VEKTORANALYS

Kursvecka 2

övningar

PROBLEM 1

Calculate the potential ϕ for the following vector field: $\overline{A} = (2xz, 2z^2, x^2 + 4yz)$

SOLUTION

Step 1. Verify that the vector field can have a potential: $\frac{\partial A_x}{\partial y} = \frac{\partial A_y}{\partial x} \quad \frac{\partial A_y}{\partial z} = \frac{\partial A_z}{\partial y} \quad \frac{\partial A_z}{\partial x} = \frac{\partial A_x}{\partial z}$

Theorem 3.5, page 24

$$\left.\begin{array}{lll} \frac{\partial A_x}{\partial y} = 0 \quad \frac{\partial A_y}{\partial x} = 0 & \Rightarrow & \frac{\partial A_x}{\partial y} = \frac{\partial A_y}{\partial x} \\ \frac{\partial A_y}{\partial z} = 4z \quad \frac{\partial A_z}{\partial y} = 4z & \Rightarrow & \frac{\partial A_y}{\partial z} = \frac{\partial A_z}{\partial y} \\ \frac{\partial A_z}{\partial x} = 2x \quad \frac{\partial A_x}{\partial z} = 2x & \Rightarrow & \frac{\partial A_z}{\partial x} = \frac{\partial A_x}{\partial z} \end{array}\right\} \Rightarrow \overline{A} \quad \textit{could have a potential}$$

Step 2. Calculate the potential

$$\overline{A} = (2xz, 2z^2, x^2 + 4yz) \qquad (a)$$

$$\overline{A} = grad\,\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) \qquad (b)$$

$$\frac{\partial \phi}{\partial x} = 2xz \Rightarrow \phi = x^2 z + F(y,z) \qquad (c)$$

From (a) and (b) $\frac{\partial \phi}{\partial y} = 2z^2$

From (c) $\frac{\partial \phi}{\partial y} = \frac{\partial F(y,z)}{\partial y}$

$$\Rightarrow \frac{\partial F(y,z)}{\partial y} = 2z^2 \Rightarrow F(y,z) = 2yz^2 + G(z)$$

$$\phi = x^2 z + 2yz^2 + G(z) \qquad (d)$$

From (a) and (b) $\frac{\partial \phi}{\partial z} = x^2 + 4yz$

From (d) $\frac{\partial \phi}{\partial z} = x^2 + 4yz + \frac{\partial G(z)}{\partial z}$

$$\Rightarrow \frac{\partial G(z)}{\partial z} = 0 \Rightarrow G(z) = const.$$

$$\boxed{\phi = x^2 z + 2yz^2 + const.}$$

PROBLEM 2

Calculate the line integral of the vector field: $\overline{A} = (2 - y, -xy, 1)$

along: (A) the curve *L*

$$\begin{cases} 4x - y^2 = 0 \\ x^2 + y^2 - z = 0 \end{cases}$$

(The same curve of problem 1 in week 1)

from the point (0,0,0) to the point (1,2,5)

(B) a straight line from point (0,0,0) to (1,2,0) and then from (1,2,0) to (1,2,5)

SOLUTION (A)

The line integral is: $\int_L \overline{A}(\overline{r}) \cdot d\overline{r} = \int_a^b \overline{A}(\overline{r}(u)) \cdot \frac{d\overline{r}}{du} du$

STEP 1: we need to find a parameterization of the curve L in order to have $\overline{r} = \overline{r}(u)$

If $u=y$ we have:

$$x(u) = \frac{u^2}{4} \quad y(u) = u \quad z(u) = \frac{u^4}{16} + u^2 \qquad \overline{r}(u) = \left(\frac{u^2}{4}, u, \frac{u^4}{16} + u^2 \right)$$

(See problem 1 week 1)

from point (0,0,0) till the point (1,2,5) $\Rightarrow$ u: a→b with a=0 and b=2

STEP 2: we calculate the integral

$$\overline{A}(\overline{r}(u)) = (2 - y(u), -x(u)y(u), 1) = \left(2 - u, -\frac{u^3}{4}, 1 \right)$$

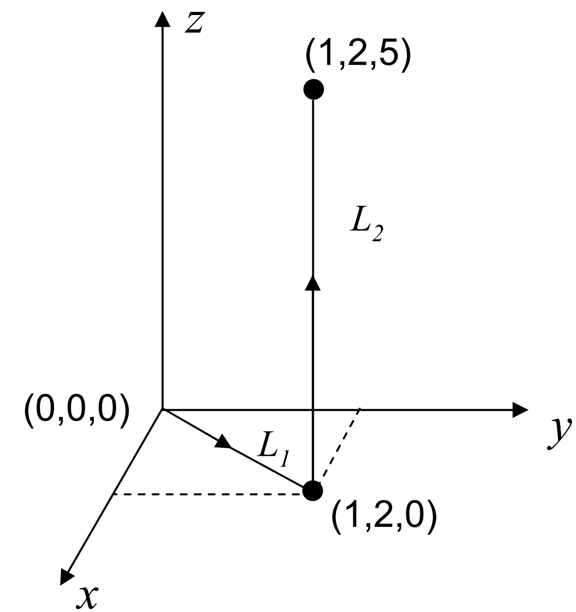
$$\frac{d\overline{r}}{du} = \left(\frac{2u}{4}, 1, \frac{4u^3}{16} + 2u \right) = \left(\frac{u}{2}, 1, \frac{u^3}{4} + 2u \right)$$

$$\Rightarrow \int_L \overline{A}(\overline{r}) \cdot d\overline{r} = \int_0^2 \left(2 - u, -\frac{u^3}{4}, 1 \right) \cdot \left(\frac{u}{2}, 1, \frac{u^3}{4} + 2u \right) du =$$

$$\int_0^2 \left(u - \frac{u^2}{2} - \frac{u^3}{4} + \frac{u^3}{4} + 2u \right) du = \left[3\frac{u^2}{2} - \frac{u^3}{6} \right]_0^2 = \frac{14}{3}$$

SOLUTION (B)

straight line $\underbrace{\text{from point (0,0,0) till (1,2,0)}}_{L_1}$ and $\underbrace{\text{then from (1,2,0) till (1,2,5)}}_{L_2}$ $\Rightarrow \int_L \overline{A}(\overline{r})\cdot d\overline{r} = \int_{L_1} \overline{A}(\overline{r})\cdot d\overline{r} + \int_{L_2} \overline{A}(\overline{r})\cdot d\overline{r}$



- We start with the first integral.

STEP 1: parameterization of L_1 : $\overline{r}(u) = (u, 2u, 0)$

u: $0 \rightarrow 1$

$$\frac{d\overline{r}}{du} = (1,2,0)$$

STEP 2: line integral calculation

$$\int_{L_1} \overline{A}(\overline{r})\cdot d\overline{r} = \int_a^b \overline{A}(\overline{r}(u))\cdot\frac{d\overline{r}}{du}du = \int_0^1 (2-2u, -2u^2, 1)\cdot(1,2,0)du =$$

$$\int_0^1 (2-2u-4u^2)du = \left[2u - u^2 - \frac{4}{3}u^3\right]_0^1 = \left(2-1-\frac{4}{3}\right) = -\frac{1}{3}$$

- We continue with the second integral.

STEP 1: parameterization of L_2 : $\overline{r}(u) = (1,2,u)$

u: $0 \rightarrow 5$

$$\frac{d\overline{r}}{du} = (0,0,1)$$

STEP 2: line integral calculation

$$\int_{L_2} \overline{A}(\overline{r})\cdot d\overline{r} = \int_a^b \overline{A}(\overline{r}(u))\cdot\frac{d\overline{r}}{du}du = \int_0^5 (2-2,-2,1)\cdot(0,0,1)du = \int_0^5 du = 5$$

- Finally: $$\int_L \overline{A}(\overline{r})\cdot d\overline{r} = \int_{L_1} \overline{A}(\overline{r})\cdot d\overline{r} + \int_{L_2} \overline{A}(\overline{r})\cdot d\overline{r} = -\frac{1}{3} + 5 = \frac{14}{3}$$

PROBLEM 3

Calculate the line integral of the vector field: $\overline{A} = \left(\frac{x}{\sqrt{x^2+y^2+5z^2}}, \frac{\sqrt{5}yz}{\sqrt{x^2+y^2+z^2}}, \frac{e^x \sin y}{1+\ln(xyz)} \right)$

Along the path L defined by:

$$L = \begin{cases} x^2 + y^2 = 4 \\ z = 1 \end{cases}$$

From the point P_1=(2,0,1) to the point P_2=(0,2,1)

SOLUTION

Parameterization of L:

$$\left.\begin{array}{l} x = 2\cos\varphi \\ y = 2\sin\varphi \\ z = 1 \end{array}\right\} \Rightarrow \quad L: \ \overline{r}(\varphi) = (2\cos\varphi, 2\sin\varphi, 1) \quad \Rightarrow \quad \frac{d\overline{r}(\varphi)}{d\varphi} = (-2\sin\varphi, 2\cos\varphi, 0)$$

$$\text{P1=(2,0,1)} \quad \Rightarrow \quad \varphi_1 = 0$$

$$\text{P2=(0,2,1)} \quad \Rightarrow \quad \varphi_2 = \frac{\pi}{2}$$

$$\int_L \overline{A} \cdot d\overline{r} = \int_{\varphi_1}^{\varphi_2} \overline{A}(\overline{r}(\varphi)) \cdot \frac{d\overline{r}}{d\varphi} d\varphi$$

$$\overline{A} = \left(\frac{x}{\sqrt{x^2+y^2+5z^2}}, \frac{\sqrt{5}yz}{\sqrt{x^2+y^2+z^2}}, \frac{e^x \sin y}{1+\ln(xyz)} \right) = \left(\frac{2\cos\varphi}{\sqrt{(2\cos\varphi)^2+(2\sin\varphi)^2+5(1)^2}}, \frac{2\sin\varphi\sqrt{5}}{\sqrt{(2\cos\varphi)^2+(2\sin\varphi)^2+(1)^2}}, \frac{e^x \sin y}{1+\ln(xyz)} \right) = \left(\frac{2\cos\varphi}{3}, \ 2\sin\varphi, \ \frac{e^x \sin y}{1+\ln(xyz)} \right)$$

$$\int_L \overline{A} \cdot d\overline{r} = \int_0^{\pi/2} \left(\frac{2\cos\varphi}{3}, \ 2\sin\varphi, \ \frac{e^x \sin y}{1+\ln(xyz)} \right) \cdot (-2\sin\varphi, 2\cos\varphi, 0)\, d\varphi = \int_0^{\pi/2} \left(-\frac{4}{3}\sin\varphi\cos\varphi + 4\sin\varphi\cos\varphi \right) d\varphi = \int_0^{\pi/2} \left(\frac{8}{3}\sin\varphi\cos\varphi \right) d\varphi =$$

$$\int_0^{\pi/2} \left(\frac{4}{3}\sin 2\varphi \right) d\varphi = -\frac{4}{3}\left[\frac{\cos 2\varphi}{2} \right]_0^{\pi/2} = -\frac{2}{3}[-1-1] = \frac{4}{3}$$

PROBLEM 4

Calculate the line integral of the vector field: $\overline{A} = \left(x^2, \quad \sinh(yz), \quad z^2\right)$ $\left(\sinh x = \dfrac{e^x - e^{-x}}{2}\right)$

Along the path L defined by:

$$L = \begin{cases} x^2 - z = 0 \\ y = 2 \end{cases}$$

From the point P_1=(-1,2,1) to the point P_2=(1,2,1).

SOLUTION

Parameterization of L:

$$\left.\begin{aligned} x &= u \\ y &= 2 \\ z &= u^2 \end{aligned}\right\} \Rightarrow \quad L: \ \overline{r}(u) = \left(u, 2, u^2\right) \quad \Rightarrow \quad \frac{d\overline{r}(u)}{du} = \left(1, 0, 2u\right)$$

$$P_1=(-1,2,1) \quad \Rightarrow \quad u_1 = -1$$
$$P_2=(1,2,1) \quad \Rightarrow \quad u_2 = +1$$

$$\int_L \overline{A} \cdot d\overline{r} = \int_{u_1}^{u_2} \overline{A}\left(\overline{r}(u)\right) \cdot \frac{d\overline{r}}{du} du$$

$$\overline{A} = \left(x^2, \quad \sinh(yz), \quad z^2\right) = \left(u^2, \quad \sinh(2u^2), \quad u^4\right)$$

$$\int_L \overline{A} \cdot d\overline{r} = \int_{-1}^{1} \left(u^2, \quad \sinh(2u^2), \quad u^4\right) \cdot \left(1, 0, 2u\right) du =$$

$$\int_{-1}^{1} \left(u^2 + 2u^5\right) du = \left[\frac{1}{3}u^3 + \frac{1}{3}u^6\right]_{-1}^{1} = \frac{2}{3}$$

PROBLEM 5

Calculate the flux of the vector field: $\overline{A} = (x^2 - y^2, (x+y)^2, (x-y)^2)$

through the surface S: $\overline{r}(u,v) = (u+v, u-v, uv)$

$$\begin{cases} u: & -1 \to 1 \\ v: & -1 \to 1 \end{cases}$$

$$\hat{n} \cdot \hat{e}_z > 0$$

SOLUTION

The flux is the integral: $\iint_S \overline{A} \cdot d\overline{S} = \int_v \int_u \overline{A}(\overline{r}(u,v)) \cdot \left(\frac{\partial \overline{r}}{\partial u} \times \frac{\partial \overline{r}}{\partial v} \right) dudv$

- $\overline{A}(\overline{r}(u,v)) = ((u+v)^2 - (u-v)^2, (u+v+u-v)^2, (u+v-u+v)^2) = (u^2 + v^2 + 2uv - (u^2 + v^2 - 2uv), 4u^2, 4v^2) =$

  $(4uv, 4u^2, 4v^2) = 4(uv, u^2, v^2)$

- $\left.\begin{matrix} \frac{d\overline{r}}{du} = (1,1,v) \\ \frac{d\overline{r}}{dv} = (1,-1,u) \end{matrix}\right\} \Rightarrow \frac{d\overline{r}}{du} \times \frac{d\overline{r}}{dv} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & v \\ 1 & -1 & u \end{vmatrix} = (u+v, v-u, -2)$ $\qquad \hat{n} \cdot \hat{e}_z > 0 \quad \Rightarrow$-S

- $\iint_{-S} \overline{A} \cdot d\overline{S} = 4\int_{-1}^{1}\int_{-1}^{1} (uv, u^2, v^2) \cdot (u+v, v-u, -2) dudv = 4\int_{-1}^{1}\int_{-1}^{1} (u^2 v + uv^2 + u^2 v - u^3 - 2v^2) dudv =$

  $$4\int_{-1}^{1} \left[\frac{u^3}{3} v + \frac{u^2}{2} v^2 + \frac{u^3}{3} v - \frac{u^4}{4} - 2uv^2 \right]_{-1}^{1} dv = 4\int_{-1}^{1} \left(\frac{2v}{3} + \frac{2v}{3} - 4v^2 \right) dv = 4\left[\frac{4v^2}{6} - \frac{4v^3}{3} \right]_{-1}^{1} = -\frac{32}{3}$$

$$\Rightarrow \iint_S \overline{A} \cdot d\overline{S} = -\iint_{-S} \overline{A} \cdot d\overline{S} = \frac{32}{3}$$

PROBLEM 6

Calculate the flux of the vector field: $\overline{A} = \left(xy, yz, xz+1\right)$

through the surface S:
$$\begin{cases} z = 4 - x^2 - y^2 \\ x^2 + y^2 \le 4 \\ \overline{n} = \hat{e}_z \quad \text{in the point } (0,0,4). \end{cases}$$

SOLUTION

The flux is the integral: $\iint_S \overline{A} \cdot d\overline{S} = \int_v \int_u \overline{A}(\overline{r}(u,v)) \cdot \left(\frac{\partial \overline{r}}{\partial u} \times \frac{\partial \overline{r}}{\partial v} \right) dudv$

Parameterization of S:

$$\left.\begin{array}{l} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = 4-\rho^2 \end{array}\right\} \Rightarrow \quad S: \quad \overline{r}(\rho,\theta) = \left(\rho\cos\theta, \rho\sin\theta, 4-\rho^2\right)$$

$$\left.\begin{array}{l} \dfrac{d\overline{r}(\rho,\theta)}{d\rho} = \left(\cos\theta, \sin\theta, -2\rho\right) \\ \dfrac{d\overline{r}(\rho,\theta)}{d\theta} = \left(-\rho\sin\theta, \rho\cos\theta, 0\right) \end{array}\right\} \Rightarrow \left(\frac{d\overline{r}(\rho)}{d\rho} \times \frac{d\overline{r}(\rho,\theta)}{d\theta} \right) = \left(2\rho^2\cos\theta, 2\rho^2\sin\theta, \rho\right)$$

$$\iint_S \overline{A} \cdot d\overline{S} = \int_0^{2\pi}\int_0^{2} \left(\rho^2\sin\theta\cos\theta, \left(4-\rho^2\right)\rho\sin\theta, \left(4-\rho^2\right)\rho\cos\theta+1\right) \cdot \left(2\rho^2\cos\theta, 2\rho^2\sin\theta, \rho\right) d\rho d\theta =$$

$$= \int_0^{2\pi}\int_0^{2} \left(2\rho^4\sin\theta\cos^2\theta + 2\left(4-\rho^2\right)\rho^3\sin^2\theta, \left(4-\rho^2\right)\rho^2\cos\theta + \rho\right) d\rho d\theta = \int_0^{2\pi} \left(\frac{64}{5}\sin\theta\cos^2\theta + \frac{32}{3}\sin^2\theta + \left(\frac{32}{3} - \frac{32}{5}\right)\cos\theta + 2 \right) d\theta =$$

$$= \left[-\frac{64}{5}\frac{1}{3}\cos^3\theta + \frac{32}{3}\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + \left(\frac{32}{3} - \frac{32}{5}\right)\sin\theta + 2\theta \right]_0^{2\pi} = \frac{32}{3}\pi + 4\pi = \frac{44}{3}\pi$$