VEKTORANALYS

Kursvecka 1

övningar

PROBLEM 1

Calculate the gradient of the following scalar field:

$$\phi(x,y) = e^{-\left(x^2+y^2\right)}$$

(a) What is the direction of the maximum increase in point P=(-1,1)?
(b) What is the maximum increase in point P=(-1,1)?

SOLUTION

(a) The direction of the maximum increase is the direction of the gradient *(theorem 1)*

$$grad\,\phi = \left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}\right) = \left(-2xe^{-\left(x^2+y^2\right)},-2ye^{-\left(x^2+y^2\right)}\right)$$

In P we have: $grad\,\phi = -2e^{-\left((-1)^2+1^2\right)}\left(-1,1\right) = -2e^{-2}\left(-1,1\right)$

The direction is: $\hat{d} = \frac{(1,-1)}{\sqrt{2}}$

(b) The maximum increase in P is the absolute value of the gradient in P *(theorem 1)*

$$\left|grad\,\phi\right| = \left|-2e^{-2}\left(-1,1\right)\right| = 2e^{-2}\sqrt{2}$$

PROBLEM 2

The scalar fields f, g, h are given by:

$$f(\overline{r}) = 2xy - y^2z^2 + 2xz$$

$$g(\overline{r}) = x^3y + y^3z - xz^3$$

$$h(\overline{r}) = x^3y^2z + x^2yz^3 + xy^3z^2$$

(a) Calculate the direction $\hat{n}$ for which the directional derivative of f and g in the point P (-1,0,1) is zero

(b) Calculate the directional derivative of h in P along the direction $\hat{n}$

SOLUTION

(a) The directional derivative of the scalar field ϕ in the direction $\hat{n}$ is:

$$\frac{d\phi}{ds} = grad\phi \cdot \hat{n}$$

Therefore, we need to find the direction $\hat{n}$ for which:

$$\begin{cases} grad(f) \cdot \hat{n} = 0 \\ grad(g) \cdot \hat{n} = 0 \end{cases}$$

Let's calculate the gradient of f and g:

$$grad\ f = \left(2y + 2z, 2x - 2yz^2, -2y^2 z + 2x\right)$$

$$grad\ g = \left(3x^2 y - z^3, x^3 + 3y^2 z, y^3 - 3xz^2\right)$$

In the point P=(-1,0,1) the gradients are:

$$\left(grad\ f\right)_P = \left(2, -2, -2\right)$$

$$\left(grad\ g\right)_P = \left(-1, -1, 3\right)$$

If $\hat{n} = (a, b, c)$ we obtain:

$$\left(grad\ f\right)_P \cdot \hat{n} = \left(2, -2, -2\right) \cdot (a, b, c) = 0$$

$$\left(grad\ g\right)_P \cdot \hat{n} = \left(-1, -1, 3\right) \cdot (a, b, c) = 0$$

$$\left.\begin{aligned} a - b - c &= 0 \\ a + b - 3c &= 0 \end{aligned}\right\} \Rightarrow \begin{cases} a = 2c \\ b = c \end{cases}$$

Therefore: $\hat{n} = (2c, c, c)$

Normalizing: $\boxed{\hat{n} = \dfrac{(2,1,1)}{\sqrt{6}}}$

(b) The directional derivative of the h in the direction $\hat{n}$ in the point P is:

$$\left(\frac{dh}{ds}\right)_P = \left(grad h\right)_P \cdot \hat{n} = (0,1,0) \cdot \frac{(2,1,1)}{\sqrt{6}} = \frac{1}{\sqrt{6}}$$

PROBLEM 3

(A) Show a simple parametric description $\overline{r} = \overline{r}(u)$ for the curve:

$$\begin{cases} 4x - y^2 = 0 & (1) \\ x^2 + y^2 - z = 0 & (2) \end{cases}$$

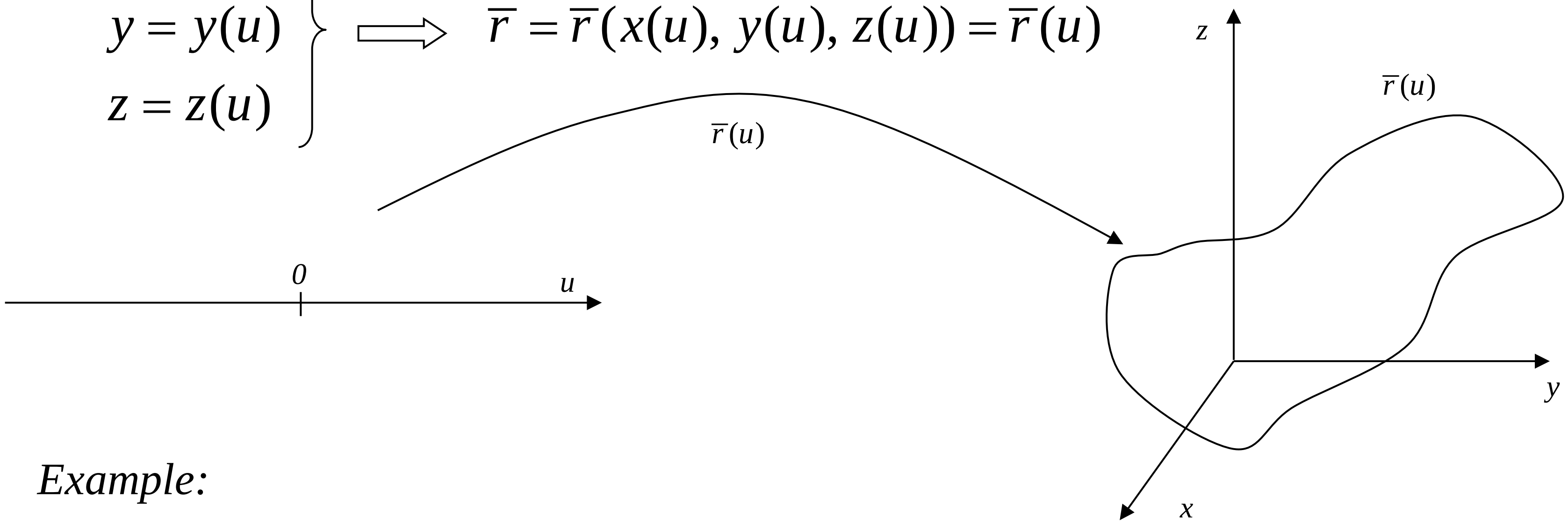
From the point (0,0,0) to the point (1,2,5)

(B) Calculate the vector tangent to the point $\left(\frac{1}{4}, 1, \frac{17}{16}\right)$

SOLUTION *(point A)*

A "parameterization" means that we have to introduce a new variable (u for example). The "old" variables x, y, and z will be dependent on u.

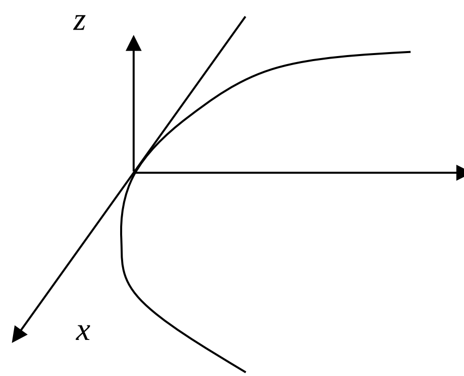
$$\left.\begin{aligned} x &= x(u) \\ y &= y(u) \\ z &= z(u) \end{aligned}\right\} \Longrightarrow \bar{r} = \bar{r}(x(u),\, y(u),\, z(u)) = \bar{r}(u)$$



Example:

A parabola located in the xy-plane

$$\left.\begin{aligned} x &= u \\ y &= u^2 \\ z &= 0 \end{aligned}\right\} \Longrightarrow \bar{r}(u) = (u, u^2, 0)$$



SOLUTION *(point A)*

$$\begin{cases} 4x - y^2 = 0 & (1) \\ x^2 + y^2 - z = 0 & (2) \end{cases}$$

From the point (0,0,0) to the point (1,2,5)

For example, we can choose: $u=y$

From equation (1) $\Rightarrow$ $u^2 = 4x \quad \Rightarrow \quad x = \dfrac{u^2}{4}$

From equation (2) $\Rightarrow$ $z = x^2 + y^2 = \dfrac{u^4}{16} + u^2$

So we obtain: $\bar{r}(u) = \left(\dfrac{u^2}{4}, u, \dfrac{u^4}{16} + u^2 \right)$

From the point (0,0,0) to the point (1,2,5)

$(0,0,0) \Rightarrow u=0$

$(1,2,5) \Rightarrow u=2$

The curve is:

$$\bar{r}(u) = \left(\frac{u^2}{4}, u, \frac{u^4}{16} + u^2 \right)$$

$$u : 0 \to 2$$

SOLUTION *(point B)*

The tangent in a point is the value of the derivative (calculated in the parameter u) in that point.

$$\bar{t} = \frac{d\bar{r}}{du} = \left(\frac{2u}{4}, 1, \frac{4u^3}{16} + 2u \right) = \left(\frac{u}{2}, 1, \frac{u^3}{4} + 2u \right)$$

The tangent has to be calculated in the point $\left(\frac{1}{4}, 1, \frac{17}{16} \right)$

since $u=y$, we have $u=1$

Therefore:

$$\bar{t} = \left(\frac{1}{2}, 1, \frac{1}{4} + 2 \right) = \left(\frac{1}{2}, 1, \frac{9}{4} \right)$$

PROBLEM 4

Consider the following surface:

$$x^2 - 2y^2 - 2z = 0 \qquad (1)$$

Calculate:

(A) The equations of the normal line to the surface in the point $P=(2,1,1)$
(B) The equation of the tangent plane to the surface in the point P

SOLUTION *(point A)*

A normal line is a line that intersects the surface in the point. The direction of the line is perpendicular to the surface.

How to calculate the direction perpendicular to a surface?

Theorem 3: The gradient of a scalar field $\phi(x,y,z)$ in the point P is orthogonal to the level surface $\phi=c$ in P.

Consider the level surface $\phi = x^2 - 2y^2 - 2z = 0$

The normal line is parallel to the gradient

$$grad\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) = (2x, -4y, -2)$$

In $P=(2,1,1)$ $\quad grad\phi = (4, -4, -2)$

The normal line can be written as: $\bar{n}_P = (4, -4, -2)$

SOLUTION *(point B)*

A tangent plane is a plane that is parallel to the surface in the point.

From "*basic*" geometry, given a vector $\overline{v} = (A, B, C)$

Then the plane $Ax + By + Cz + D = 0$ is perpendicular to $\overline{v}$

Therefore, using $\overline{v} = \overline{n}_P = (4, -4, -2)$

we have that the plane $4x - 4y - 2z + D = 0$ is perpendicular to $\overline{n}_P$

D is chosen in order that the plane passes through the point $P=(2,1,1)$:

$$4 \cdot 2 - 4 \cdot 1 - 2 \cdot 1 + D = 0 \quad \Rightarrow \quad D=-2$$

$2x - 2y - z - 1 = 0$ passes through P and is tangent to the surface