VEKTORANALYS

Kursvecka 2

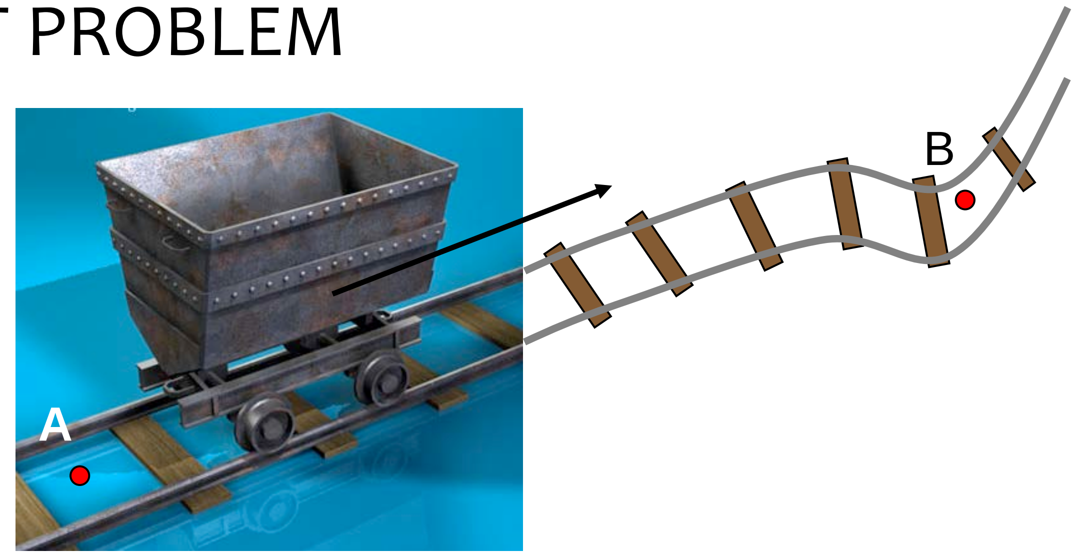
LINE INTEGRAL
and
FLUX INTEGRAL

Kapitel 4-5
Sidor 29-50

TARGET PROBLEM

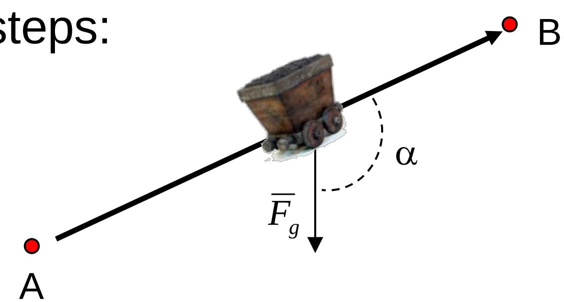
A person is pushing a mine cart along a path L on a hill.

Calculate the "work" done to move the cart from A to B.

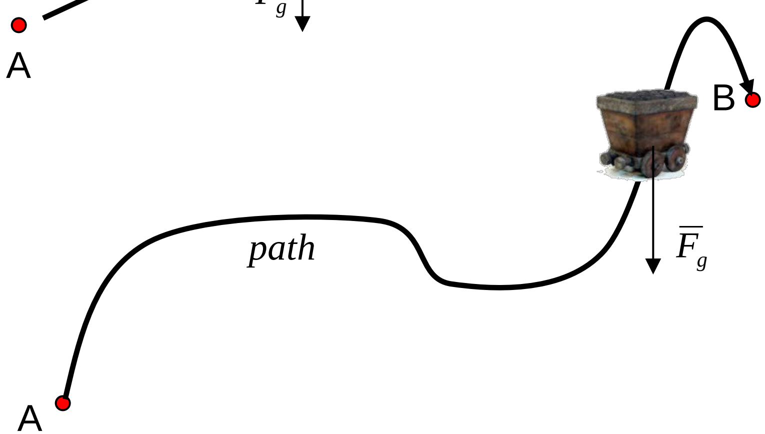


We will arrive to the final answer in two steps:

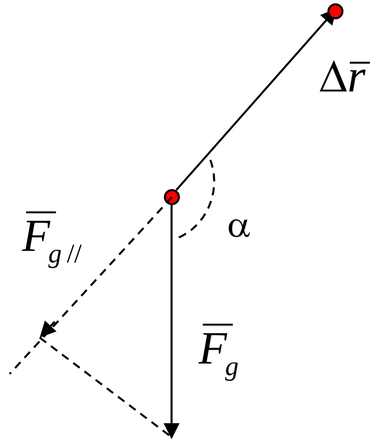
1- The slope is constant



2- the slope is not constant



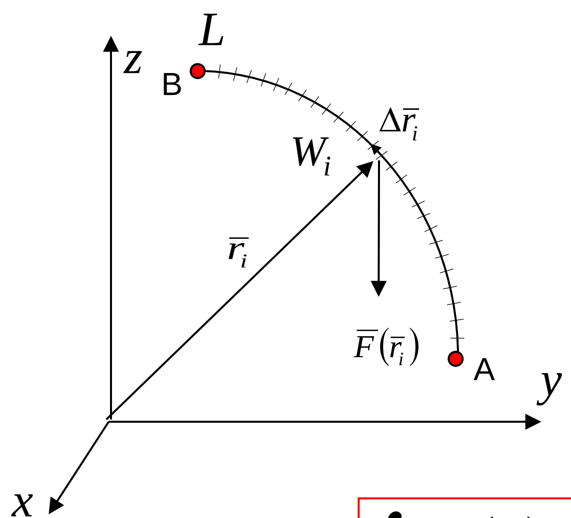
Step 1:



$$W = \left|\overline{F}_{g//}\right|\left|\Delta\overline{r}\right| = \left|\overline{F}_g\right|\left|\Delta\overline{r}\right|\cos\alpha = \overline{F}\cdot\Delta\overline{r}$$

(to be precise, we should change sign to calculate the work done by the person)

Step 2:



$$\left.\begin{aligned} W &= \sum_i W_i \\ W_i &\approx \overline{F}(\overline{r}_i)\cdot\Delta\overline{r}_i \end{aligned}\right\} \Rightarrow W \approx \sum_i \overline{F}(\overline{r}_i)\cdot\Delta\overline{r}_i$$

For *"very small"* segments:

$$W = \lim_{\Delta\overline{r}_i\to 0}\sum_i \overline{F}(\overline{r}_i)\cdot\Delta\overline{r}_i \equiv \int_L \overline{F}(\overline{r})\cdot d\overline{r}$$

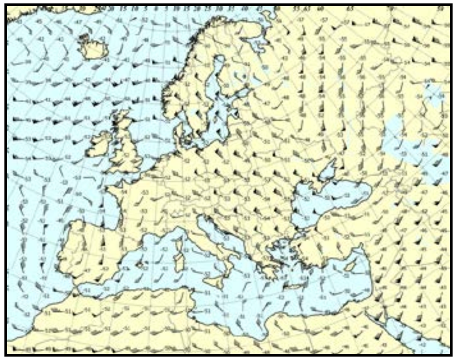
$\boxed{\int_L \overline{F}(\overline{r})\cdot d\overline{r}}$ is the line integral of $\overline{F}$ along the path L

We need to:

- introduce a **VECTOR FIELD**, $\overline{F}(\overline{r})$
- Define the **infinitesimal displacement** $d\overline{r}$ along the path L

VECTOR FIELD

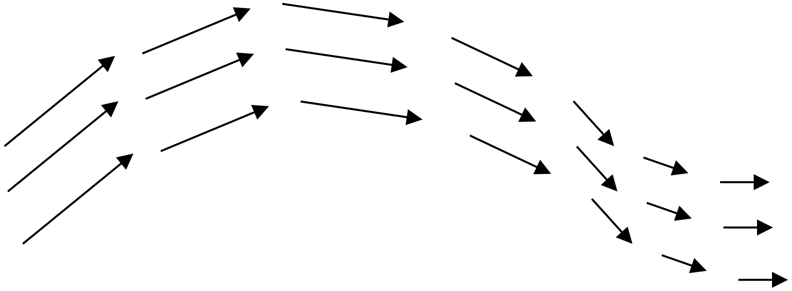
A **vector field** associates a vector $\bar{A}$(x,y,z) to each point (x,y,z) of the space.



Examples: - velocity distribution in a fluid
- magnetic field around a magnet
- electric field around an electric charge

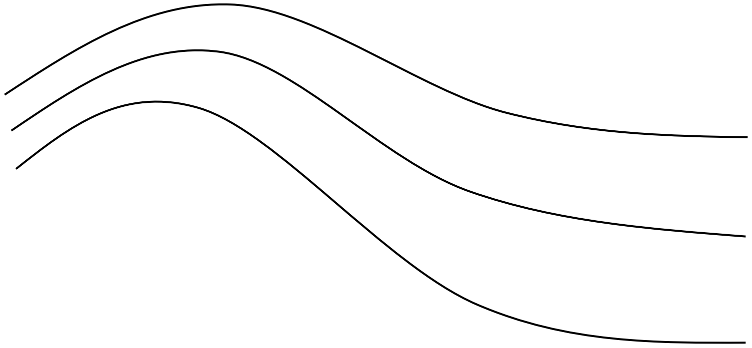
Two typical ways to represent a vector field:

1- Arrow field



- The arrow length is proportional to the field amplitude
- The arrow direction shows the field direction

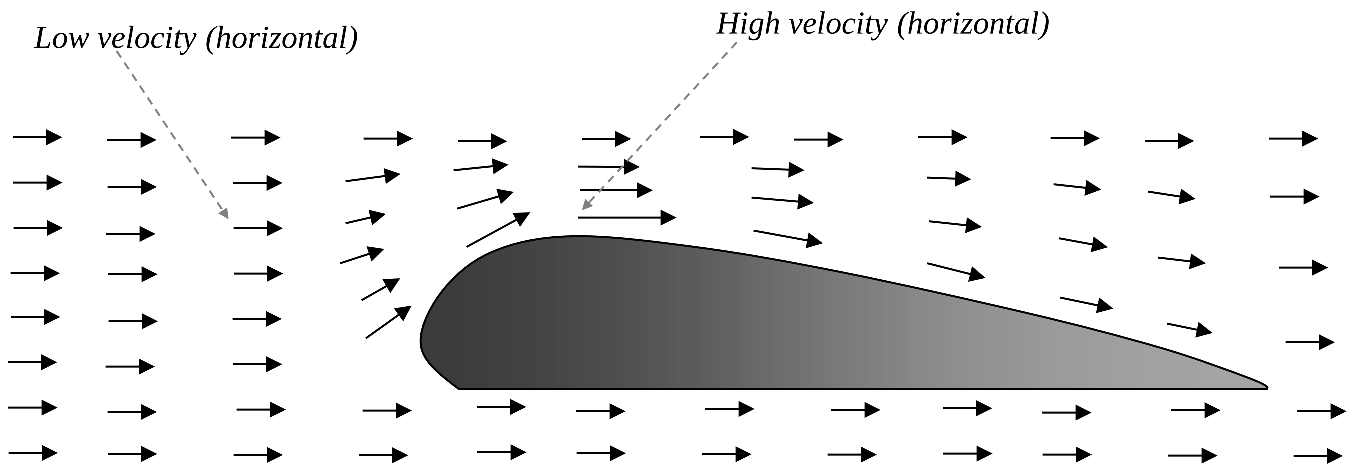
2- Line field

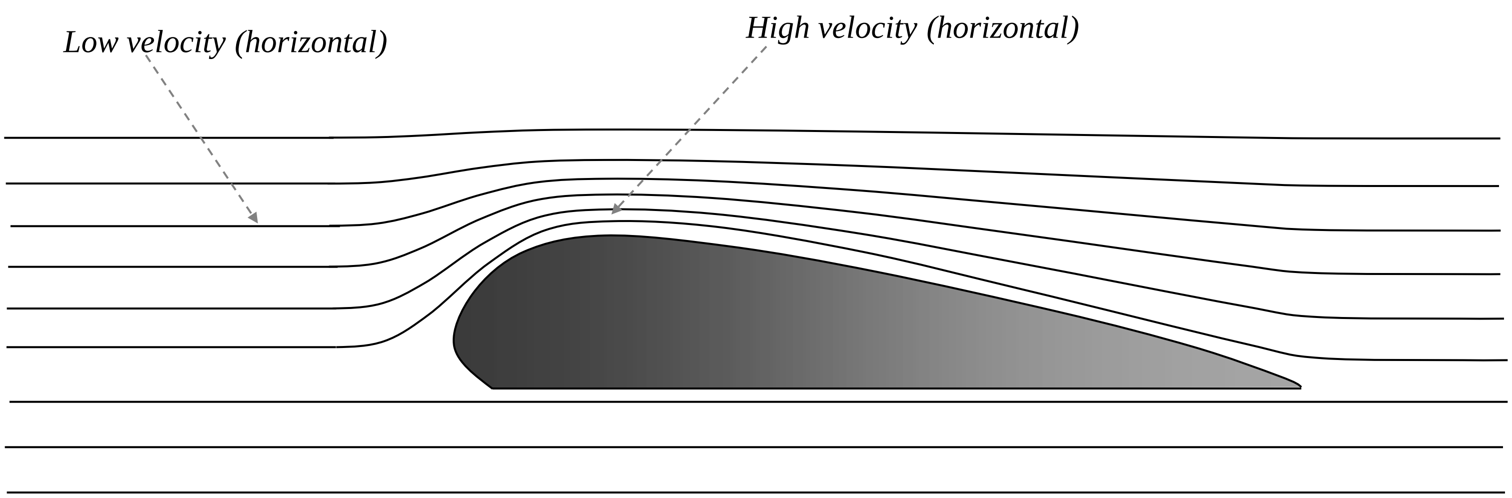


- The tangent to the curves is parallel to the vector field in all points.
- The density of the lines is proportional to the strength of the field.

VECTOR FIELD

The airplane wing example (velocity field of air around a wing)

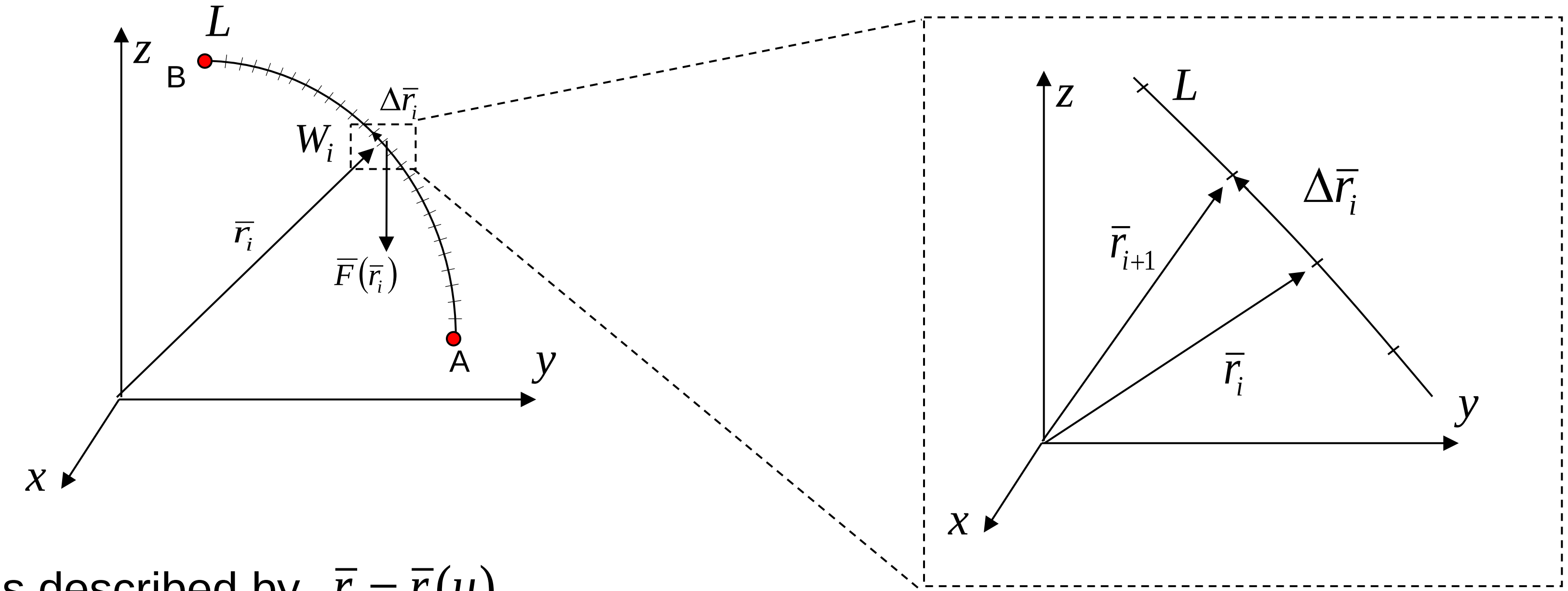




EXERCISE: 1- plot the vector field $\bar{F}(\bar{r}) = (x, 0)$

2- write $\bar{F}(\bar{r})$ on the curve defined by: $\bar{r}(u) = (u, u^2)$

$d\bar{r}$ AND THE LINE INTEGRAL



L is described by $\bar{r} = \bar{r}(u)$

$$\left.\begin{aligned} \bar{r}_i &= \bar{r}(u_i) \\ \bar{r}_{i+1} &= \bar{r}(u_{i+1}) \\ u_{i+1} &= u_i + \Delta u \end{aligned}\right\} \Rightarrow d\bar{r} = \lim_{\Delta u \to 0} \Delta\bar{r}_i = \lim_{\Delta u \to 0} \left[\bar{r}(u_{i+1}) - \bar{r}(u_i)\right] = \lim_{\Delta u \to 0} \frac{\left[\bar{r}(u_{i+1}) - \bar{r}(u_i)\right]}{\Delta u} \Delta u = \frac{d\bar{r}}{du} du$$

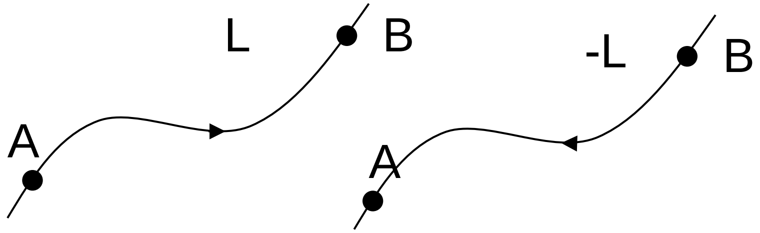
So, the line integral
can be calculated as:

$$\boxed{\int_L \bar{F}(\bar{r}) \cdot d\bar{r} = \int_a^b \bar{F}(\bar{r}(u)) \cdot \frac{d\bar{r}}{du} du}$$

EXERCISE: Calculate $\int_L \bar{F}(\bar{r}) \cdot d\bar{r}$ with $\bar{F}(\bar{r}) = (x, 0)$ and L defined by $\bar{r}(u) = (u, u^2)$
$u : 0 \to 1$

LINE INTEGRAL (some useful properties)

THEOREM 1 *(4.2 in the textbook)*



$$\boxed{\int_{-L} \overline{F}(\overline{r}) \cdot d\overline{r} = -\int_{L} \overline{F}(\overline{r}) \cdot d\overline{r}} \qquad (3)$$

PROOF

If all line elements change sign then also the integral will change sign.

DEFINITION

The line of integral of $\overline{A}$ along a closed curve C is called
circulation of a $\overline{A}$ along C:

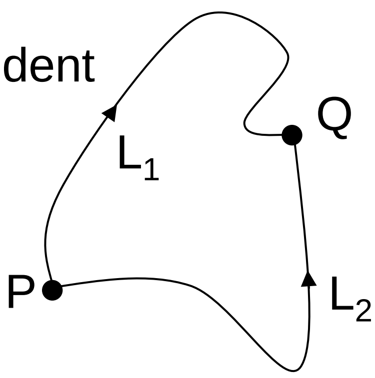
$$\boxed{\oint_{C} \overline{A}(\overline{r}) \cdot d\overline{r}} \qquad (4)$$

LINE INTEGRAL

DEFINITION: A vector field $\overline{A}$ is called <u>conservative</u> if: $\oint_C \overline{A}(\overline{r})\cdot d\overline{r} = 0$

THEOREM 2 *(4.3 in the textbook)*

The circulation of $\overline{A}$ along a close curve C is zero if and only if for all points P and Q the line integral of $\overline{A}$ from P to Q is independent from the integration path between P and Q.



PROOF

Assume that L_1 and L_2 are two curves from P to Q.
Then $L=L_1-L_2$ is a closed curve.

(1) The circulation is zero $\Rightarrow$ the line integral from P to Q is independent from the path

$$\left.\begin{aligned}&\oint_L \overline{A}(\overline{r})\cdot d\overline{r} = 0\\&\oint_L \overline{A}(\overline{r})\cdot d\overline{r} = \int_{L_1-L_2} \overline{A}(\overline{r})\cdot d\overline{r} = \int_{L_1} \overline{A}(\overline{r})\cdot d\overline{r} - \int_{L_2} \overline{A}(\overline{r})\cdot d\overline{r}\end{aligned}\right\} \Rightarrow \int_{L_1} \overline{A}(\overline{r})\cdot d\overline{r} = \int_{L_2} \overline{A}(\overline{r})\cdot d\overline{r}$$

The line integral is independent from the integration path!

(2) The line integral from P to Q is independent from the path $\Rightarrow$ the circulation is zero.

$$\left.\begin{aligned}&\int_{L_1} \overline{A}(\overline{r})\cdot d\overline{r} = \int_{L_2} \overline{A}(\overline{r})\cdot d\overline{r}\\&\oint_L \overline{A}(\overline{r})\cdot d\overline{r} = \int_{L_1-L_2} \overline{A}(\overline{r})\cdot d\overline{r} = \int_{L_1} \overline{A}(\overline{r})\cdot d\overline{r} - \int_{L_2} \overline{A}(\overline{r})\cdot d\overline{r}\end{aligned}\right\} \Rightarrow \oint_L \overline{A}(\overline{r})\cdot d\overline{r} = 0$$

The circulation is zero

LINE INTEGRAL

THEOREM 3 *(4.4 in the textbook)*

If $\overline{A} = grad\phi$:

$$\boxed{\int_P^Q \overline{A}(r)\cdot d\overline{r} = \phi(Q) - \phi(P)} \qquad (5)$$

So the line integral is independent from the integration path and depends only on the starting point and on the ending point

PROOF

If $\overline{r}(u)$ is a curve from P to Q then, using the chain rule for the partial derivative:

$$\int_{\overline{r}(u)} \overline{A}(\overline{r})\cdot d\overline{r} = \int_p^q grad\phi \cdot \frac{d\overline{r}}{du} du = \int_p^q \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)\cdot\left(\frac{dx}{du}, \frac{dy}{du}, \frac{dz}{du}\right) du =$$

$$= \int_p^q \left(\frac{\partial\phi}{\partial x}\frac{dx}{du} + \frac{\partial\phi}{\partial y}\frac{dy}{du} + \frac{\partial\phi}{\partial z}\frac{dz}{du}\right) du = \int_p^q \frac{d}{du}\phi(\overline{r}(u)) du = \phi(Q) - \phi(P)$$

Or, easier: $\int_L \overline{A}(\overline{r})\cdot d\overline{r} = \int_L grad\phi \cdot d\overline{r} = \int_L d\phi = \phi(B) - \phi(A)$

OTHER KINDS OF LINE INTEGRALS

- It is possible to combine scalar and vector line elements in many different ways along a curve L and thus get different kinds of line integrals

- Some examples:

$$\int_L \phi(\overline{r})ds \tag{6}$$

where $d\overline{r} = \hat{e}ds$

$$\int_L \phi(\overline{r})d\overline{r} \tag{7}$$

$$\int_L A(\overline{r}) \times d\overline{r} \tag{8}$$

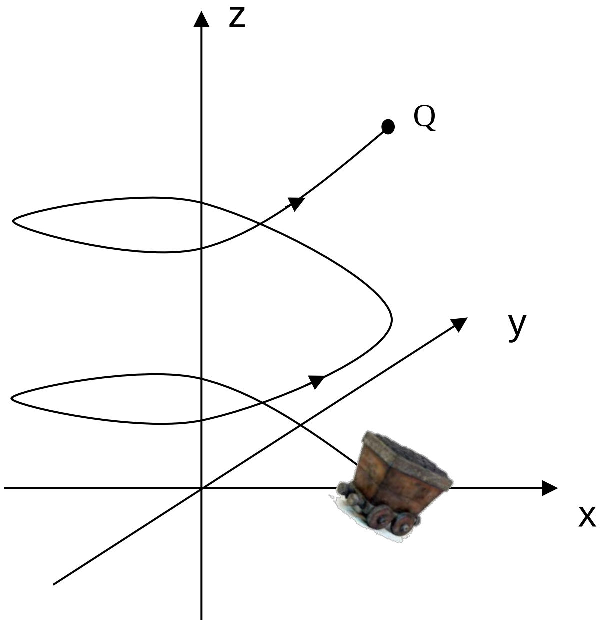
- To calculate the integrals:

$$\overline{r} \to \overline{r}(u)$$

$$L \to [a,b] \quad where\, u : a \to b$$

$$d\overline{r} = \frac{d\overline{r}}{du}du \quad or \quad ds = \left|\frac{d\overline{r}}{du}\right|du$$

EXAMPLE



The force is: $\overline{F} = (-yz, xz, -1)$

The path is L: $\overline{r} = (\cos u, \sin u, u)$ with $u{:}0 \to 4\pi$

Calculate the work from P=(1,0,0) till Q=(1,0,4π)

from definition (2)

$$W = \int_L \overline{F}(\overline{r}) \cdot d\overline{r} = \int_{a=0}^{b=4\pi} \overline{F}(\overline{r}(u)) \frac{d\overline{r}}{du} du$$

$$\left.\begin{aligned} \overline{F}(\overline{r}(u)) &= (-(\sin u)u, (\cos u)u, -1) \\ \frac{d\overline{r}}{du} &= (-\sin u, \cos u, 1) \end{aligned}\right\} \Rightarrow \overline{F}(\overline{r}(u)) \cdot \frac{d\overline{r}}{du} = (u\sin^2 u + u\cos^2 u - 1) =$$

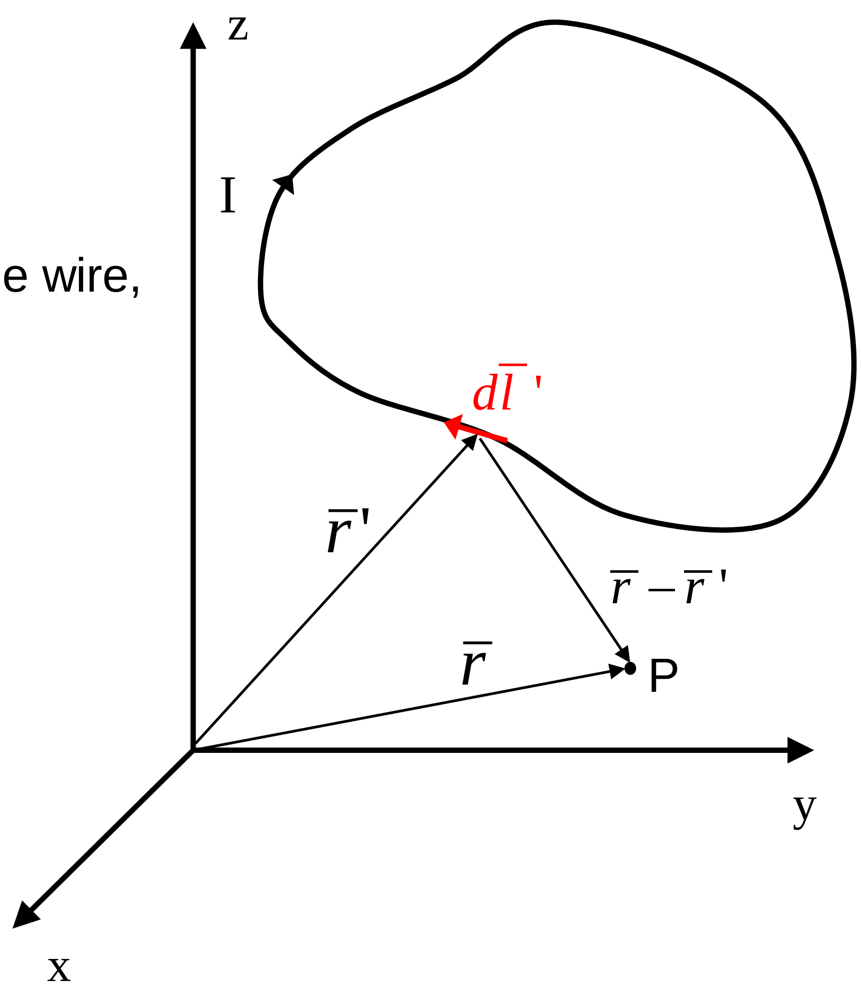
$$= u(\sin^2 u + \cos^2 u) - 1 = u - 1$$

$$W = \int_L \overline{F}(\overline{r}) \cdot d\overline{r} = \int_0^{4\pi} (u-1) du = \left[\frac{u^2}{2} - u\right]_0^{4\pi} = 8\pi^2 - 4\pi$$

PRACTICAL EXAMPLE: **THE BIOT-SAVART LAW**

The magnetic field in a point P of a steady line current is given by the Biot-Savart law:

$$\bar{B}(\bar{\mathrm{r}}) = \frac{\mu_0 I}{4\pi} \oint_L \frac{d\bar{l}\,' \times (\bar{r} - \bar{r}\,')}{\left|\bar{r} - \bar{r}\,'\right|^3}$$



Where $d\bar{l}\,'$ is an infinitesimal length along the wire,

$\bar{r}$ is the position vector of the point P and

$\bar{r}\,'$ is a vector from the origin to $d\bar{l}\,'$

Therefore, $\bar{r} - \bar{r}\,'$ is a vector from $d\bar{l}\,'$ to P

PRACTICAL EXAMPLE: **THE BIOT-SAVART LAW**

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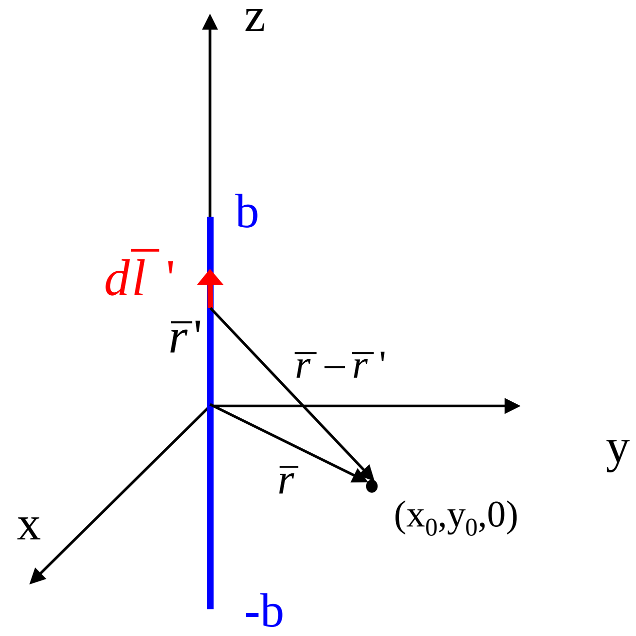
Calculate the magnetic field in $(\mathrm{x}_0,\mathrm{y}_0,0)$ produced by a straight wire with current I and length $2\mathrm{b}$



SOLUTION:

$$\begin{cases} x = 0 \\ y = 0 \\ z = u \end{cases} \Rightarrow \quad \overline{l}\,(\mathrm{u}) = \left(0,0,u\right) \qquad \textit{with} \quad u : -b \rightarrow +b$$

$$\frac{d\overline{l}\,'}{du} du = \left(0,0,1\right) du = du\,\hat{e}_z$$

$$\overline{r} - \overline{r}\,' = \left(x_0, y_0, -u\right) \quad \Rightarrow \quad \left|\overline{r} - \overline{r}\,'\right| = \sqrt{x_0^2 + y_0^2 + u^2} = \sqrt{r_c^2 + u^2}$$

$r_c^2 = x_0^2 + y_0^2$ *is the distance from the origin to P*

$$d\overline{l}\,' \times \left(\overline{r} - \overline{r}\,'\right) = du\,\hat{e}_z \times \left(x_0, y_0, -u\right) = \left(-y_0, x_0, 0\right) du$$

$$\overline{B}(\overline{\mathrm{r}}) = \frac{\mu_0 I}{4\pi} \int_{-b}^{b} \frac{\left(-y_0, x_0, 0\right) du}{\left(x_0^2 + y_0^2 + u^2\right)^{3/2}} = \frac{\mu_0 I}{4\pi} \left(-y_0, x_0, 0\right) \int_{-b}^{b} \frac{du}{\left(r_c^2 + u^2\right)^{3/2}} = \frac{\mu_0 I}{4\pi} \left(-y_0, x_0, 0\right) \left[\frac{u}{r_c^2 \sqrt{r_c^2 + u^2}}\right]_{-b}^{b} = \frac{\mu_0 I}{4\pi} \frac{2b}{r_c^2 \sqrt{r_c^2 + b^2}} \left(-y_0, x_0, 0\right)$$

$$\left|\overline{B}(\overline{\mathrm{r}})\right| = \frac{\mu_0 I}{4\pi r_c} \frac{2b}{\sqrt{r_c^2 + b^2}}$$

WHICH STATEMENT IS WRONG?

1- The image area of a vector field $\overline{A}$ is composed of vectors (yellow)

2- The line integral $\int \overline{F} \cdot d\overline{r}$ is a scalar (red)

3- The sign of the line integral $\int \overline{F} \cdot d\overline{r}$ depends on the integration path (green)

4- The gradient of a vector field can be written as: $\mathrm{grad}\,\overline{\mathrm{A}}$ (blue)

TARGET PROBLEM

We are making cranberry juice.
After cranberries are squeezed,
It is better to filter the juice!

How much juice flows trough the cloth each second?

We need :

(1) to understand how to calculate the **flux**
of a **VECTOR FIELD** $\bar{v}(x, y, z)$

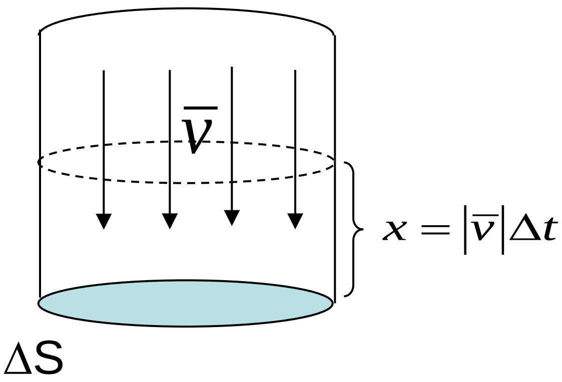
(2) a method to integrate the flux over the whole surface.

THE FLUX

In the juice example, the flux F is the volume of the fluid ΔV that flows through the surface in the time Δt.

$$F = \frac{\Delta V}{\Delta t}$$

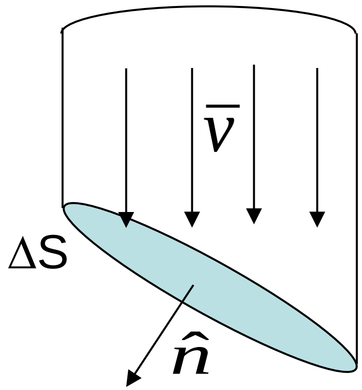
STEP 1: - the fluid velocity is perpendicular to the surface
- the surface is not curved



$$\Delta V = x\Delta S = |\overline{v}|\Delta t \Delta S$$

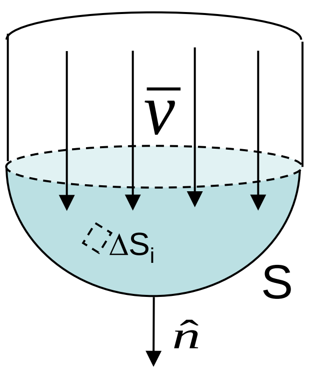
$$F = \frac{\Delta V}{\Delta t} = |\overline{v}|\Delta S$$

STEP 2: - the fluid velocity is NOT perpendicular to the surface
- the surface is not curved



$$F = |\overline{v}_\perp|\Delta S = \overline{v}\cdot\hat{n}\Delta S = \overline{v}\cdot\Delta\overline{S}$$

STEP 3: - the surface is curved

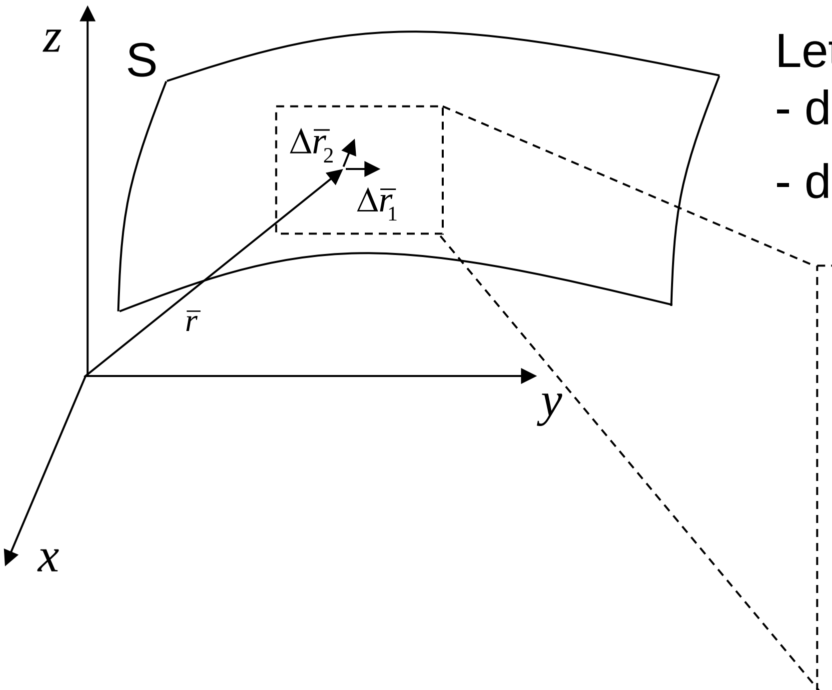


$$F = \sum_i F_i = \lim_{\Delta S_i \to 0}\sum_i \overline{v}_i \cdot \Delta\overline{S}_i \equiv \int_S \overline{v}\cdot d\overline{S}$$

$\boxed{\int_S \overline{v}\cdot d\overline{S}}$ is the flux integral of $\overline{v}$ on the surface S

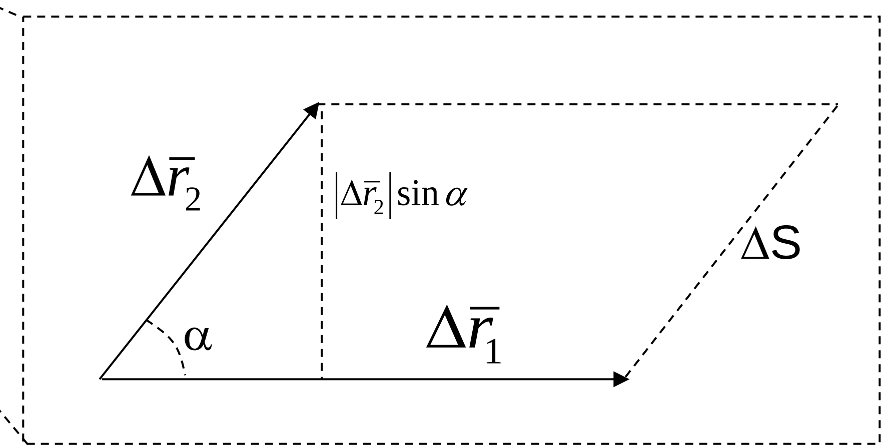
$d\overline{S}$ AND FLUX INTEGRAL

• Assume that the surface S is parameterized by $\overline{r} = \overline{r}(u,v)$



Let's consider two displacements,

- due a change in u: $\overline{r}_1 = \overline{r}(u+\Delta u, v) \Rightarrow \Delta\overline{r}_1 = \overline{r}_1 - \overline{r}$
- due a change in v: $\overline{r}_2 = \overline{r}(u, v+\Delta v) \Rightarrow \Delta\overline{r}_2 = \overline{r}_2 - \overline{r}$



The area ΔS is $\Delta S = |\Delta\overline{r}_2|\sin\alpha|\Delta\overline{r}_1| = |\Delta\overline{r}_1 \times \Delta\overline{r}_2|$

$\hat{n}$ is perpendicular to S. But also $\Delta\overline{r}_1 \times \Delta\overline{r}_2$ is perpendicular to S

$$\Rightarrow \Delta\overline{S} = \hat{n}\Delta S = \Delta\overline{r}_1 \times \Delta\overline{r}_2$$

$d\overline{S}$ AND FLUX INTEGRAL

$$d\overline{S} = \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \Delta\overline{S} = \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \Delta\overline{r}_1 \times \Delta\overline{r}_2$$

$$d\overline{r}_1 = \lim_{\Delta u \to 0} \Delta\overline{r}_1 = \lim_{\Delta u \to 0} \overline{r}(u+\Delta u, v) - \overline{r}(u,v) =$$

$$= \lim_{\Delta u \to 0} \frac{\overline{r}(u+\Delta u, v) - \overline{r}(u,v)}{\Delta u} \Delta u = \frac{\partial \overline{r}(u,v)}{\partial u} du$$

in the same way:

$$d\overline{r}_2 = \frac{\partial \overline{r}(u,v)}{\partial v} dv$$

$$\Rightarrow \boxed{d\overline{S} = \frac{\partial \overline{r}}{\partial u} \times \frac{\partial \overline{r}}{\partial v} du dv}$$

So, the flux integral of the vector field $\overline{v}$ on the surface $\overline{S}$ can be calculated as:

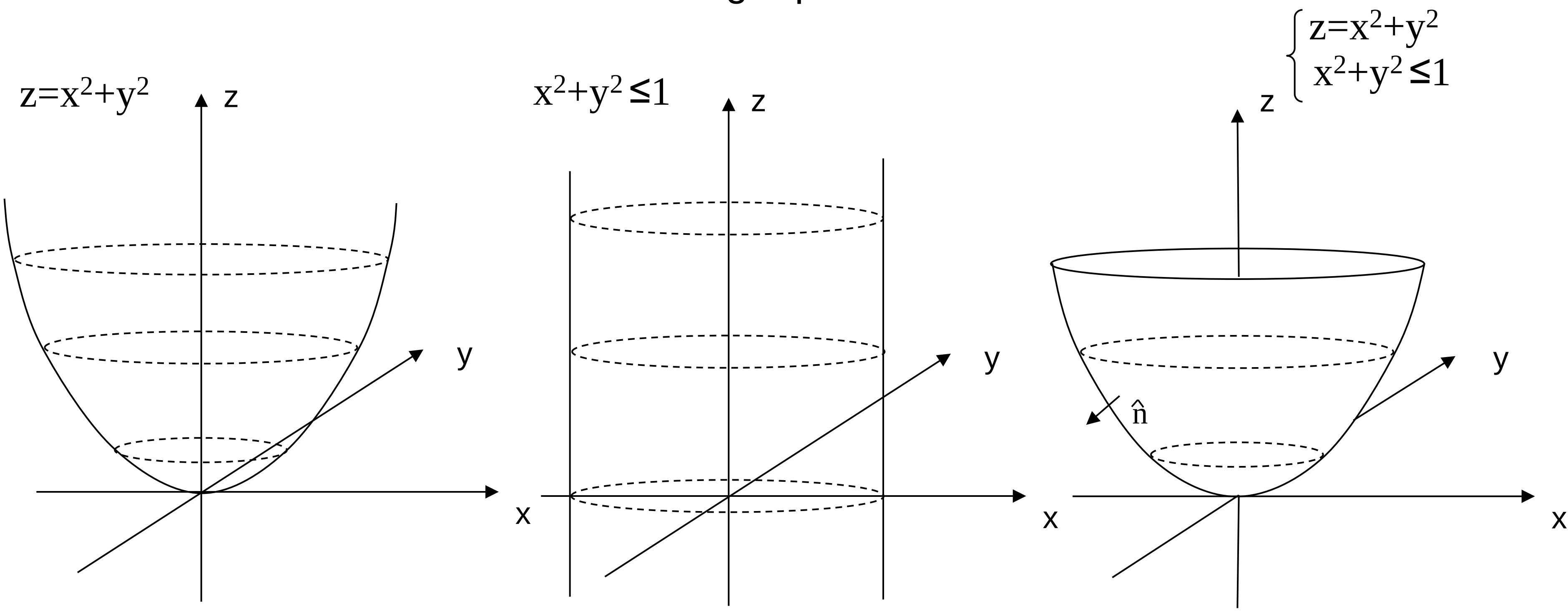
$$\boxed{\int_S \overline{v} \cdot d\overline{S} = \int_u \int_v \overline{v}(\overline{r}(u,v)) \cdot \left(\frac{\partial \overline{r}}{\partial u} \times \frac{\partial \overline{r}}{\partial v} \right) du dv}$$

EXAMPLE

Calculate the flux of the vector field $\bar{A} = (xy, 0, z^2)$
through the surface S: $z=x^2+y^2$ $x^2+y^2 \leq 1$
$\hat{n} \cdot \hat{e}_z < 0$

This defines in which direction we will calculate the flux.
$\hat{n}$ is chosen so that z-component is negative.

SOLUTION: 1- figure
2- Parameterization of $\bar{S}$
3- Flux calculation using equation 10



EXAMPLE

Parameterization of $\overline{S}$ $\quad z=x^2+y^2$

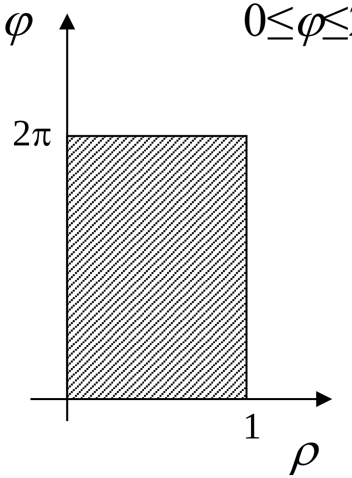
$x^2+y^2 \leq 1$

$$\overline{r}(\rho,\varphi) \quad \begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = x^2 + y^2 = (\rho\sin\varphi)^2 + (\rho\sin\varphi)^2 = \rho^2 \end{cases}$$

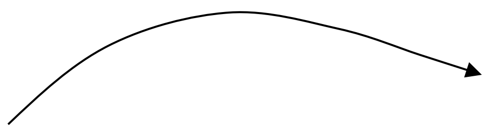
ρ and φ are polar coordinates

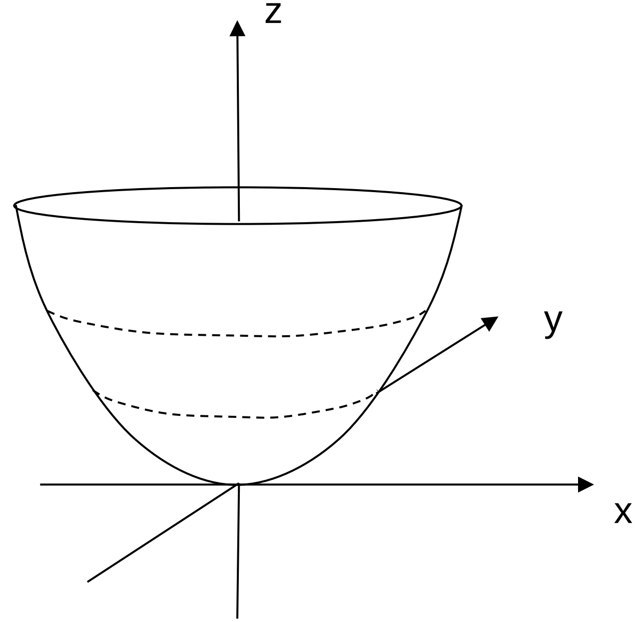
$0 \leq \rho \leq 1$

$0 \leq \varphi \leq 2\pi$



$\overline{r}(\rho,\varphi)$





Parameterization of the vector field: $\quad \overline{A} = (xy, 0, z^2) = (\rho^2 \sin\varphi\cos\varphi,\ 0,\ \rho^4)$

EXAMPLE

Flux calculation using equation 10

$$\iint_S \overline{A}\cdot d\overline{S} = \iint_S \overline{A}(\overline{r}(\rho,\varphi))\cdot\left(\frac{\partial\overline{r}}{\partial\rho}\times\frac{\partial\overline{r}}{\partial\varphi}\right)d\rho d\varphi$$

$$\left.\begin{aligned}\frac{\partial\overline{r}}{\partial\rho} &= \left(\cos\varphi,\sin\varphi,2\rho\right)\\ \frac{\partial\overline{r}}{\partial\varphi} &= \left(-\rho\sin\varphi,\rho\cos\varphi,0\right)\end{aligned}\right\} \Rightarrow \left(\frac{\partial\overline{r}}{\partial\rho}\times\frac{\partial\overline{r}}{\partial\varphi}\right) = \begin{vmatrix}\hat{e}_x & \hat{e}_y & \hat{e}_z\\ \cos\varphi & \sin\varphi & 2\rho\\ -\rho\sin\varphi & \rho\cos\varphi & 0\end{vmatrix} =$$

$$(-2\rho^2\cos\varphi,-2\rho^2\sin\varphi,\rho\cos^2\varphi+\rho\sin^2\varphi) =$$

$$(-2\rho^2\cos\varphi,-2\rho^2\sin\varphi,\rho)$$

Note that $\left(\frac{\partial\overline{r}}{\partial\rho}\times\frac{\partial\overline{r}}{\partial\varphi}\right)$ has a positive z-component, while the flux was in the other direction.

So we ordinary solve the integral but then we <u>change the sign</u> in the answer!

EXAMPLE

$$\iint_{-S} \overline{A}\cdot d\overline{S} = \iint_{-S} \overline{A}(\overline{r}(\rho,\varphi))\cdot\left(\frac{\partial\overline{r}}{\partial\rho}\times\frac{\partial\overline{r}}{\partial\varphi}\right)d\rho d\varphi =$$

$$\int_0^{2\pi}\int_0^1\left(\rho^2\sin\varphi\,\cos\varphi,0,\rho^4\right)\cdot(-2\rho^2\cos\varphi,-2\rho^2\sin\varphi,\rho)d\rho d\varphi =$$

$$\int_0^{2\pi}\int_0^1\left(-2\rho^4\sin\varphi\,\cos^2\varphi+0+\rho^5\right)d\rho d\varphi =$$

$$\int_0^{2\pi}\left[-\frac{2}{5}\rho^5\sin\varphi\,\cos^2\varphi+\frac{1}{6}\rho^6\right]_0^1 d\varphi = \int_0^{2\pi}\left(-\frac{2}{5}\sin\varphi\,\cos^2\varphi+\frac{1}{6}\right)d\varphi =$$

$$\left[-\frac{2}{5}\left(-\frac{\cos^3\varphi}{3}\right)+\frac{1}{6}\varphi\right]_0^{2\pi} = \frac{\pi}{3}$$

But we must change sign! The answer is $-\dfrac{\pi}{3}$

WHICH STATEMENT IS WRONG?

1- The flux integral is a scalar (yellow)

2- Flux integrals can be calculated also on a closed surface. (red)

3- The perpendicular to the integration surface points out from an arbitrary side. (green)

4- The flux through a membrane can be calculated with flux integrals. (blue)