VEKTORANALYS

Kursvecka 1

GRADIENTEN

Kapitel 1-3
Sidor 3-28

TARGET PROBLEM

A mosquito is flying around in the room.
How does she find us during the night?
The mosquito must know:

(A) How the temperature $T(x,y,z)$ changes along the flying direction

(B) In which direction she must fly to be in a warmer place as soon as possible

We need to:

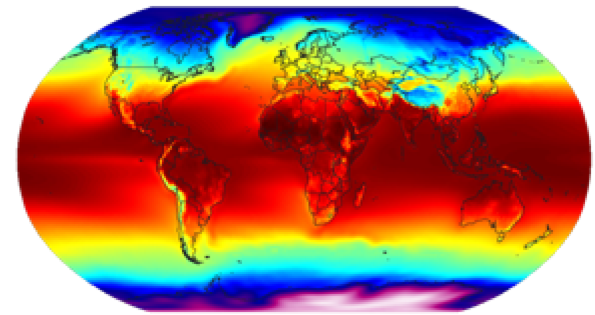
(1) introduce a **SCALAR FIELD**, $T(x,y,z)$

(2) measure the **change of the scalar field** $T(x,y,z)$ in R^3 (derivative)

(3) find the **direction** for which **the change** of $T(x,y,z)$ **is maximum**

SCALAR FIELD AND VECTOR FIELD

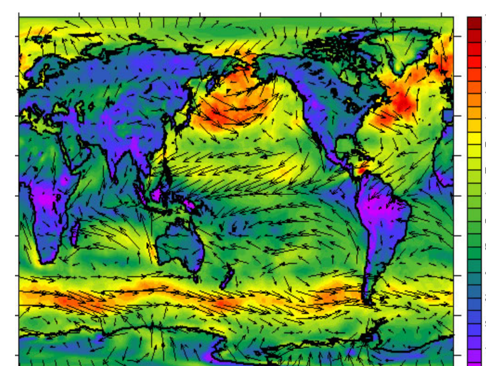
A **scalar field** associates a real number $\phi(x,y,z)$ to each point (x,y,z) of the space.



Examples: - temperature distribution in the space
- pressure distribution in a fluid
- potential around an electric charge

A **vector field** associates a vector $\bar{\mathbf{A}}(x,y,z)$ to each point (x,y,z) of the space.

Examples: - velocity distribution in a fluid
- magnetic field around a magnet
- electric field around an electric charge



To solve our problem, today we will focus on scalar fields

LEVEL SURFACES

- Level surfaces are useful to visualize a scalar field.
- What is a level surface?

A surface on which the scalar field $\phi(x,y,z)$ is constant:

$$\phi(x,y,z)=c \tag{1}$$

- To create an "image" of the scalar field $\phi(x,y,z)$ we can consider a family of level surfaces:

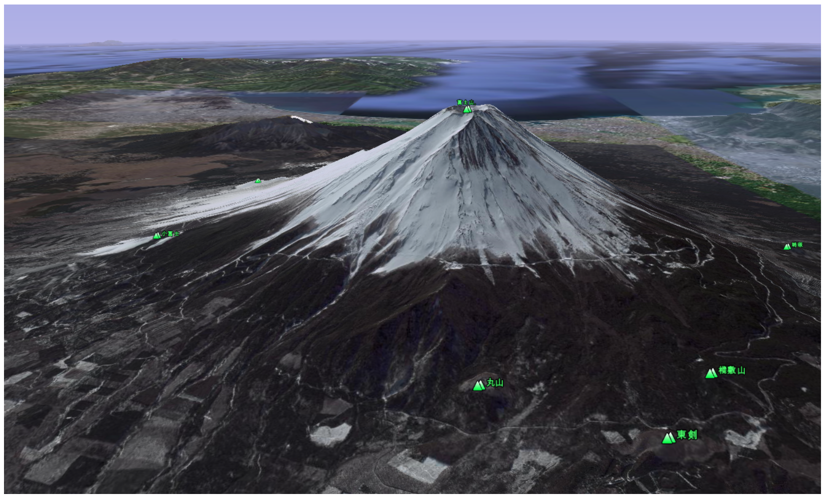
$$\phi(x,y,z)=c+nh \tag{2}$$

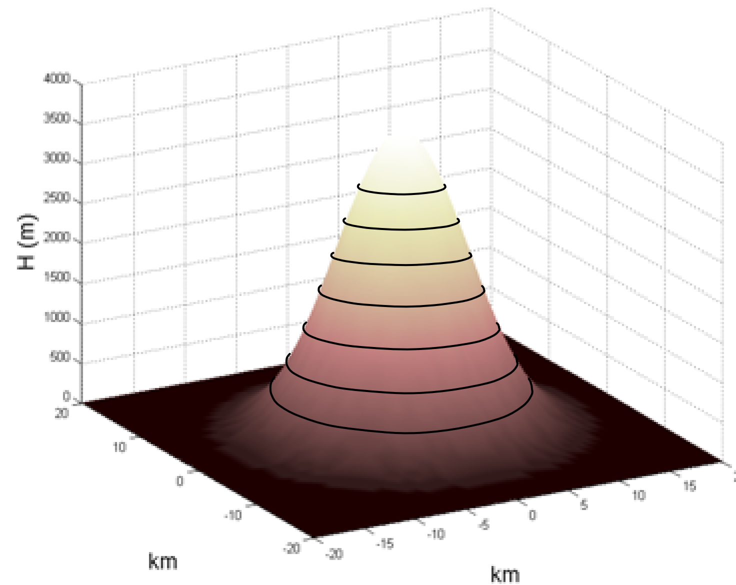
where h is a constant and $n=0, \pm 1, \pm 2, \pm 3, \ldots$
To improve the details of the "image", h must be reduced.

- In two dimension the scalar field is $\phi(x,y) \Rightarrow$ we have level curves!

EXAMPLE

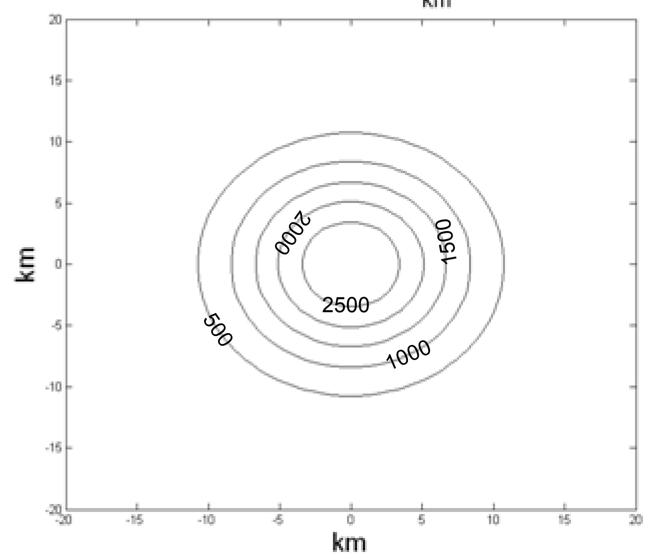
- Assume that H(x,y) is a scalar field corresponding to the height of a mountain





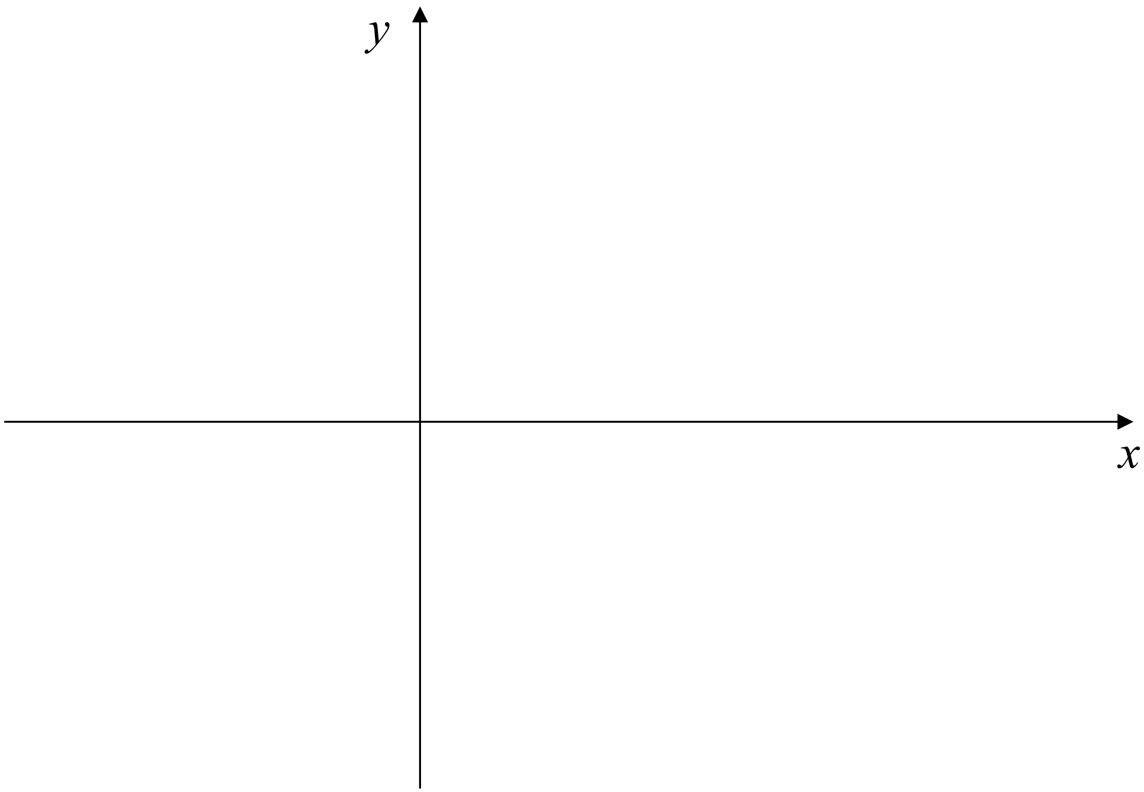
Let's consider a family of level curves:

$$H(x,y)=c+nh$$

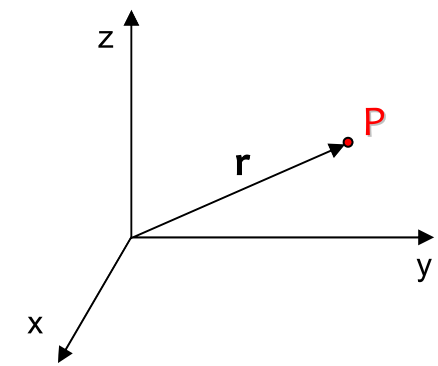


EXERCISE

Plot the level curves of the scalar field: $\phi = \dfrac{x^2}{4} + y^2$



POSITION VECTOR



- The vector from the origin to the point P=(x,y,z) is called position vector $\overline{\mathbf{r}}$
- Note that $\overline{\mathbf{r}}$ depends on the choice of the coordinate system
- $\overline{\mathbf{r}}$ can be expressed with different notations:

$$\overline{\mathbf{r}}=\overline{\mathbf{r}}(x,y,z) \qquad \overline{\mathbf{r}}=(x,y,z) \qquad \overline{\mathbf{r}}=x\hat{e}_x+y\hat{e}_y+z\hat{e}_z$$

- The differential of a position vector can be written as a vector whose components are the differential of each position vector component:

$$\overline{\mathbf{r}}=x\hat{e}_x+y\hat{e}_y+z\hat{e}_z \qquad (3)$$

$$\Downarrow$$

$$d\overline{\mathbf{r}}=\hat{e}_x\,dx + \hat{e}_y\,dy + \hat{e}_z\,dz \qquad (4)$$

THE GRADIENT

- Assume that $\phi(x,y,z)$ is a continuous and derivable scalar field
- DEFINITION:

$$grad\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \qquad (5)$$

the gradient of a scalar field is a vector field!
This definition assume a right-handed cartesian coordinate system.

EXERCISE: calculate the gradient of the vector field: $\phi = \frac{x^2}{4} + y^2$ and plot $grad\phi$ in the point $P=(2,0)$ and in the point $P=(0,-1)$

- Scalar field differential:

$$d\phi = grad\phi \cdot d\bar{r} \qquad (6)$$

This expression can be used as a coordinate-free definition of the gradient

- Let's introduce :
  - the amplitude of the position vector differential, ds, and
  - the direction ê

$$d\bar{\mathbf{r}} = \hat{e}\, ds \qquad (7)$$

Equations (6) and (7) give:

Directional derivative

$$\frac{d\phi}{ds} = grad\phi \cdot \hat{e} \qquad (8)$$

The rate of variation of ϕ in a given direction corresponds to the component of the vector gradient in that direction

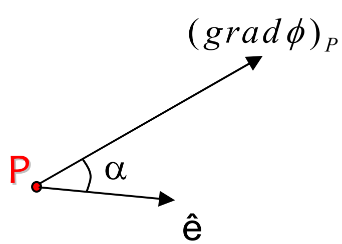
Answer to question (A)!!

THE GRADIENT

THEOREM 1 *(3.1 in the textbook)*

The gradient in the point P is a vector that points to the direction in which the growth of ϕ in P is the highest.
The maximum increase of ϕ per unit length is $|(grad\phi)_P|$

PROOF



a- let's calculate the derivative in the direction ê *Eq. (8)*

$$\frac{d\phi}{ds} = grad\phi \cdot \hat{e} = |grad\phi| \cos\alpha$$

b- this is maximum when:

cos α =1

which implies:

α=0 (ê // $grad\phi$) and $\frac{d\phi}{ds} = |grad\phi|$

Answer to question (B)!!

THE GRADIENT

THEOREM 2 *(3.2 in the textbook)*

The gradient in the point P is zero if ϕ has a maximum or a minimum in P

PROOF

From Equation (8): $$\frac{d\phi}{ds} = grad\,\phi \cdot \hat{e}$$

a- ϕ has a maximum or a minimum in P $\Rightarrow$ $d\phi/ds=0$

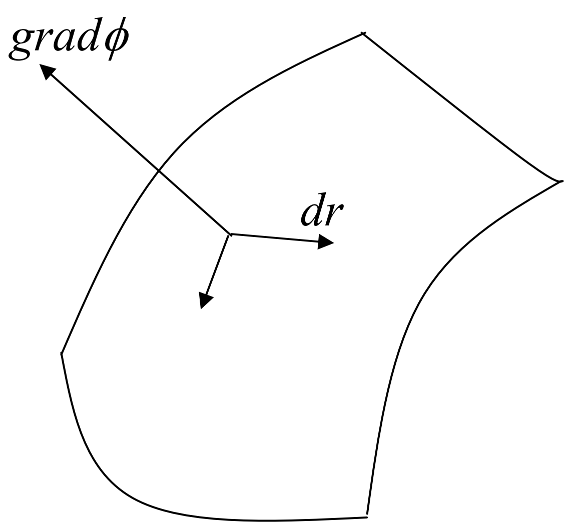
b- Using equation (8), $d\phi/ds=0$ implies $\mathrm{grad}\,\phi=0$

THE GRADIENT

THEOREM 3 *(3.3 in the textbook)*

The gradient of a scalar field $\phi(x,y,z)$ in the point P is orthogonal to the level surface ϕ=c in P.

PROOF



a- Let's do a small movement $d\bar{r}$ along the level surface

b- Remember that on the level surface ϕ is constant:

$$d\phi=0$$

c- Then, using equation (6):

$$d\phi = grad\,\phi \cdot d\bar{r} = 0$$

d- This implies that $grad\phi$ is perpendicular to $d\bar{r}$

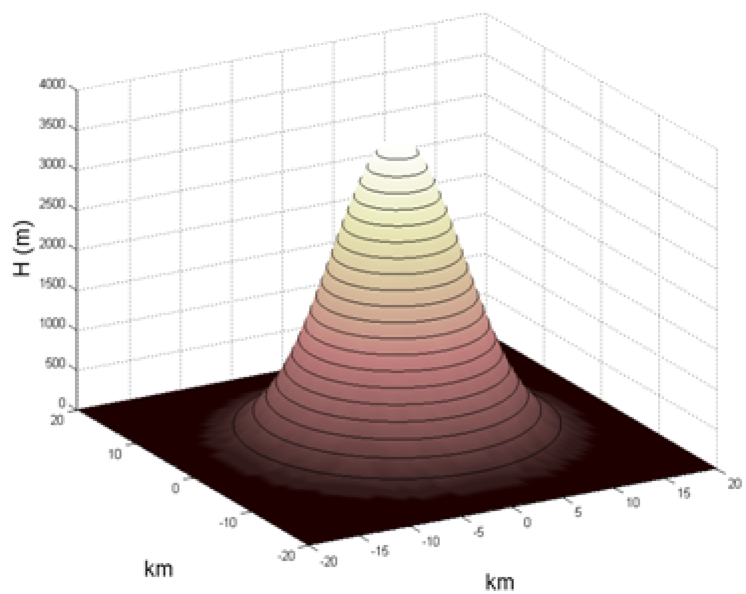
e- $grad\phi$ is perpendicular to each $d\bar{r}$ on the level surface $grad\phi$ is perpendicular to the level surface

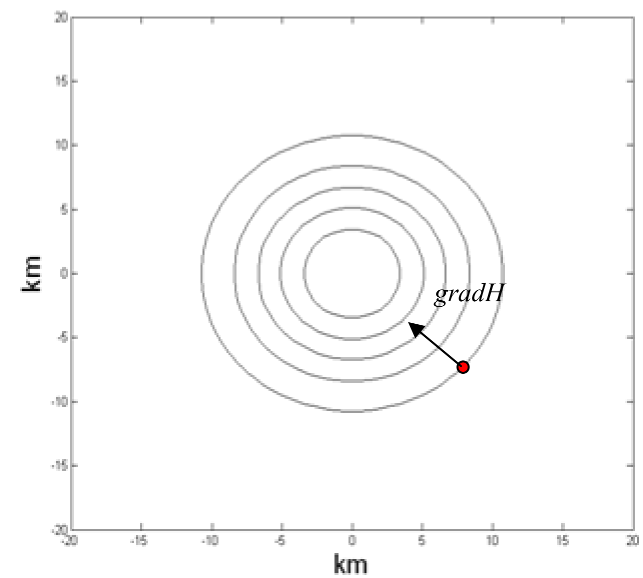
2D-EXAMPLE

- Assume that H(x,y) is a scalar field corresponding to the height of a mountain
- In 2D the gradient is: $$gradH = \left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y} \right)$$

(remember that H must be a continuous and derivable scalar field)

- Theorems 1, 2 and 3 are valid also in two dimensions.
- $gradH$ is a vector field that:
  - in each point is orthogonal to the level curve in that point and
  - always points along the direction in which the height grows faster





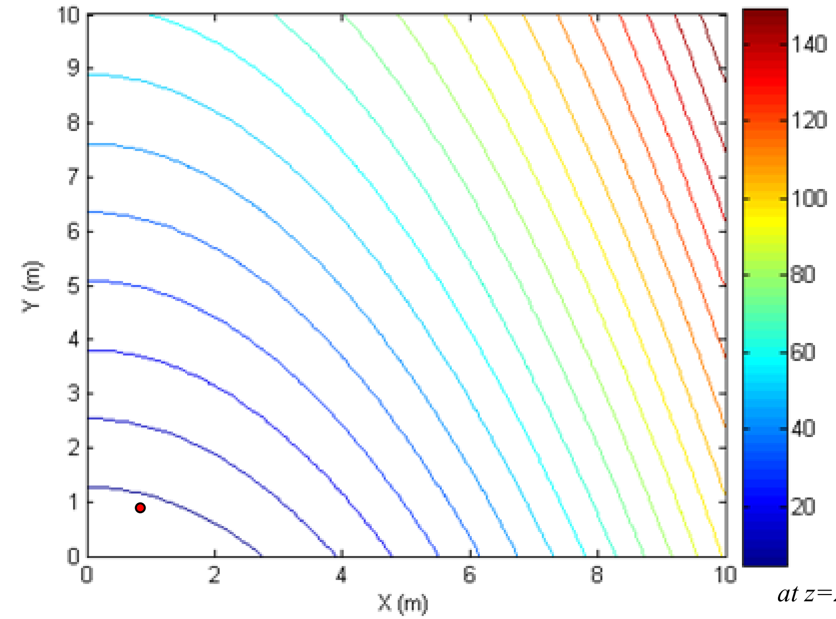
TARGET PROBLEM

A mosquito is flying around in the room.

The temperature is described by the scalar field:

$$T(x,y,z)=x^2+2yz-z \quad [^\circ C]$$

The mosquito is in the point P=(1,1,2)



(a) In which direction the mosquito will fly to be in a warmer place as quick as possible?

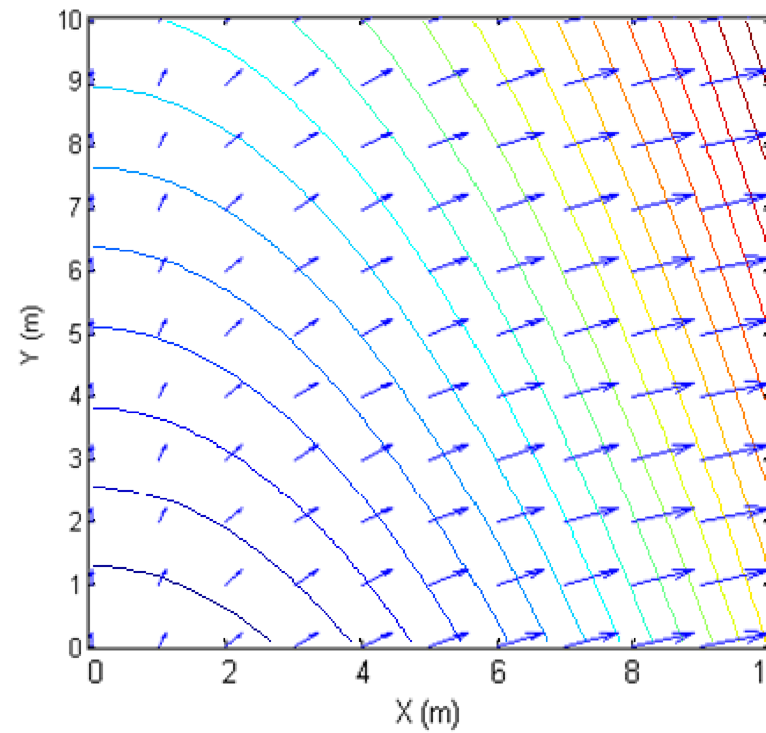
(b) How much would the temperature change if the mosquito flies with velocity 3m/s in direction (-2,2,1)?

TARGET PROBLEM

(a) In which direction the mosquito will fly to be warm as quick as possible?

We use **theorem 1:** The gradient in the point P is a vector that points to the direction in which the scalar field in P has the highest growth.

at z=2



From definition (5):

$$gradT = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right)$$

$$T(x, y, z) = x^2 + 2yz - z$$

$$\frac{\partial T}{\partial x} = 2x, \qquad \frac{\partial T}{\partial y} = 2z, \qquad \frac{\partial T}{\partial z} = 2y - 1$$

$$gradT = (2x, 2z, 2y - 1)$$

The mosquito is in P=(1,1,2)

$$(gradT)_{P=(1,1,2)} = (2\cdot 1, 2\cdot 2, 2\cdot 1 - 1) = (2,4,1)$$

The mosquito will fly in direction (2,4,1)

TARGET PROBLEM

(b) How much would the temperature change if the mosquito flies with velocity 3m/s in direction (-2,2,1)?

We must calculate $\frac{dT}{dt}$ where t is the time

Using equation (6):

$$\frac{dT}{dt} = gradT \cdot \frac{d\bar{r}}{dt} = gradT \cdot \hat{e}\frac{ds}{dt}$$

where

$$\left\{ \begin{aligned} &\frac{ds}{dt} = |\bar{v}| = 3m/s \\ &\hat{e} = \frac{\bar{v}}{|\bar{v}|} = \frac{(-2,2,1)}{|(-2,2,1)|} = \frac{(-2,2,1)}{\sqrt{(-2)^2+2^2+1^2}} = \frac{(-2,2,1)}{3} \end{aligned} \right\} \Rightarrow \frac{d\bar{r}}{dt} = \frac{(-2,2,1)}{3}\cdot 3 = (-2,2,1)$$

$$\frac{dT}{dt} = gradT \cdot \frac{d\bar{r}}{dt} = (2,4,1)\cdot(-2,2,1) = 5 \quad \text{[C/s]}$$

WHICH STATEMENT IS WRONG?

1- A scalar field associates a real number to a point in space (yellow)

2- A scalar field can be written as $\overline{\mathbf{A}}=\overline{\mathbf{A}}(x,y,z)$ (red)

3- If ϕ is a scalar then $grad\,\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$ in R^3 (green)

4- The ϕ increase in a given direction depends on the directional derivative (blue)