 \cdot \hat{e}_z = d\overline{S}_2 \cdot \hat{e}_z$$

$$dxdy = -dS_1\hat{n}_1 \cdot \hat{e}_z = -d\overline{S}_1 \cdot \hat{e}_z$$

THE GAUSS' THEOREM

PROOF

$$\iiint_V div\bar{A}dV = \iiint_V \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) dxdydz =$$

$$\iiint_V \frac{\partial A_x}{\partial x} dxdydz + \iiint_V \frac{\partial A_y}{\partial y} dxdydz + \iiint_V \frac{\partial A_z}{\partial z} dxdydz$$

Let's calculate the last term:

$$\iiint_V \frac{\partial A_z}{\partial z} dxdydz = \iint_{S_p} dxdy \int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial A_z}{\partial z} dz = \iint_{S_p} \left[A_z(x,y,f_2(x,y)) - A_z(x,y,f_1(x,y)) \right] dxdy =$$

$dxdy$ is the projection on S_p of the small element surfaces on dS_1 and dS_2.

Therefore: $dxdy = -\hat{e}_z \cdot \hat{n}_1 dS_1 = \hat{e}_z \cdot \hat{n}_2 dS_2$

$$= \iint_{S_2} A_z(x,y,f_2(x,y))\hat{e}_z \cdot \hat{n}_2 dS_2 + \iint_{S_1} A_z(x,y,f_1(x,y))\hat{e}_z \cdot \hat{n}_1 dS_1 = \iint_S A_z \hat{e}_z \cdot \hat{n} dS$$

Which means:

$$\iiint_V \frac{\partial A_z}{\partial z} dV = \iint_S A_z \hat{e}_z \cdot \hat{n} dS \quad (3)$$

THE GAUSS' THEOREM

PROOF

In the same way we get:

$$\iiint\limits_V \frac{\partial A_x}{\partial x} dV = \iint\limits_S A_x \hat{e}_x \cdot \hat{n} dS \qquad (4)$$

$$\iiint\limits_V \frac{\partial A_y}{\partial y} dV = \iint\limits_S A_y \hat{e}_y \cdot \hat{n} dS \qquad (5)$$

Adding together equations (3), (4) and (5) we finally obtain:

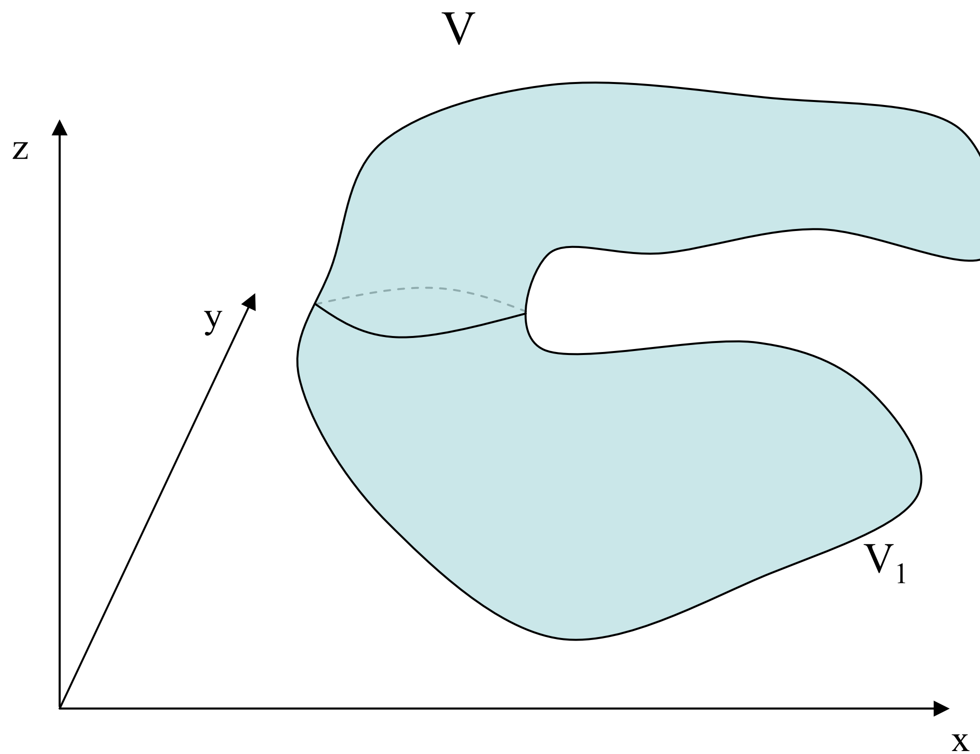
$$\iiint\limits_V div\bar{A} dV = \iiint\limits_V \frac{\partial A_x}{\partial x} dxdydz + \iiint\limits_V \frac{\partial A_y}{\partial y} dxdydz + \iiint\limits_V \frac{\partial A_z}{\partial z} dxdydz =$$

$$\iint\limits_S A_x \hat{e}_x \cdot \hat{n} dS + \iint\limits_S A_y \hat{e}_y \cdot \hat{n} dS + \iint\limits_S A_z \hat{e}_z \cdot \hat{n} dS = \iint\limits_S \bar{A} \cdot d\bar{S}$$

THE GAUSS' THEOREM

PROOF

What if we consider a more complicated volume?



We divide the volume V in smaller and "simpler" volumes

$$V = V_1 + V_2 + \ldots = \sum_i V_i$$

$$\iiint_V divA dV = \sum_i \iiint_{V_i} divA dV =$$

$$\sum_i \iint_{S_i} A \cdot dS = \iint_S A \cdot dS$$

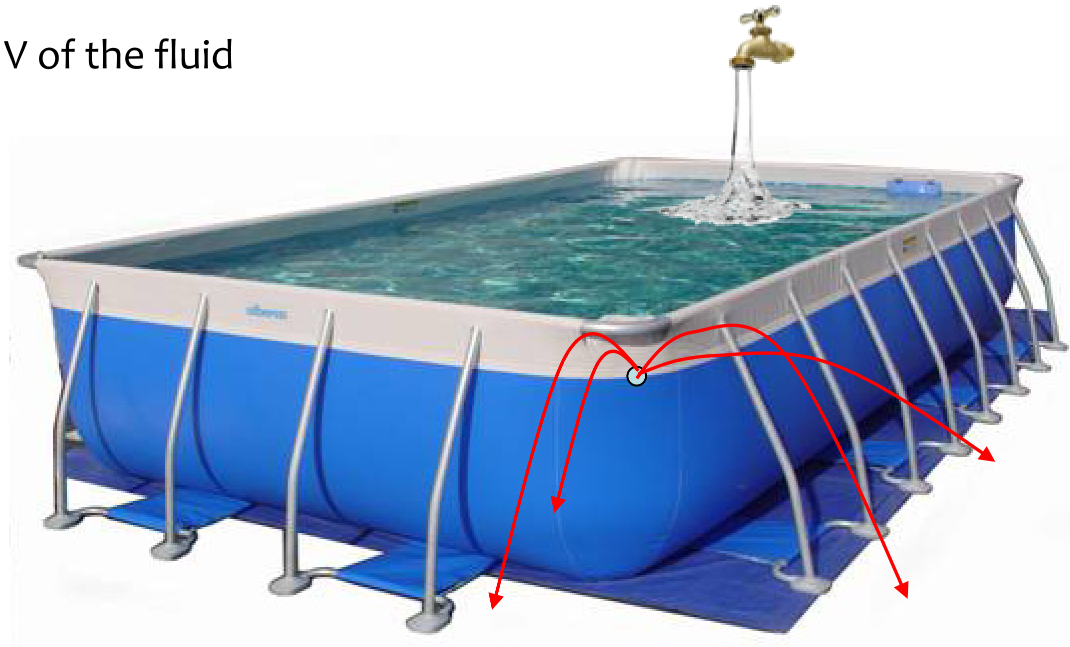
PHYSICAL INTERPRETATION

Suppose that $\overline{v}(\overline{r})$ is the velocity field of a fluid (homogeneous and incompressible)

Let's apply the Gauss' theorem to a volume V of the fluid

$$\iint_S \overline{v} \cdot d\overline{S} = \iiint_V div(\overline{v}) dV$$

It is the fluid volume per second [m^3/s] that moves out *(in)* from the closed surface S



If there are no sinks and no sources, then the amount of fluid that enters in S is equal to the amount of fluid that exits from S. This implies that the flow $\iint_S \overline{v} \cdot d\overline{S}$ is zero. Therefore, $div(\overline{v}) = 0$

$div(\overline{v}) = 0 \Rightarrow$ No sink and no source

$div(\overline{v}) < 0 \Rightarrow$ flux is destroyed and there is a sink

$div(\overline{v}) > 0 \Rightarrow$ flux is created and there is a source

TARGET PROBLEM

Do magnetic monopoles exist?

If $\iint_S \bar{B} \cdot d\bar{S} \neq 0$ then magnetic monopoles exist!!

We can use the Gauss' theorem and the Maxwell equation to calculate $\iint_S \bar{B} \cdot d\bar{S}$

Gauss ⇒ $\iint_S \bar{B} \cdot d\bar{S} = \iiint_V div\bar{B}dV$

Maxwell ⇒ $div\bar{B} = 0$

$$\left.\begin{array}{l} \iint_S \bar{B} \cdot d\bar{S} = \iiint_V div\bar{B}dV \\ div\bar{B} = 0 \end{array}\right\} \Rightarrow \iint_S \bar{B} \cdot d\bar{S} = 0$$

Magnetic monopoles do NOT exist

WHICH STATEMENT IS WRONG?

1- The divergence of a vector field is a scalar (yellow)

2- The divergence is related to a measurement of the flux (red)

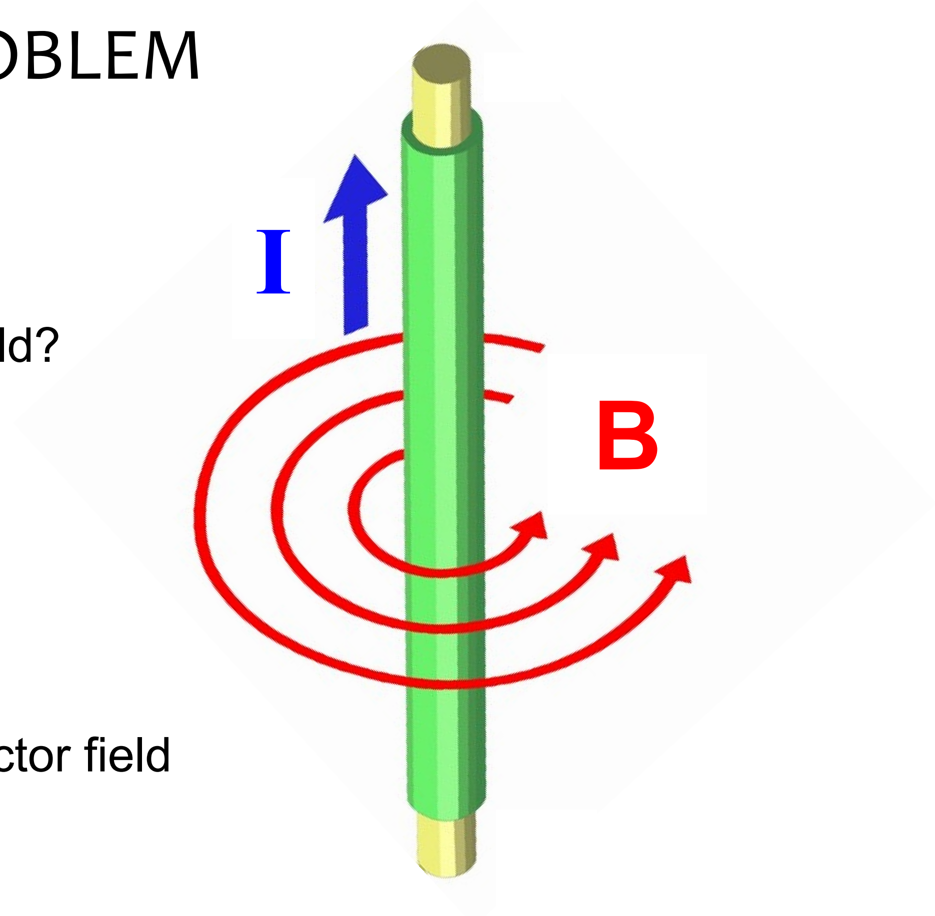
3- The Gauss' theorem translates a surface integral
into a volume integral (green)

4- The Gauss' theorem can be applied also to a non closed surface (blue)

VEKTORANALYS

STOKES’ THEOREM

TARGET PROBLEM



- A current **I** flows in a conductor
- How can we calculate the magnetic field?

We need:

- Definition of the "curl" (or rotor) of a vector field

$$rot\,\overline{A}$$

- The Stokes' theorem

$$\oint_L \overline{A} \cdot dr = \iint_S rot\overline{A} \cdot d\overline{S}$$

- A law that relates the current with the magnetic field: the fourth Maxwell's equation (with static electric field): $rot\overline{B} = \mu_0 \overline{j}$

THE CURL $rot\,\overline{A}$

DEFINITION (in cartesian coordinate)

$$rot\,\overline{A} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \quad \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \quad \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

rot stands for “rotation”

In fact, the curl is a measure of how much the direction of a vector field changes in space, i.e. how much the field “rotates”.

In every point of the space, $rot\overline{A}$ is a vector whose length and direction characterize the rotation of the field $\overline{A}$.

The direction is the axis of rotation of $\overline{A}$

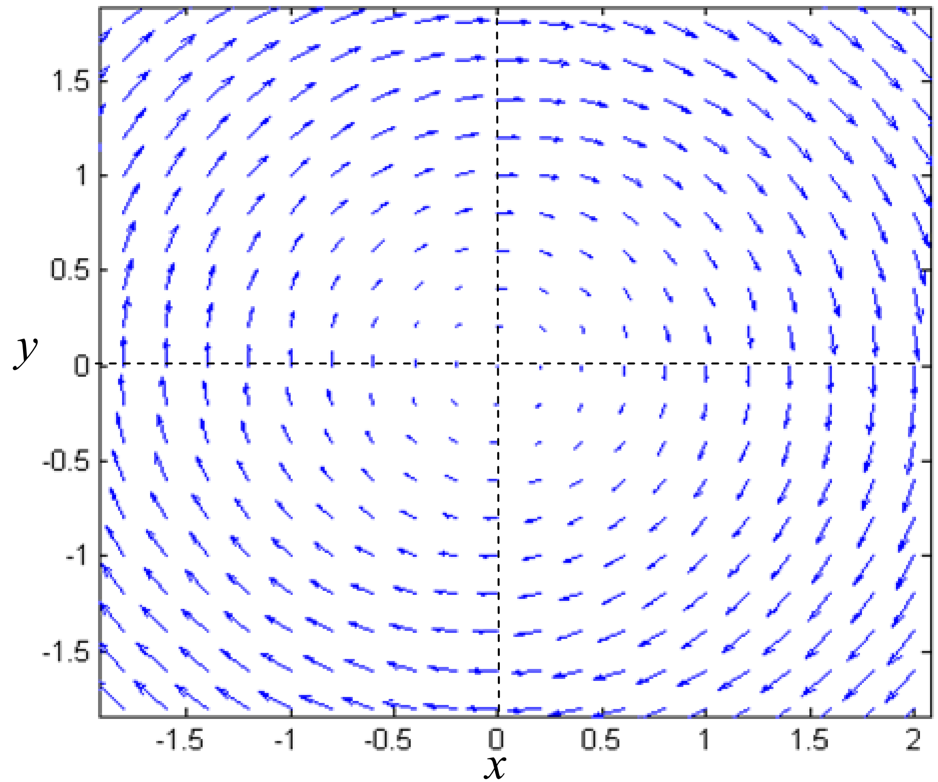
The magnitude is the magnitude of rotation of $\overline{A}$

THE CURL $rot\bar{A}$

EXAMPLE

$$\bar{A}(x,y,z) = (y,-x,0)$$

Evident rotation ⇒



$$rot\bar{A} = (0,0,-2)$$

Direction: the direction is the axis of rotation, i.e. perpendicular to the plane of the figure

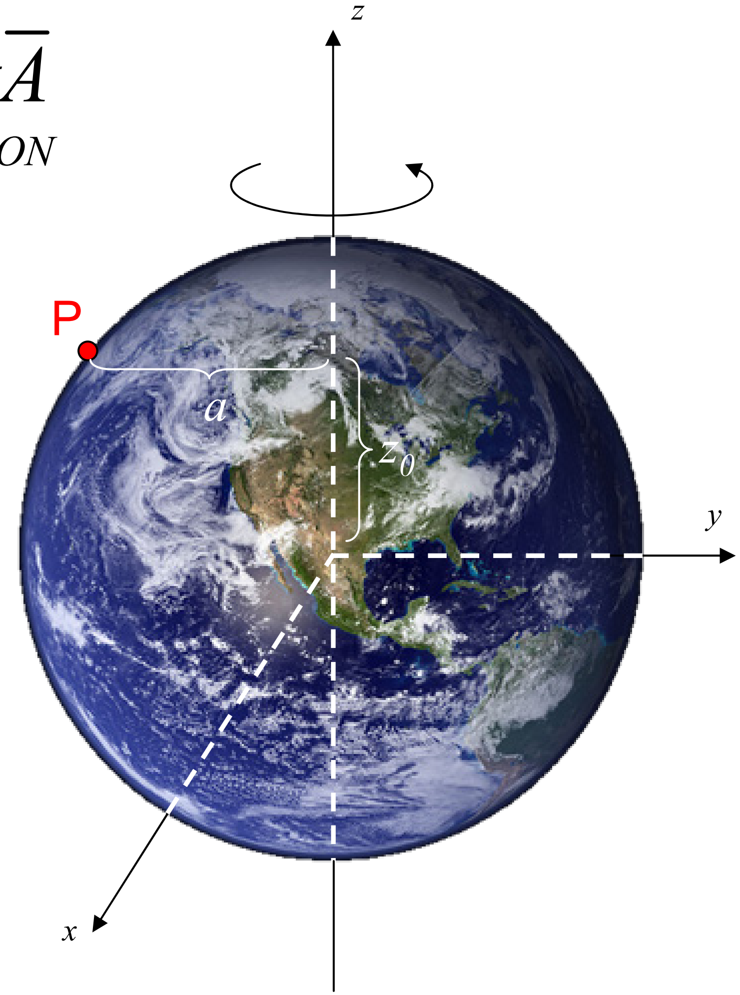
The sign (negative, in this case) is determined by the right-hand rule

Magnitude: the amount of rotation

In this example, it is constant and independent of the position, i.e. the amount of rotation is the same at any point.

THE CURL $rot\bar{A}$

PHYSICAL INTERPRETATION



Consider the rotation of a rigid body around the z-axis

The coordinates of a point P on the body located at the distance a from the z-$axis$ and at $z=z_0$ changes in time:

$$x(t) = a\cos\omega t$$

$$y(t) = a\sin\omega t$$

$$z = z_0$$

The velocity of the point P is:

$$\left.\begin{array}{l} v_x(t) = -a\omega\sin\omega t = -\omega y(t) \\ v_y(t) = a\omega\cos\omega t = \omega x(t) \\ v_z = 0 \end{array}\right\} \Rightarrow \bar{v} = (-\omega y, \omega x, 0)$$

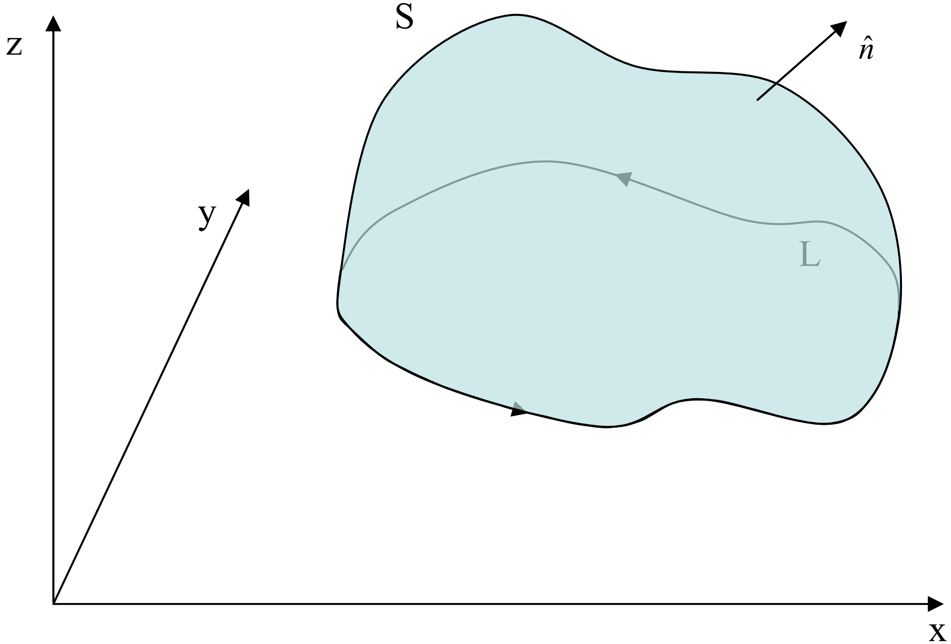
Therefore $rot\,\bar{v} = (0, 0, 2\omega)$

$$\bar{\omega} = \frac{1}{2} rot\,\bar{v}$$

THE STOKES' THEOREM

$$\oint_L \overline{A} \cdot d\overline{r} = \iint_S rot\overline{A} \cdot d\overline{S}$$

where $\overline{A}$ is a vector field, ***L* is a closed curve** and S is a surface whose boundary is defined by L.
$\overline{A}$ must be continuously differentiable on S



THE STOKES' THEOREM

PROOF

Five steps:

1. We divide S in "many" "smaller" *(infinitesimal)* surfaces:

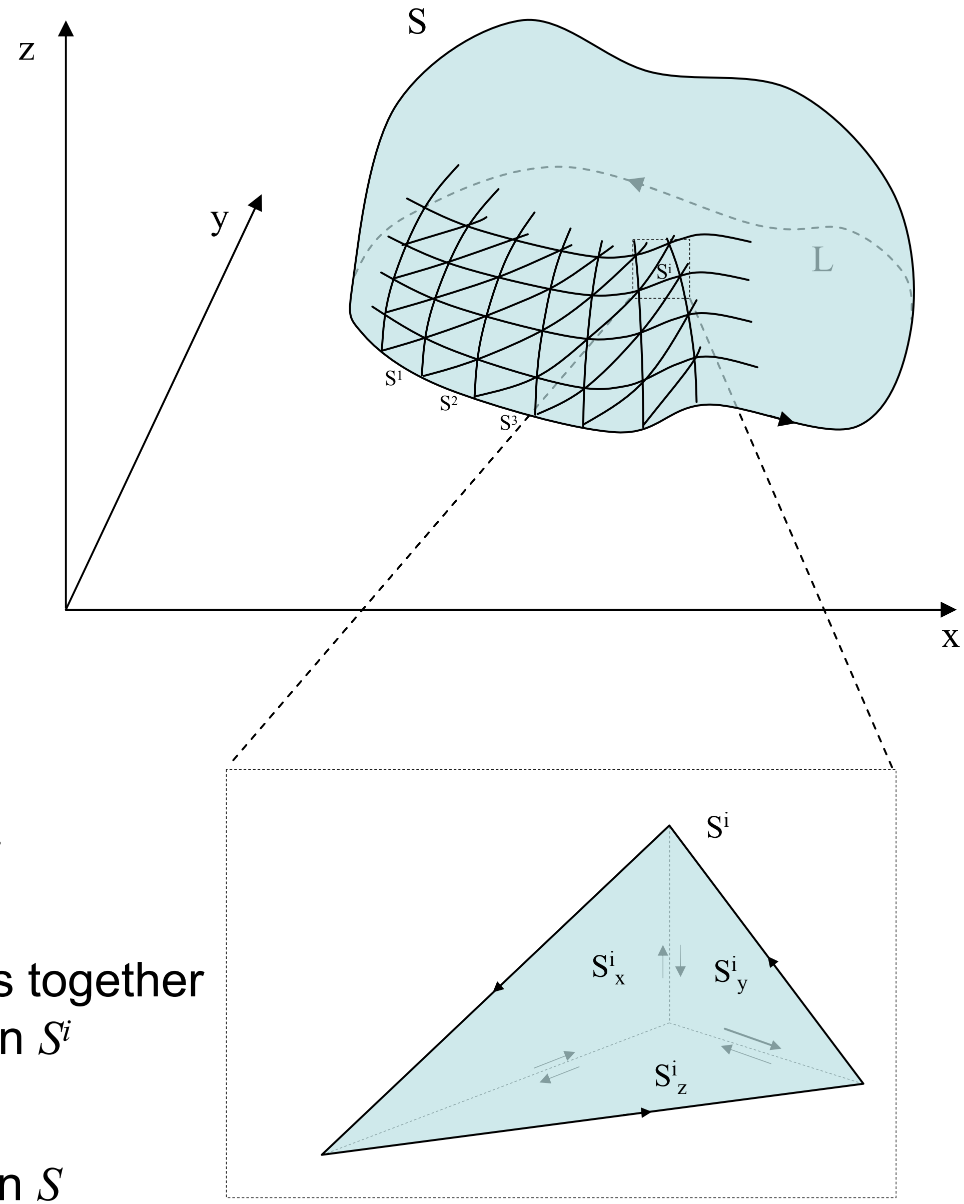
$$S = \sum_i S^i$$

2. We project S^i on:
   the xy-plane S^i_z
   the yz-plane S^i_x
   the xz-plane S^i_y

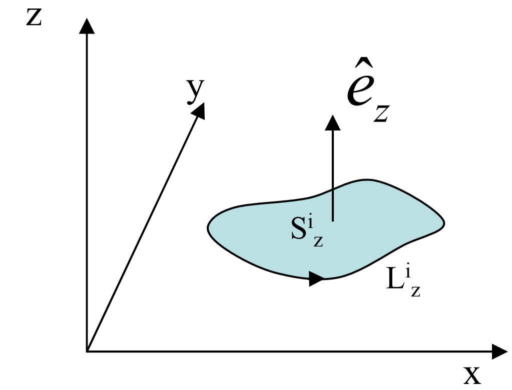
3. We prove the Stokes' theorem on S^i_z,
   (the only difficult part)

4. We add the results for the projections together and we obtain the Stokes' theorem on S^i

5. We add the results for S^i together and we obtain the Stokes' theorem on S



THE STOKES' THEOREM



PROOF
Let's consider the plane surface S^i_z
located in the xy-plane (i.e. z=constant=z_0)
with boundary defined by the curve L^i_z

Let's calculate $\oint_{L^i_z} \bar{A} \cdot d\bar{r}$

$$\oint_{L^i_z} \bar{A} \cdot d\bar{r} = \oint_{L^i_z} \underbrace{A_x(x,y,z_0)dx}_{\text{Term 1}} + \underbrace{A_y(x,y,z_0)dy}_{\text{Term 2}} + \underbrace{A_z(x,y,z_0)dz}_{\text{Term 3}}$$

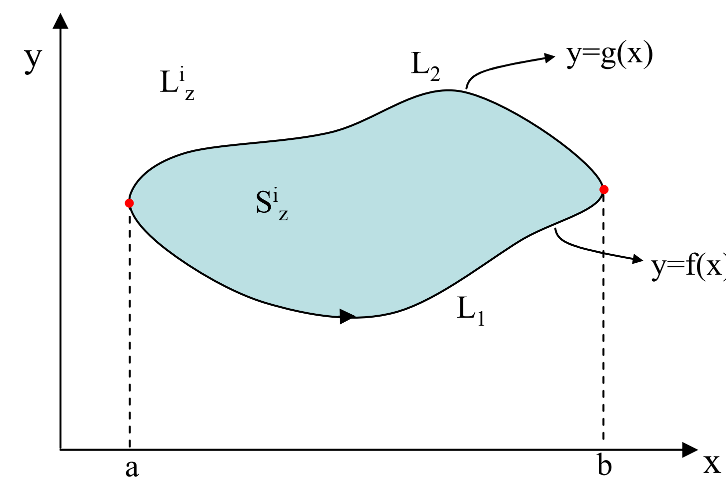
Term 3 =0 *(z=constant!⇒ dz=0)*

Term 1

$$\oint_{L^i_z} A_x(x,y,z_0)dx = \oint_{L_1+L_2} A_x(x,y,z_0)dx =$$

$$\int_{L_1} A_x(x,y,z_0)dx + \int_{L_2} A_x(x,y,z_0)dx =$$

$$\int_a^b A_x(x,f(x),z_0)dx + \int_b^a A_x(x,g(x),z_0)dx =$$



THE STOKES' THEOREM

PROOF

$$= \int_a^b A_x(x, f(x), z_0)dx - \int_a^b A_x(x, g(x), z_0)dx = \int_a^b \left[A_x(x, f(x), z_0) - A_x(x, g(x), z_0) \right] dx =$$

$$\int_a^b \int_{g(x)}^{f(x)} \frac{\partial A_x(x, y, z_0)}{\partial y} dxdy = -\int_a^b \int_{f(x)}^{g(x)} \frac{\partial A_x}{\partial y} dxdy = -\iint_{S^i_z} \frac{\partial A_x}{\partial y} dxdy$$

Therefore we get:

Term 1 $$\oint_{L^i_z} A_x(x, y, z_0)dx = -\iint_{S^i_z} \frac{\partial A_x}{\partial y} dxdy$$

In a similar way:

Term 2 $$\oint_{L^i_z} A_y(x, y, z_0)dx = \iint_{S^i_z} \frac{\partial A_y}{\partial x} dxdy$$

It is the z-component of $rot\bar{A}$!!

Adding Term 1, Term 2 and Term 3:

$$\oint_{L^i_z} \bar{A} \cdot d\bar{r} = \iint_{S^i_z} \left(\overbrace{\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}} \right) dxdy$$

THE STOKES' THEOREM

So can rewrite it as:

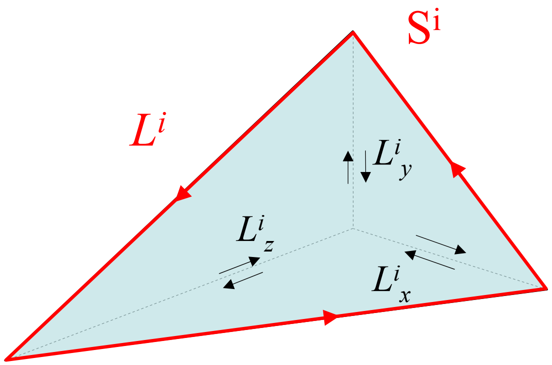
$$\oint_{L^i{}_z} \bar{A}\cdot d\bar{r} = \iint_{S^i{}_z} (rot\bar{A})_z\, dxdy = \iint_{S^i} (rot\bar{A})_z\, \hat{e}_z \cdot d\bar{S}$$

$$dxdy = \hat{e}_z \cdot \hat{n} dS = \hat{e}_z \cdot d\bar{S}$$

In a similar way we have:

$$\oint_{L^i{}_y} \bar{A}\cdot d\bar{r} = \iint_{S^i} (rot\bar{A})_y\, \hat{e}_y \cdot d\bar{S}$$

$$\oint_{L^i{}_x} \bar{A}\cdot d\bar{r} = \iint_{S^i} (rot\bar{A})_x\, \hat{e}_x \cdot d\bar{S}$$



Now let's add everything together:

$$\oint_{L^i{}_x} \bar{A}\cdot d\bar{r} + \oint_{L^i{}_y} \bar{A}\cdot d\bar{r} + \oint_{L^i{}_z} \bar{A}\cdot d\bar{r} = \oint_{L^i} \bar{A}\cdot d\bar{r}$$

$$\iint_{S^i} (rot\bar{A})_x\, \hat{e}_x \cdot d\bar{S} + \iint_{S^i} (rot\bar{A})_y\, \hat{e}_y \cdot d\bar{S} + \iint_{S^i} (rot\bar{A})_z\, \hat{e}_z \cdot d\bar{S} = \iint_{S^i} rot\bar{A}\cdot d\bar{S}$$

THE STOKES' THEOREM

PROOF

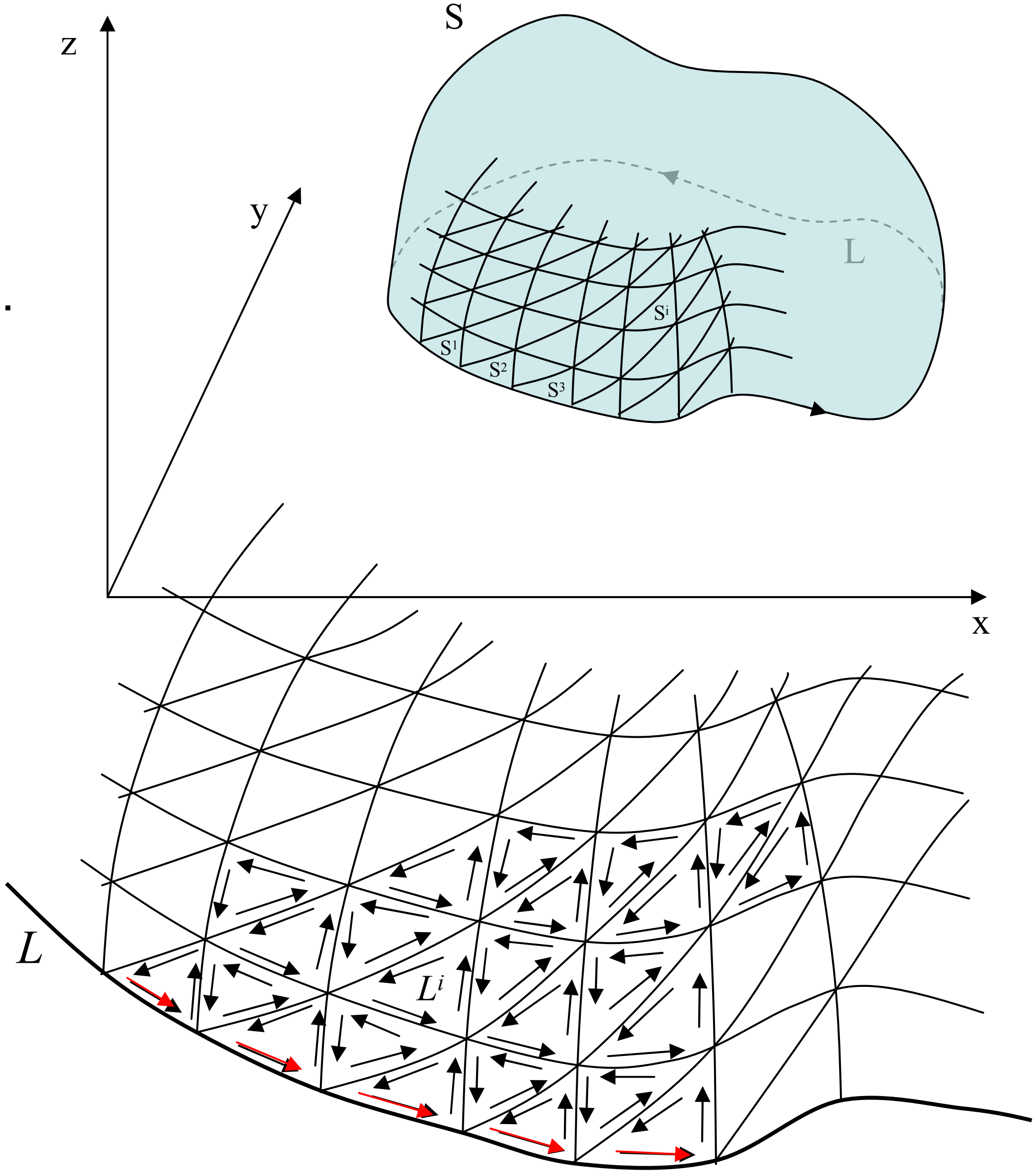
$$\oint_{L^i} \overline{A} \cdot d\overline{r} = \iint_{S^i} rot\overline{A} \cdot d\overline{S}$$

But we are interested in the whole S. So we add these small contributions altogether:

$$\underbrace{\sum_i \iint_{S^i} rot\overline{A} \cdot d\overline{S}}_{} = \iint_S rot\overline{A} \cdot d\overline{S}$$

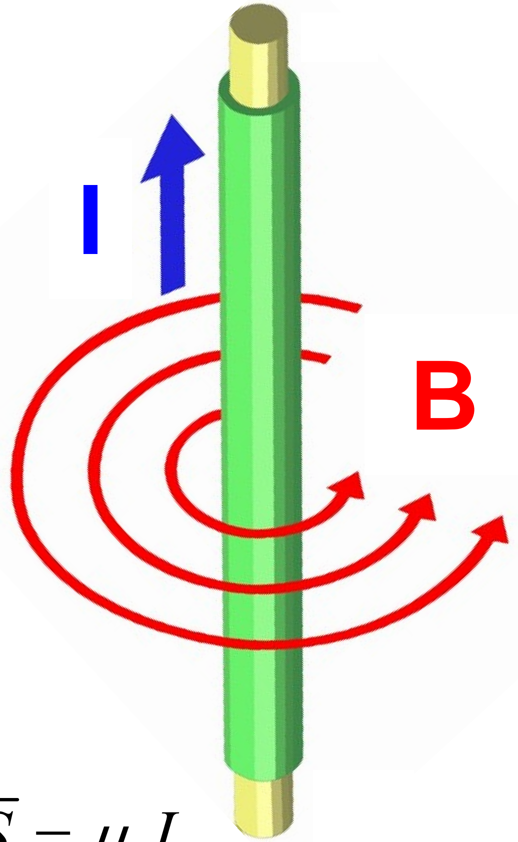
$$\overbrace{\sum_i \int_{L^i} \overline{A} \cdot d\overline{r}}^{=} = \int_L \overline{A} \cdot d\overline{r}$$

$$\oint_L \overline{A} \cdot d\overline{r} = \iint_S rot\,\overline{A} \cdot d\,\overline{S}$$



TARGET PROBLEM

Now we can easily calculate the magnetic field $\overline{B}$ at a distance a from the conductor!



Ampere's law $\quad rot\overline{B} = \mu_0 \overline{j}$

Where $\overline{j}$ is the current density:

$$\oint_L \overline{B} \cdot d\overline{r} = \iint_S rot\,\overline{B} \cdot d\overline{S} = \iint_S \mu_0 \overline{j} \cdot d\overline{S} = \mu_0 \iint_S \overline{j} \cdot d\overline{S} = \mu_0 I$$

Stokes

Ampere

$$I = \iint_S \overline{j} \cdot d\overline{S}$$

THE GREEN FORMULA IN THE PLANE

THEOREM *(7.1 in the textbook)*

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint_L \left(Pdx + Qdy \right)$$

PROOF

We can start from Stokes' theorem $\oint_L \overline{A} \cdot d\overline{r} = \iint_S rot\,\overline{A} \cdot d\overline{S}$

$$\oint_L \overline{A} \cdot d\overline{r} = \oint \left(A_x dx + A_y dy + A_z dz \right) = \oint \left(A_x dx + A_y dy \right)$$

But we are in a plane,
so we can assume A=(A_x,A_y,0)

$$\iint_S rot\,\overline{A} \cdot d\overline{S} = \iint_S \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \underbrace{\hat{e}_z \cdot \hat{e}_z}_{=1} dxdy$$

$$rot\,\overline{A} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & 0 \end{vmatrix}$$

$$\iint_D \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dxdy = \oint_L \left(A_x dx + A_y dy \right)$$

which is the Green formula
for $P=A_x$ and $Q=A_y$

CURL FREE FIELD AND SCALAR POTENTIAL

DEFINITION: A vector field $\overline{A}$ is "curl free" if $rot\overline{A}=0$. *Sometimes called "irrotational"*

THEOREM *(7.5 in the textbook)* $rot\overline{A}=0 \;\Leftrightarrow$ has a scalar potential ϕ, $\overline{A}=grad\,\phi$.

PROOF

(1) $rot\;\overline{A}=0$

$$\oint_L \overline{A}\cdot d\overline{r} = \iint_S rot\,\overline{A}\cdot d\,\overline{S} = 0$$

If the circulation is zero, then the field is conservative and has a scalar potential. *See theorem 4.5 in the textbook.*

(2) $\overline{A}=grad\,\phi$

$$rot\;\overline{A} = rot\;grad\,\phi = rot\left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z}\right) = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \left(\frac{\partial}{\partial y}\frac{\partial\phi}{\partial z}-\frac{\partial}{\partial z}\frac{\partial\phi}{\partial y},\ldots,\ldots\right) = (0,0,0)$$

(or see a proof on next week)

SOLENOIDAL FIELD AND VECTOR POTENTIAL

DEFINITION: A vector field $\overline{B}$ is called **solenoidal** if $div\overline{B} = 0$

DEFINITION: The vector field $\overline{B}$ has a vector potential $\overline{A}$ if , $\overline{B} = rot\overline{A}$

THEOREM *(7.7 in the textbook)* $\overline{B}$ has a vector potential $\overline{A}$, $\overline{B} = rot\overline{A} \Leftrightarrow div\overline{B} = 0$

PROOF

(1) $\overline{B}$ has a vector potential $\Rightarrow \overline{B} = rot\overline{A} \Rightarrow div\overline{B} = div\left(rot\overline{A}\right) = 0$

(2) $div\overline{B} = 0$

Let's try to find a solution $\overline{A}$ to the equation $\overline{B} = rot\overline{A}$

We start looking for a particular solution A* of this kind:

$$\overline{A}^* = \left(A_x^*(x,y,z),\, A_y^*(x,y,z),\, 0\right)$$

CURL FREE FIELD AND SCALAR POTENTIAL

PROOF

Assuming $\overline{B} = rot\overline{A}$ we obtain:

$$-\frac{\partial A_y^*}{\partial z} = B_x \quad \Rightarrow \quad A_y^*(x,y,z) = -\int_{z_0}^{z} B_x(x,y,z)dz + F(x,y)$$

$$\frac{\partial A_x^*}{\partial z} = B_y \quad \Rightarrow \quad A_x^*(x,y,z) = \int_{z_0}^{z} B_y(x,y,z)dz + G(x,y)$$

$$\frac{\partial A_y^*}{\partial x} - \frac{\partial A_x^*}{\partial y} = B_z \quad \Rightarrow \quad -\int_{z_0}^{z} \frac{\partial B_x}{\partial x} dz + \frac{\partial F}{\partial x} - \int_{z_0}^{z} \frac{\partial B_y}{\partial y} dz - \frac{\partial G}{\partial y} = B_z$$

But $div\overline{B} = 0 \Rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = -\frac{\partial B_z}{\partial z}$

$$\underbrace{\int_{z_0}^{z} \frac{\partial B_z}{\partial z} dz}_{= B_z(x,y,z) - B_z(x,y,z_0)} + \frac{\partial F}{\partial x} - \frac{\partial G}{\partial y} = B_z \quad \Rightarrow \quad \frac{\partial F}{\partial x} - \frac{\partial G}{\partial y} = B_z(x,y,z_0)$$

A solution to this equation is:

$$\begin{cases} F(x,y) = 0 \\ G(x,y) = -\int_{y_0}^{y} B_z(x,y,z_0)dy \end{cases}$$

$$\overline{A}^* = \left(\int_{z_0}^{z} B_y(x,y,z)dz - \int_{y_0}^{y} B_z(x,y,z_0)dy, \quad -\int_{z_0}^{z} B_x(x,y,z)dz, \quad 0 \right)$$

The general solution can be found using $\overline{B} = rot\overline{A}$:

$$rot\left(\overline{A} - \overline{A}^*\right) = \overline{B} - \overline{B} = 0 \quad \Rightarrow \quad \overline{A} - \overline{A}^* = grad\psi \quad \Rightarrow \quad \overline{A} = \overline{A}^* + grad\psi$$

WHICH STATEMENT IS WRONG?

1- The curl of a vector field is a scalar (yellow)

2- The curl is related to the line integral of a field along a closed surface (red)

3- Stokes' theorem translates a line integral into a surface integral (green)

4- The Stokes' theorem can be applied only to a closed curve (blue)