VEKTORANALYS

Kursvecka 2

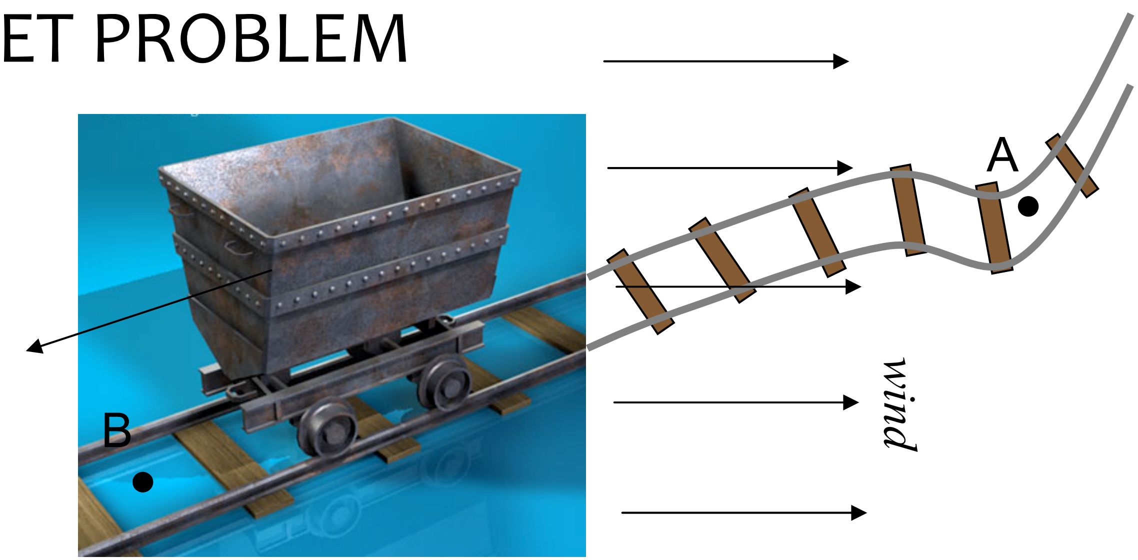
LINE INTEGRAL
and
FLUX INTEGRAL

Kapitel 4-5
Sidor 29-50

TARGET PROBLEM

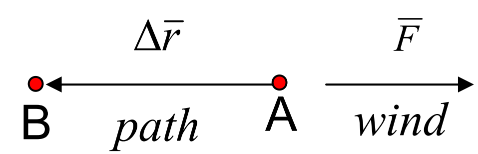
We want to push a mine cart
along a path L from A to B.
But the wind is blowing.

How much energy is needed?
(i.e. how much is the 'work'?)

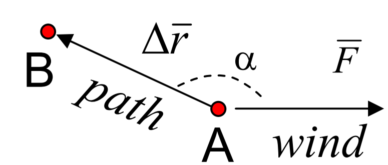


We will arrive to the final answer in three steps:

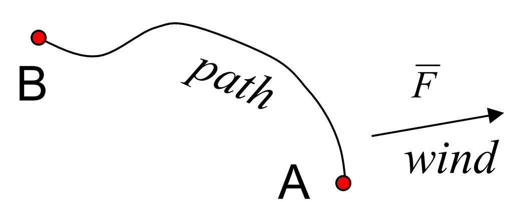
1- Path: straight line
Direction: the path L has the same direction of the wind



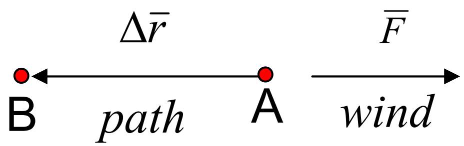
2- Path: straight line
Direction: the path L has not the same direction of the wind



3- Path: curved line
Wind not constant

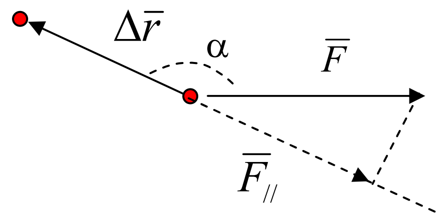


Step 1:



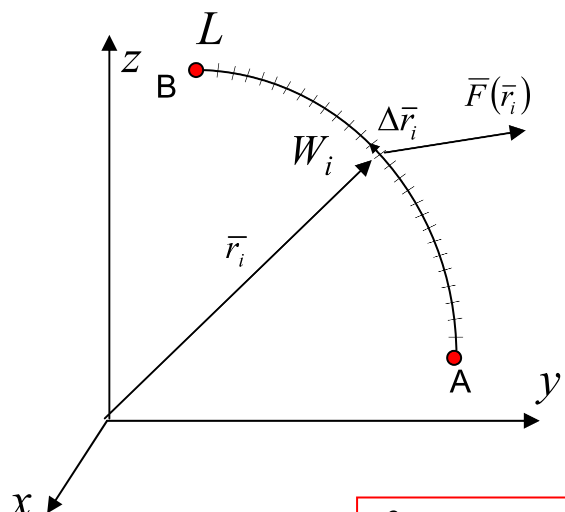
$$W = |\overline{F}||\Delta\overline{r}|$$

Step 2:



$$W = |\overline{F}_{//}||\Delta\overline{r}| = |\overline{F}||\Delta\overline{r}|\cos\alpha = \overline{F}\cdot\Delta\overline{r}$$

Step 3:



$$\left.\begin{array}{l} W = \sum_i W_i \\ W_i \approx \overline{F}(\overline{r}_i)\cdot\Delta\overline{r}_i \end{array}\right\} \Rightarrow W \approx \sum_i \overline{F}(\overline{r}_i)\cdot\Delta\overline{r}_i$$

For *”very small”* segments:

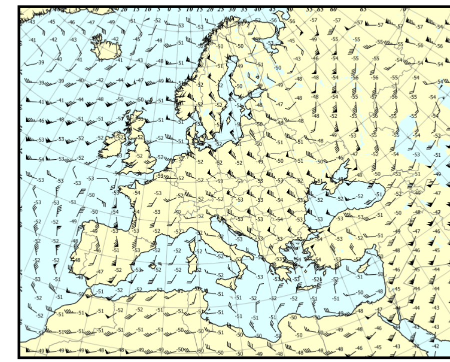
$$W = \lim_{\Delta\overline{r}_i \to 0} \sum_i \overline{F}(\overline{r}_i)\cdot\Delta\overline{r}_i \equiv \int_L \overline{F}(\overline{r})\cdot d\overline{r}$$

$\int_L \overline{F}(\overline{r})\cdot d\overline{r}$ is the line integral of $\overline{F}$ along the path L

We need to:
- introduce a **VECTOR FIELD**, $\overline{F}(\overline{r})$
- Define the **infinitesimal displacement** $d\overline{r}$ along the path L

VECTOR FIELD

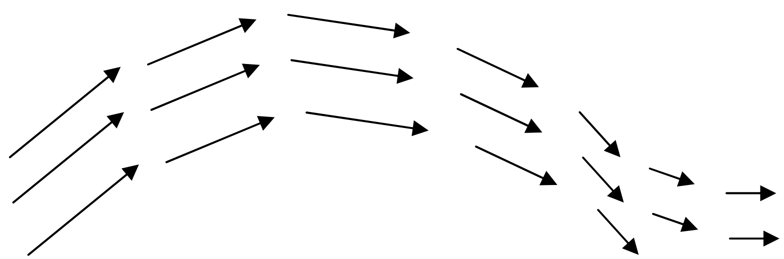
A **vector field** associates a vector $\bar{\mathbf{A}}(x,y,z)$ to each point (x,y,z) of the space.



Examples:
- velocity distribution in a fluid
- magnetic field around a magnet
- electric field around an electric charge

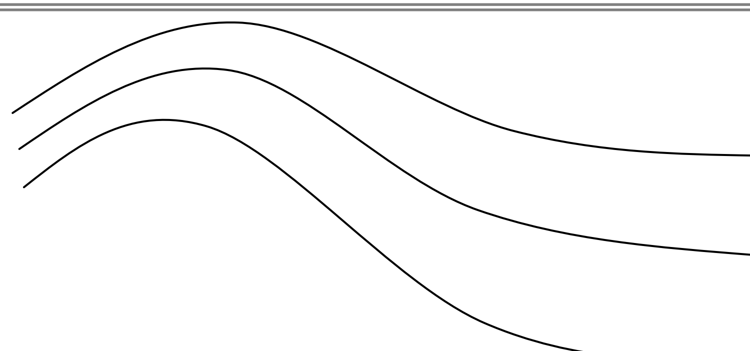
Two typical ways to represent a vector field:

1- Arrow field



- The arrow length is proportional to the field amplitude
- The arrow direction shows the field direction

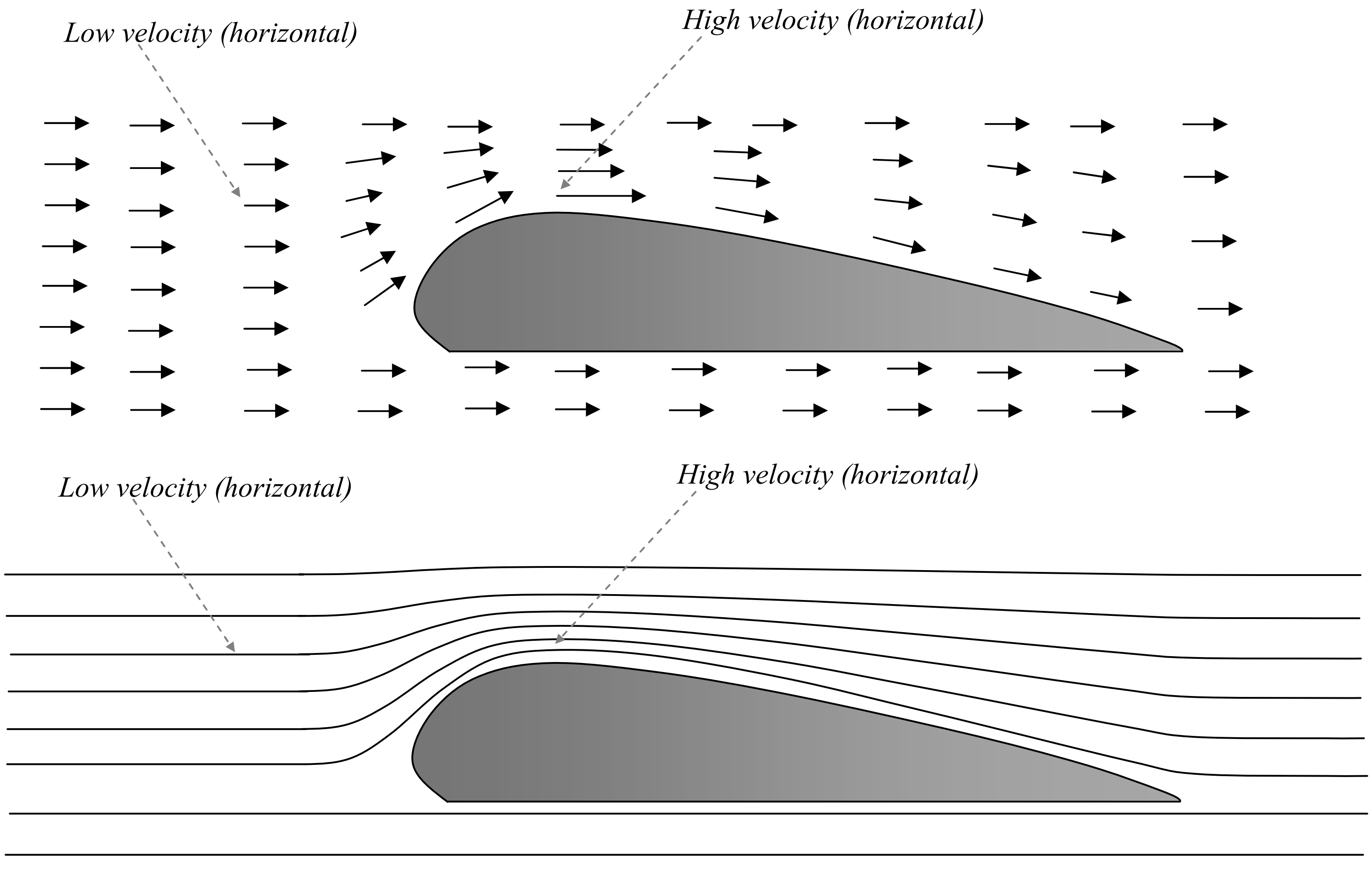
2- Line field



- The tangent to the curves is parallel to the vector field in all points.
- The density of the lines is proportional to the strength of the field.

VECTOR FIELD

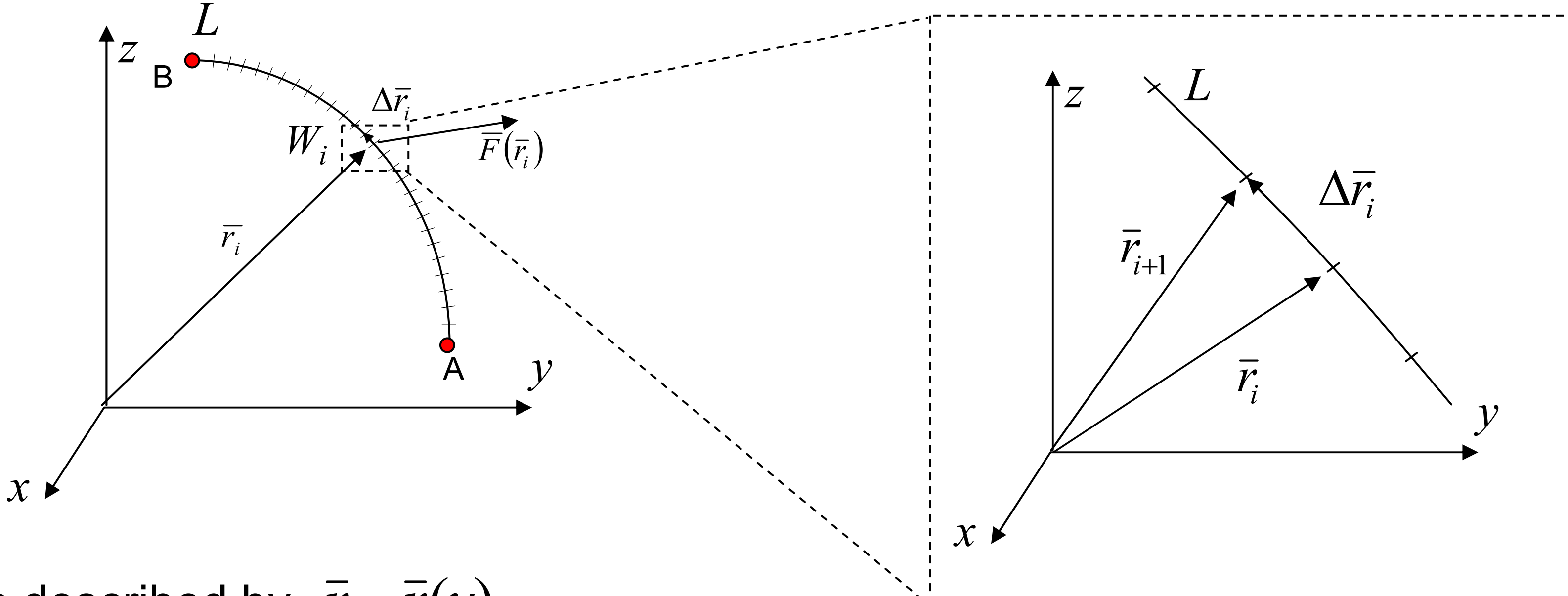
The airplane wing example (velocity field of air around a wing)



EXERCISE: 1- plot the vector field $\overline{F}(\overline{r}) = (x, 0)$

2- write $\overline{F}(\overline{r})$ on the curve defined by: $\overline{r}(u) = (u, u^2)$

$d\bar{r}$ AND THE LINE INTEGRAL



L is described by $\bar{r} = \bar{r}(u)$

$$\left.\begin{aligned} \bar{r}_i &= \bar{r}(u_i) \\ \bar{r}_{i+1} &= \bar{r}(u_{i+1}) \\ u_{i+1} &= u_i + \Delta u \end{aligned}\right\} \Rightarrow d\bar{r} = \lim_{\Delta u \to 0} \Delta\bar{r}_i = \lim_{\Delta u \to 0} \left[\bar{r}(u_{i+1}) - \bar{r}(u_i)\right] = \lim_{\Delta u \to 0} \frac{\left[\bar{r}(u_{i+1}) - \bar{r}(u_i)\right]}{\Delta u} \Delta u = \frac{d\bar{r}}{du} du$$

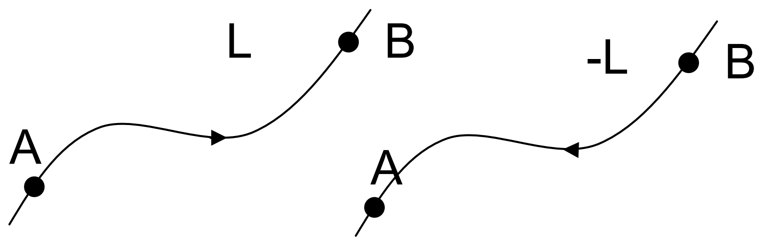
So, the line integral can be calculated as:

$$\int_L \bar{F}(\bar{r}) \cdot d\bar{r} = \int_a^b \bar{F}(\bar{r}(u)) \cdot \frac{d\bar{r}}{du} du$$

EXERCISE: Calculate $\int_L \bar{F}(\bar{r}) \cdot d\bar{r}$ with $\bar{F}(\bar{r}) = (x, 0)$ and L defined by $\bar{r}(u) = (u, u^2)$, $u : 0 \to 1$

LINE INTEGRAL (some useful properties)

THEOREM 1 *(4.2 in the textbook)*



$$\int_{-L} \overline{F}(\overline{r}) \cdot d\overline{r} = -\int_{L} \overline{F}(\overline{r}) \cdot d\overline{r} \qquad (3)$$

PROOF

If all line elements change sign then also the integral will change sign.

DEFINITION

The line of integral of $\bar{A}$ along a closed curve C is called circulation of a $\bar{A}$ along C:

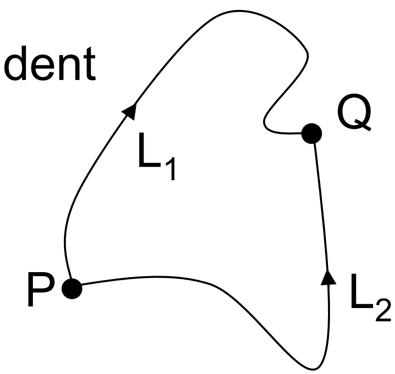
$$\oint_{C} \overline{A}(\overline{r}) \cdot d\overline{r} \qquad (4)$$

LINE INTEGRAL

DEFINITION: A vector field $\bar{A}$ is called conservative if: $\oint_C \bar{A}(\bar{r}) \cdot d\bar{r} = 0$

THEOREM 2 *(4.3 in the textbook)*

The circulation of $\bar{A}$ along a close curve L is zero if and only if for all points P and Q the line integral of A from P to Q is independent from the integration path between P and Q.



PROOF

Assume that L_1 and L_2 are two curves from P to Q.
Then $L = L_1 - L_2$ is a closed curve.

(1) The circulation is zero $\Rightarrow$ the line integral from P to Q is independent from the path

$$\left.\begin{array}{l} \oint_L \bar{A}(\bar{r}) \cdot d\bar{r} = 0 \\ \oint_L \bar{A}(\bar{r}) \cdot d\bar{r} = \int_{L_1 - L_2} \bar{A}(\bar{r}) \cdot d\bar{r} = \int_{L_1} \bar{A}(\bar{r}) \cdot d\bar{r} - \int_{L_2} \bar{A}(\bar{r}) \cdot d\bar{r} \end{array}\right\} \Rightarrow \int_{L_1} \bar{A}(\bar{r}) \cdot d\bar{r} = \int_{L_2} \bar{A}(\bar{r}) \cdot d\bar{r}$$

The line integral is independent from the integration path!

(2) The line integral from P to Q is independent from the path $\Rightarrow$ the circulation is zero.

$$\left.\begin{array}{l} \int_{L_1} \bar{A}(\bar{r}) \cdot d\bar{r} = \int_{L_2} \bar{A}(\bar{r}) \cdot d\bar{r} \\ \oint_L \bar{A}(\bar{r}) \cdot d\bar{r} = \int_{L_1 - L_2} \bar{A}(\bar{r}) \cdot d\bar{r} = \int_{L_1} \bar{A}(\bar{r}) \cdot d\bar{r} - \int_{L_2} \bar{A}(\bar{r}) \cdot d\bar{r} \end{array}\right\} \Rightarrow \oint_L \bar{A}(\bar{r}) \cdot d\bar{r} = 0$$

The circulation is zero

LINE INTEGRAL

THEOREM 3 *(4.4 in the textbook)*

If $\bar{A} = grad\,\phi$:

$$\int_P^Q \bar{A}(\bar{r}) \cdot d\bar{r} = \phi(Q) - \phi(P) \qquad (5)$$

So the line integral is independent from the integration path and depends only on the starting point and on the ending point

PROOF

If $\bar{r}(u)$ is a curve from P to Q then, using the chain rule for the partial derivative:

$$\int_{\bar{r}(u)} \overline{A}(\overline{r}) \cdot d\overline{r} = \int_p^q grad\,\phi \cdot \frac{d\overline{r}}{du} du = \int_p^q \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \cdot \left(\frac{dx}{du}, \frac{dy}{du}, \frac{dz}{du} \right) du =$$

$$= \int_p^q \left(\frac{\partial \phi}{\partial x}\frac{dx}{du} + \frac{\partial \phi}{\partial y}\frac{dy}{du} + \frac{\partial \phi}{\partial z}\frac{dz}{du} \right) du = \int_p^q \frac{d}{du}\phi(\overline{r}(u))du = \phi(Q) - \phi(P)$$

Or, easier: $\int_L \overline{A}(\overline{r}) \cdot d\overline{r} = \int_L grad\,\phi \cdot d\overline{r} = \int_L d\phi = \phi(B) - \phi(A)$

OTHER KINDS OF LINE INTEGRALS

- It is possible to combine scalar and vector line elements in many different ways along a curve L and thus get different kinds of line integrals

- Some examples:

$$\int_L \phi(\bar{r})ds \tag{6}$$

where $d\bar{r}=\hat{e}ds$

$$\int_L \phi(\bar{r})d\bar{r} \tag{7}$$

$$\int_L \bar{A}(\bar{r})\times d\bar{r} \tag{8}$$

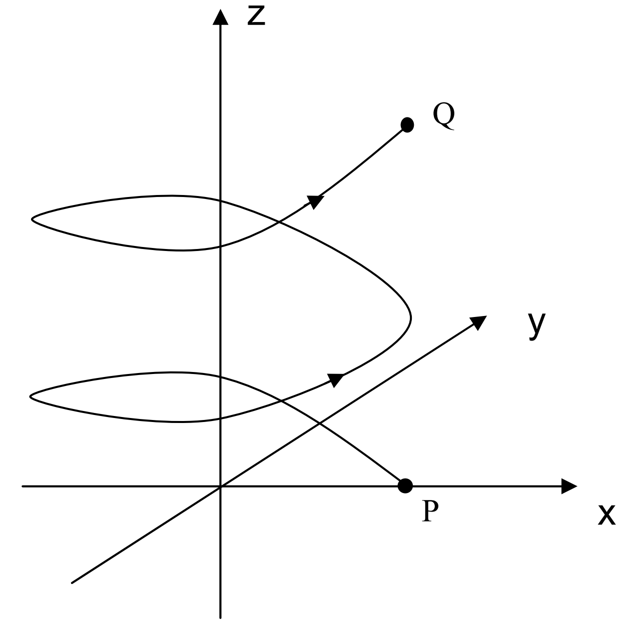
- To calculate the integrals:

$$\bar{r} \rightarrow \bar{r}(u)$$

$$L \rightarrow [a,b] \quad where\, u : a \rightarrow b$$

$$d\bar{r} = \frac{d\bar{r}}{du}du \quad or \quad ds = \left|\frac{d\bar{r}}{du}\right|du$$

EXAMPLE



The force is: $\bar{F}=(-yz,xz,-1)$

The path is L: $\bar{r}=(\cos u,\ \sin u,\ u)$ with $u{:}\,0 \rightarrow 4\pi$

Calculate the work from P=(1,0,0) till Q=(1,0,4π)

from definition (2)

$$W=\int_L \bar{F}(\bar{r})\cdot d\bar{r}=\int_{a=0}^{b=4\pi}\bar{F}(\bar{r}(u))\frac{d\bar{r}}{du}du$$

$$\left.\begin{aligned}\bar{F}(\bar{r}(u))&=(-(\sin u)u,(\cos u)u,-1)\\ \frac{d\bar{r}}{du}&=(-\sin u,\cos u,1)\end{aligned}\right\}\Rightarrow\quad \bar{F}(\bar{r}(u))\cdot\frac{d\bar{r}}{du}=(u\sin^2 u+u\cos^2 u-1)=$$

$$=u(\sin^2 u+\cos^2 u)-1=u-1$$

$$W=\int_L \bar{F}(\bar{r})\cdot d\bar{r}=\int_0^{4\pi}(u-1)du=\left[\frac{u^2}{2}-u\right]_0^{4\pi}=8\pi^2-4\pi$$

WHICH STATEMENT IS WRONG?

1- The image area of a vector field $\overline{A}$ is composed of vectors (yellow)

2- The line integral $\int \overline{F} \cdot d\overline{r}$ is a scalar (red)

3- The sign of the line integral $\int \overline{F} \cdot d\overline{r}$ depends on the integration path (green)

4- The gradient of a vector field can be written as: $\mathrm{grad}\ \overline{\mathrm{A}}$ (blue)

TARGET PROBLEM

We are making cranberry juice.
After cranberries are squeezed,
It is better to filter the juice!

How much juice flows trough the cloth each second?

We need :

(1) to understand how to calculate the **flux** of a **VECTOR FIELD** $\bar{v}(x, y, z)$

(2) a method to integrate the flux over the whole surface.

THE FLUX

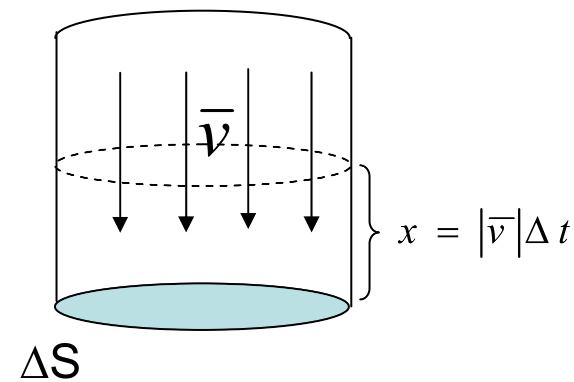
In the juice example, the flux F is the volume of the fluid ΔV that flows through the surface in the time Δt.

$$F = \frac{\Delta V}{\Delta t}$$

STEP 1: - the fluid velocity is perpendicular to the surface
- the surface is not curved

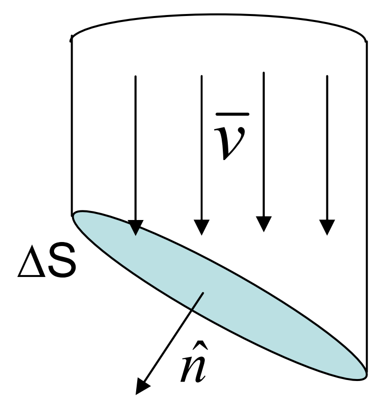
$$\Delta V = x\Delta S = |\bar{v}|\Delta t \Delta S$$

$$F = \frac{\Delta V}{\Delta t} = |\bar{v}|\Delta S$$



STEP 2: - the fluid velocity is NOT perpendicular to the surface
- the surface is not curved

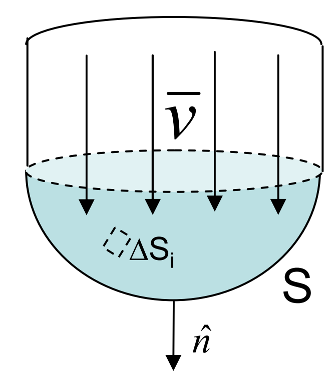
$$F = |\bar{v}_{//}|\Delta S = \bar{v}\cdot\hat{n}\Delta S = \bar{v}\cdot\Delta\bar{S}$$



STEP 3: - the surface is curved

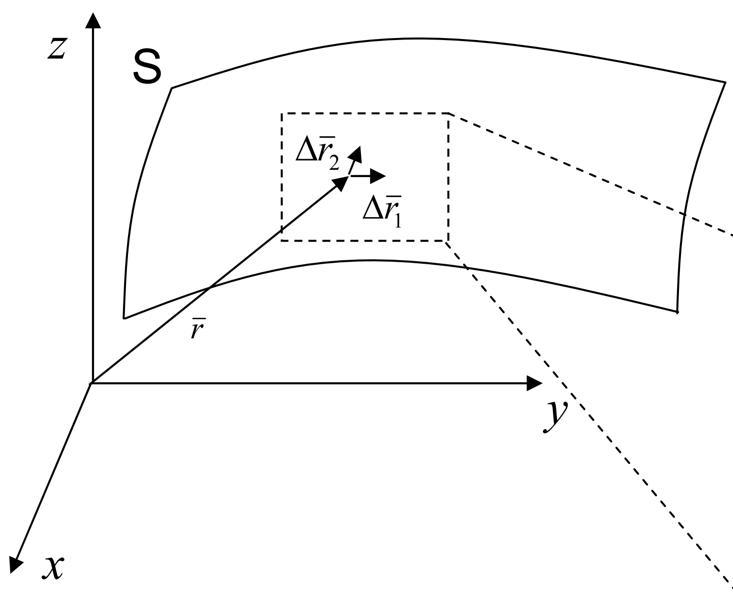
$$F = \sum_i F_i = \lim_{\Delta S_i \to 0} \sum_i \bar{v}_i \cdot \Delta\bar{S}_i \equiv \int_S \bar{v}\cdot d\bar{S}$$

$\int_S \bar{v}\cdot d\bar{S}$ is the flux integral of $\bar{v}$ on the surface S



$d\bar{S}$ AND FLUX INTEGRAL

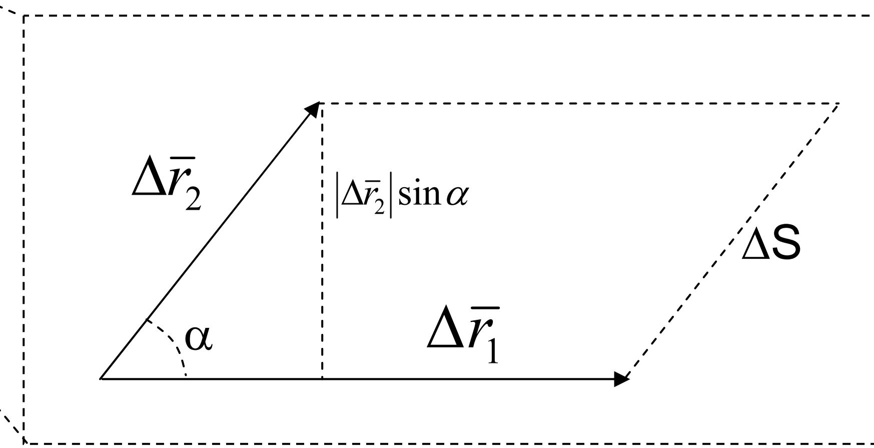
- Assume that the surface S is parameterized by $\bar{r} = \bar{r}(u,v)$



Let's consider two displacements,

- due a change in u: $\bar{r}_1 = \bar{r}(u+\Delta u, v) \Rightarrow \Delta\bar{r}_1 = \bar{r}_1 - \bar{r}$

- due a change in v: $\bar{r}_2 = \bar{r}(u, v+\Delta v) \Rightarrow \Delta\bar{r}_2 = \bar{r}_2 - \bar{r}$



The area ΔS is $\Delta S = |\Delta\bar{r}_2|\sin\alpha|\Delta\bar{r}_1| = |\Delta\bar{r}_1 \times \Delta\bar{r}_2|$

$\hat{n}$ is perpendicular to S. But also $\Delta\bar{r}_1 \times \Delta\bar{r}_2$ is perpendicular to S

$$\Rightarrow \Delta\bar{S} = \hat{n}\Delta S = \Delta\bar{r}_1 \times \Delta\bar{r}_2$$

$d\overline{S}$ AND FLUX INTEGRAL

$$d\overline{S} = \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \Delta\overline{S} = \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \Delta\overline{r}_1 \times \Delta\overline{r}_2$$

$$d\overline{r}_1 = \lim_{\Delta u \to 0} \Delta\overline{r}_1 = \lim_{\Delta u \to 0} \overline{r}(u+\Delta u, v) - \overline{r}(u,v) =$$

$$= \lim_{\Delta u \to 0} \frac{\overline{r}(u+\Delta u, v) - \overline{r}(u,v)}{\Delta u} \Delta u = \frac{\partial \overline{r}(u,v)}{\partial u} du$$

in the same way:

$$d\overline{r}_2 = \frac{\partial \overline{r}(u,v)}{\partial v} dv$$

$$\Rightarrow d\overline{S} = \frac{\partial \overline{r}}{\partial u} \times \frac{\partial \overline{r}}{\partial v} du dv$$

So, the flux integral of the vector field $\overline{v}$ on the surface S can be calculated as:

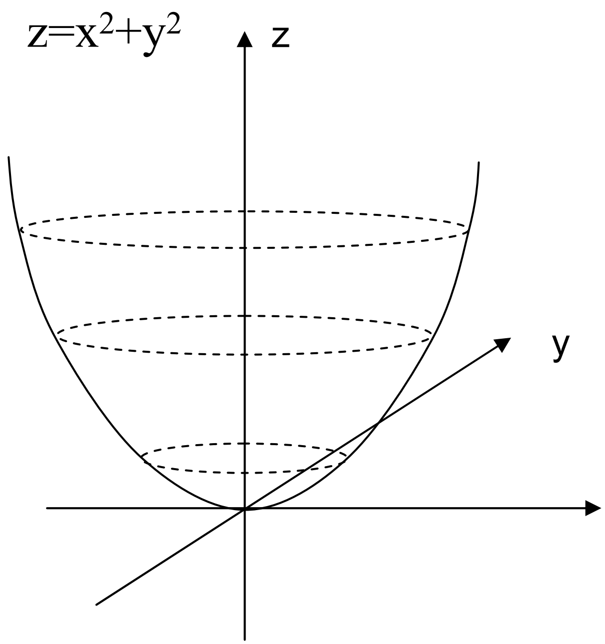
$$\int_S \overline{v} \cdot d\overline{S} = \int_u \int_v \overline{v}\left(\overline{r}(u,v)\right) \cdot \left(\frac{\partial \overline{r}}{\partial u} \times \frac{\partial \overline{r}}{\partial v} \right) du dv$$

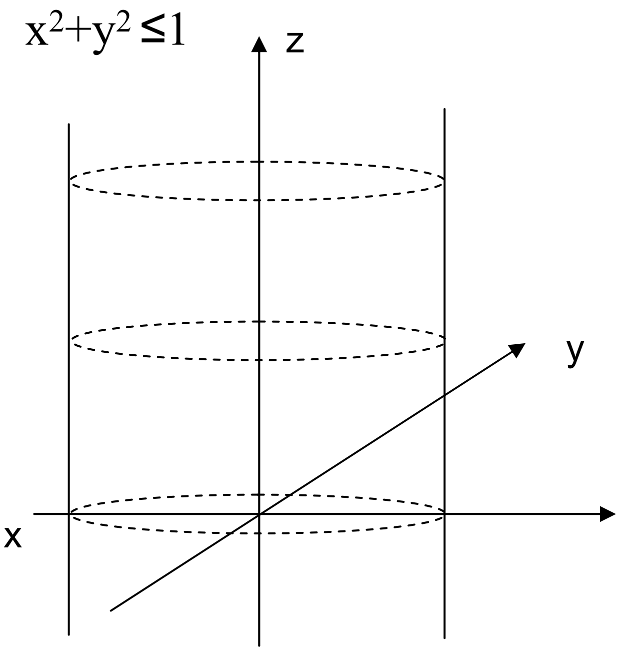
EXAMPLE

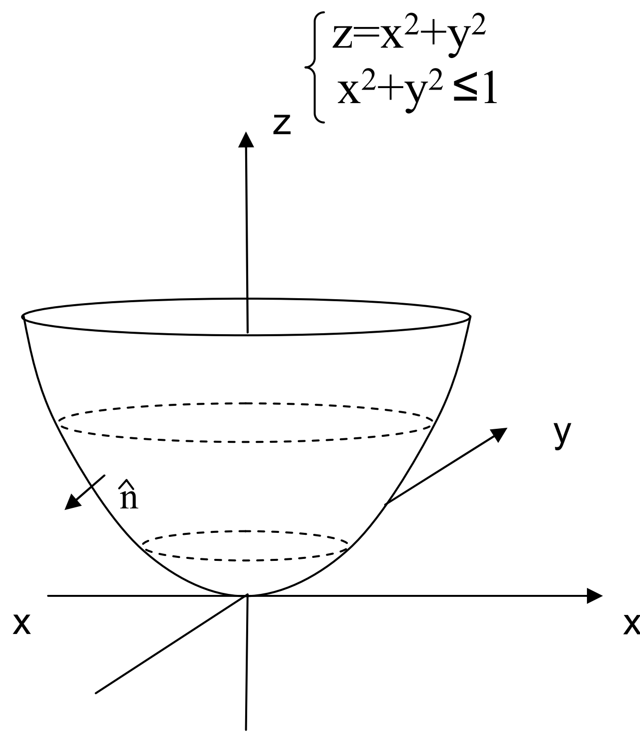
Calculate the flux of the vector field $\overline{A}=(xy,0,z^2)$
through the surface S: $z=x^2+y^2$ $x^2+y^2\leq 1$
$\hat{n}\cdot\hat{e}_z<0$

This defines in which direction we will calculate the flux.
$\hat{n}$ is chosen so that z-component is negative.

SOLUTION: 1- figure
2- Parameterization of S
3- Flux calculation using equation 10







EXAMPLE

Parameterization of S $z=x^2+y^2$

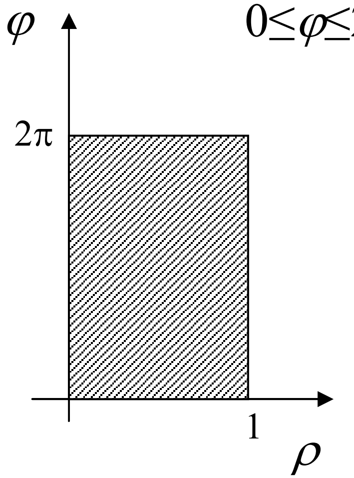
$x^2+y^2 \leq 1$

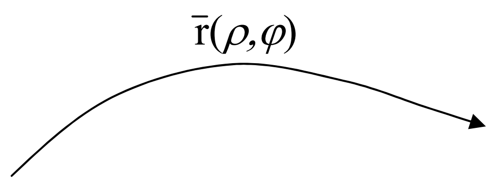
$\bar{r}(\rho,\varphi)$
$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = x^2 + y^2 = (\rho\sin\varphi)^2 + (\rho\sin\varphi)^2 = \rho^2 \end{cases}$$

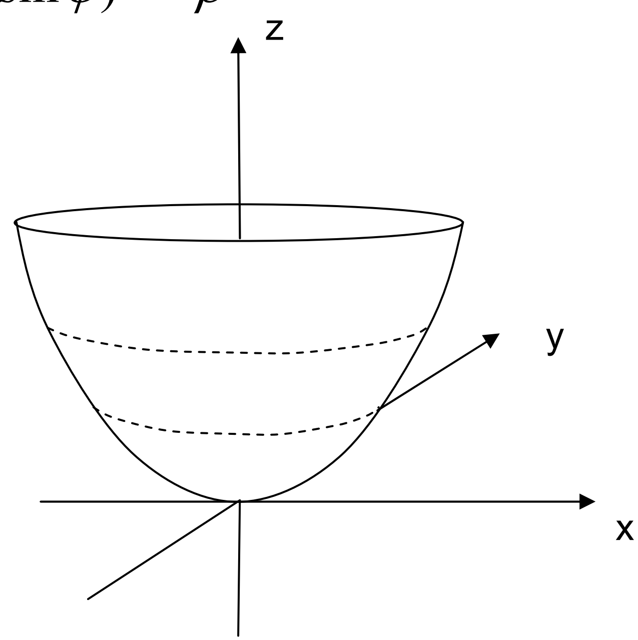
ρ and φ are polar coordinates

$0 \leq \rho \leq 1$

$0 \leq \varphi \leq 2\pi$







Parameterization of the vector field: $\bar{A}=(xy,0,z^2)=(\rho^2\sin\varphi\cos\varphi,\ 0,\ \rho^4)$

EXAMPLE

Flux calculation using equation 10

$$\iint_S \bar{A}\cdot d\bar{S} = \iint_S \bar{A}(\bar{r}(\rho,\varphi))\cdot\left(\frac{\partial\bar{r}}{\partial\rho}\times\frac{\partial\bar{r}}{\partial\varphi}\right)d\rho d\varphi$$

$$\left.\begin{aligned}\frac{\partial\bar{r}}{\partial\rho} &= (\cos\varphi, \sin\varphi, 2\rho)\\ \frac{\partial\bar{r}}{\partial\varphi} &= (-\rho\sin\varphi, \rho\cos\varphi, 0)\end{aligned}\right\}\Rightarrow\left(\frac{\partial\bar{r}}{\partial\rho}\times\frac{\partial\bar{r}}{\partial\varphi}\right)=\begin{vmatrix}\hat{e}_x & \hat{e}_y & \hat{e}_z\\ \cos\varphi & \sin\varphi & 2\rho\\ -\rho\sin\varphi & \rho\cos\varphi & 0\end{vmatrix}=$$

$$(-2\rho^2\cos\varphi, -2\rho^2\sin\varphi, \rho\cos^2\varphi+\rho\sin^2\varphi)=$$

$$(-2\rho^2\cos\varphi, -2\rho^2\sin\varphi, \rho)$$

Note that $\left(\frac{\partial\bar{r}}{\partial\rho}\times\frac{\partial\bar{r}}{\partial\varphi}\right)$ has a positive z-component,
while the flux was in the other direction.

So we ordinary solve the integral but then we <u>change the sign</u> in the answer!

EXAMPLE

$$\iint_{-S} \overline{A}\cdot d\overline{S} = \iint_{-S} \overline{A}(\overline{r}(\rho,\varphi))\cdot\left(\frac{\partial\overline{r}}{\partial\rho}\times\frac{\partial\overline{r}}{\partial\varphi}\right)d\rho d\varphi =$$

$$\int_0^{2\pi}\int_0^1\left(\rho^2\sin\varphi\ \cos\varphi,0,\rho^4\right)\cdot(-2\rho^2\cos\varphi,-2\rho^2\sin\varphi,\rho)d\rho d\varphi =$$

$$\int_0^{2\pi}\int_0^1\left(-2\rho^4\sin\varphi\ \cos^2\varphi+0+\rho^5\right)d\rho d\varphi =$$

$$\int_0^{2\pi}\left[-\frac{2}{5}\rho^5\sin\varphi\ \cos^2\varphi+\frac{1}{6}\rho^6\right]_0^1 d\varphi = \int_0^{2\pi}\left(-\frac{2}{5}\sin\varphi\ \cos^2\varphi+\frac{1}{6}\right)d\varphi =$$

$$\left[-\frac{2}{5}\left(-\frac{\cos^3\varphi}{3}\right)+\frac{1}{6}\varphi\right]_0^{2\pi} = \frac{\pi}{3}$$

But we must change sign! The answer is $-\frac{\pi}{3}$

WHICH STATEMENT IS WRONG?

1- The flux integral is a scalar (yellow)

2- Flux integrals can be calculated also on a closed surface. (red)

3- The perpendicular to the integration surface points out from an arbitrary side. (green)

4- The flux through a membrane can be calculated with flux integrals. (blue)