VEKTORANALYS

Kursvecka 1

GRADIENTEN

Kapitel 1-3
Sidor 3-28

TARGET PROBLEM

A mosquito is flying in the room.
How does the mosquito find us in the dark?
A theory is: the mosquito flies toward the warmest region of the room

The mosquito must know:

(A) How the temperature T(x,y,z) changes along the flying direction

(B) In which direction she must fly to be in a warmer place as soon as possible

We need to:

(1) introduce a **SCALAR FIELD**, T(x,y,z)

(2) measure the **change of the scalar field** T(x,y,z) in R^3 (i.e. the derivative)

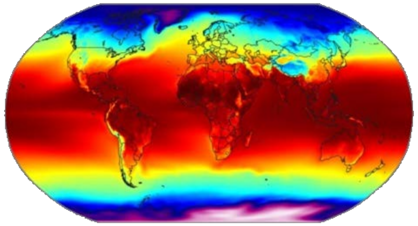
(3) find the **direction** for which **the change** of T(x,y,z) **is maximum**

SCALAR FIELD AND VECTOR FIELD

A scalar quantity is said to be a **field** if it is a function of position

A **scalar field** associates a real number ϕ(x,y,z) to each point (x,y,z) of the space.

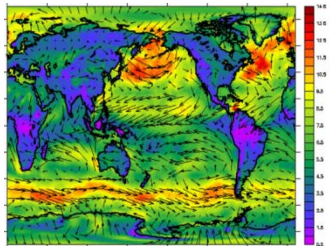
Examples: - temperature distribution in the space
- pressure distribution in a fluid
- potential around an electric charge



A vector quantity is said to be a **field** if it is a function of position

A **vector field** associates a vector $\bar{A}$(x,y,z) to each point (x,y,z) of the space.

Examples: - velocity distribution in a fluid
- magnetic field around a magnet
- electric field around an electric charge



To solve our problem, today we will focus on scalar fields

LEVEL SURFACES

- Level surfaces are useful to visualize a scalar field.
- What is a level surface?

A surface on which the scalar field $\phi(x,y,z)$ is constant:

$$\phi(x,y,z)=c \qquad (1)$$

- To create an "image" of the scalar field $\phi(x,y,z)$ we can consider a family of level surfaces:

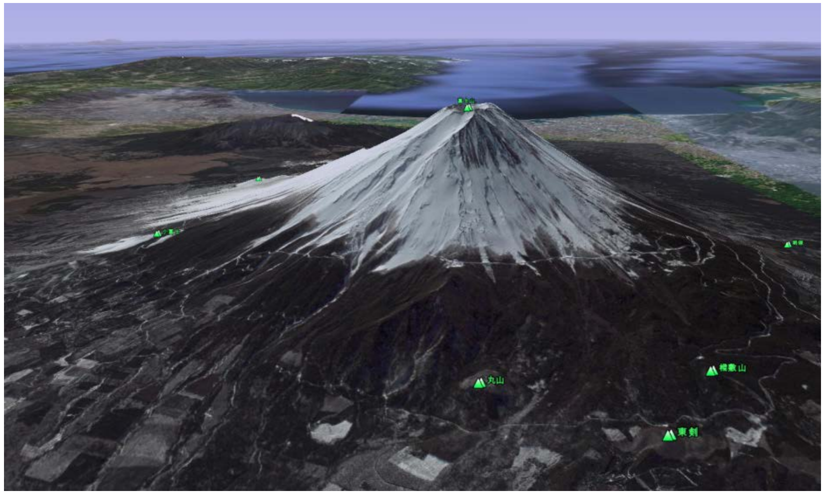
$$\phi(x,y,z)=c+nh \qquad (2)$$

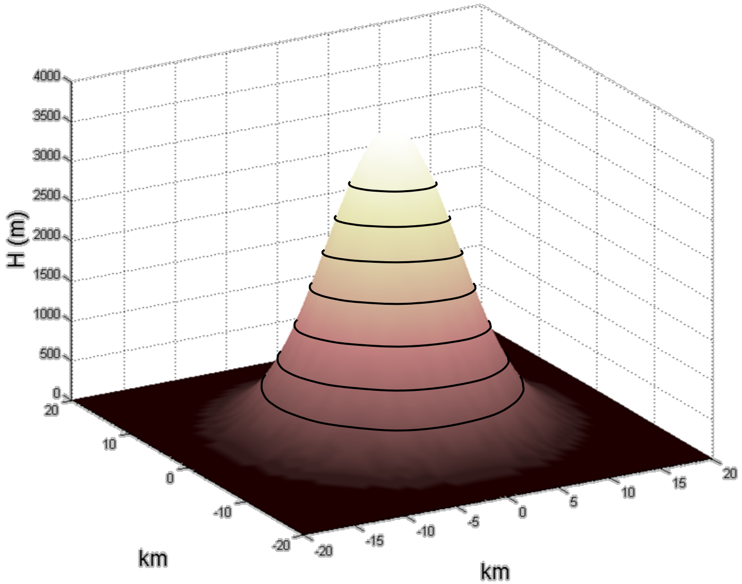
where h is a constant and $n=0, \pm1, \pm2, \pm3,\ldots$
To improve the details of the "image", you can decrease h.

- In two dimensions the scalar field is $\phi(x,y) \Rightarrow$ we have level curves!

EXAMPLE

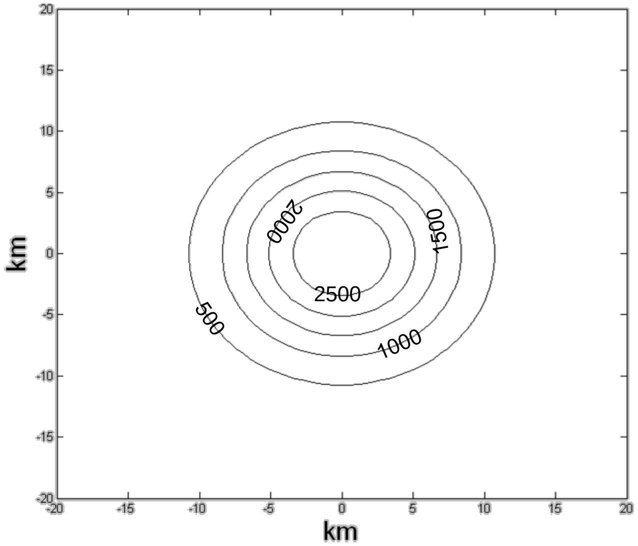
- Assume that H(x,y) is a scalar field corresponding to the height of a mountain





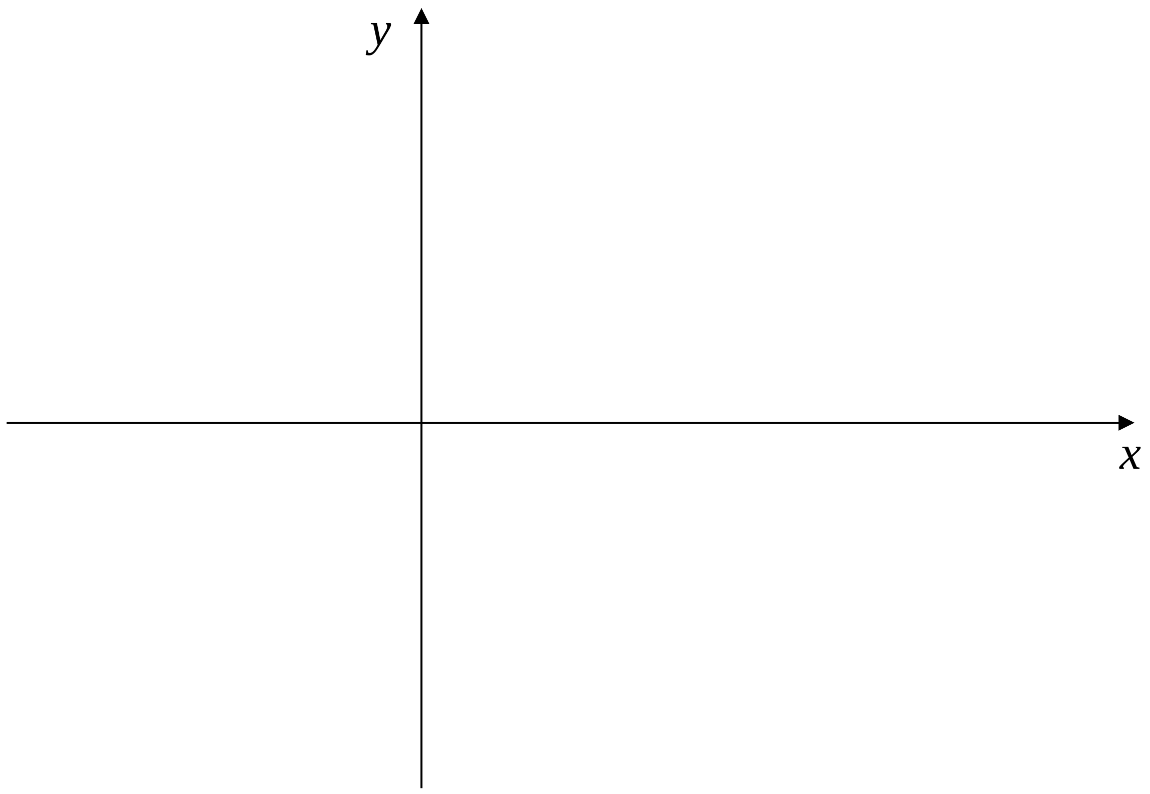
Let's consider a family of level curves:

$$H(x,y)=c+nh$$

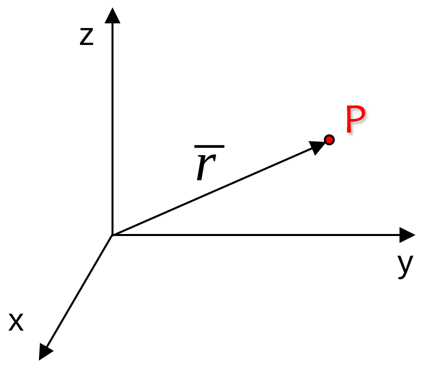


EXERCISE

Plot the level curves of the scalar field: $\phi = \dfrac{x^2}{4} + y^2$



POSITION VECTOR



- The vector from the origin to the point P=(x,y,z) is called position vector $\overline{r}$
- Note that $\overline{r}$ depends on the choice of the coordinate system
- $\overline{r}$ can be expressed with different notations:

$$\overline{r} = \overline{r}(x, y, z) \qquad \overline{r} = (x, y, z) \qquad \overline{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$

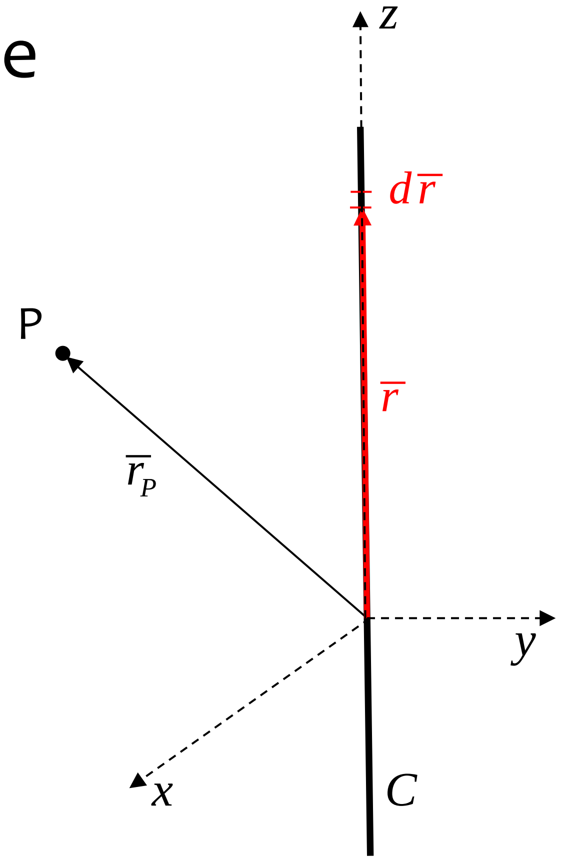
- The differential of a position vector can be written as a vector whose components are the differential of each position vector component:

$$\overline{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z \qquad (3)$$

$$\Downarrow$$

$$d\overline{r} = dx\hat{e}_x + dy\hat{e}_y + dz\hat{e}_z \qquad (4)$$

POSITION VECTOR: example



The electrostatic potential in the point P (x_p, y_p, z_p) produced by an electrically charged wire *"C"* with charge density τ is:

$$\phi(\overline{r}_P) = \int_C \frac{\tau}{4\pi} \frac{|d\overline{r}|}{|\overline{r}_P - \overline{r}|}$$

where $\overline{r}_P = (x_P, y_P, z_P)$

We will not solve the integral (see next week).

But how can we express the term $\frac{|d\overline{r}|}{|\overline{r}_P - \overline{r}|}$?

$$\overline{r} = (0, 0, z) = z\hat{e}_z$$

$$d\overline{r} = (0, 0, dz) = dz\hat{e}_z \quad \Rightarrow \quad |d\overline{r}| = dz$$

$$\overline{r}_P = (x_P, y_P, z_P) \quad \Rightarrow \quad \overline{r}_P - \overline{r} = (x_P, y_P, z_P - z) \quad \Rightarrow \quad |\overline{r}_P - \overline{r}| = \sqrt{x_P^2 + y_P^2 + (z_P - z)^2}$$

$$\frac{|d\overline{r}|}{|\overline{r}_P - \overline{r}|} = \frac{dz}{\sqrt{\rho_P^2 + (z_P - z)^2}}$$

where ρ_P is the distance of P from the curve C

THE GRADIENT

- Assume that $\phi(\bar{r})$ is a continuous and derivable scalar field
- DEFINITION:

$$grad\,\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \tag{5}$$

IMPORTANT: the gradient of a scalar field is a vector field!

EXERCISE: calculate the gradient of the vector field: $\phi = \frac{x^2}{4} + y^2$ and plot $grad\,\phi$ in the point $P=(2,0)$ and in the point $P=(0,-1)$

- Scalar field differential:

$$d\phi = grad\,\phi \cdot d\bar{r} \tag{6}$$

- Let's introduce :
  - the amplitude of the position vector differential, *ds*, and
  - the direction ê (ê is a *unit vector, i.e.* |ê|=*1*)

$$d\bar{r} = \hat{e}\,ds \tag{7}$$

Equations (6) and (7) give:

Directional derivative

$$\frac{d\phi}{ds} = grad\,\phi \cdot \hat{e} \tag{8}$$

The rate of variation of ϕ in a given direction corresponds to the component of the vector gradient in that direction

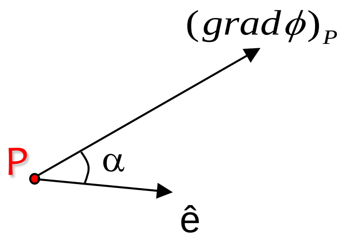
Answer to question (A)!!

THE GRADIENT

THEOREM 1 *(3.1 in the textbook)*

The gradient in the point P is a vector that points to the direction in which the growth of ϕ in P is the highest.
The maximum increase of ϕ per unit length is $\left|(grad\phi)_P\right|$

PROOF



a- let's calculate the derivative in the direction ê *Eq. (8)*

$$\frac{d\phi}{ds} = grad\phi \cdot \hat{e} = \left|grad\phi\right| \cos\alpha$$

b- this is maximum when:

$\cos\alpha = 1$

which implies:

$\alpha = 0$ $(\hat{e} \,//\, grad\phi)$ and $\frac{d\phi}{ds} = \left|grad\phi\right|$

Answer to question (B)!!

THE GRADIENT

THEOREM 2 *(3.2 in the textbook)*

The gradient in the point P is zero if ϕ has a maximum or a minimum in P

PROOF

From Equation (8): $$\frac{d\phi}{ds} = grad\,\phi \cdot \hat{e}$$

ϕ has a maximum or a minimum in P $\Rightarrow$ $d\phi/ds=0$

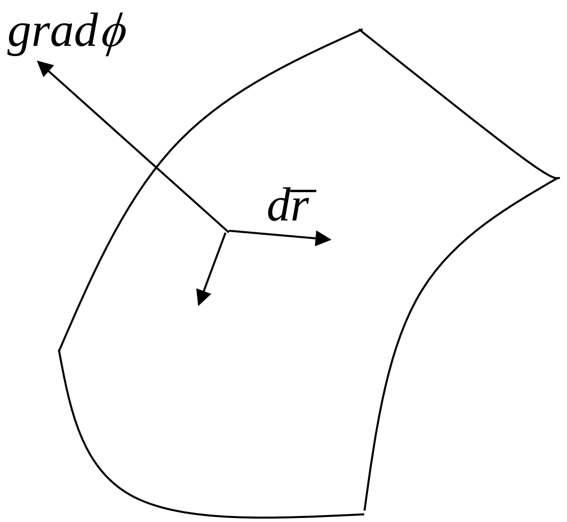
using equation (8), $d\phi/ds=0$ implies $\mathrm{grad}\,\phi=0$

THE GRADIENT

THEOREM 3 *(3.3 in the textbook)*

The gradient of a scalar field $\phi(x,y,z)$ in the point P is orthogonal to the level surface $\phi=c$ in P.

PROOF



a- Let's do a small movement $d\bar{r}$ along the level surface

b- Remember that on the level surface ϕ is constant:

$$d\phi=0$$

c- Then, using equation (6):

$$d\phi = grad\phi \cdot d\bar{r} = 0$$

d- This implies that $grad\phi$ is perpendicular to $d\bar{r}$

e- $grad\phi$ is perpendicular to each $d\bar{r}$ on the level surface $grad\phi$ is perpendicular to the level surface

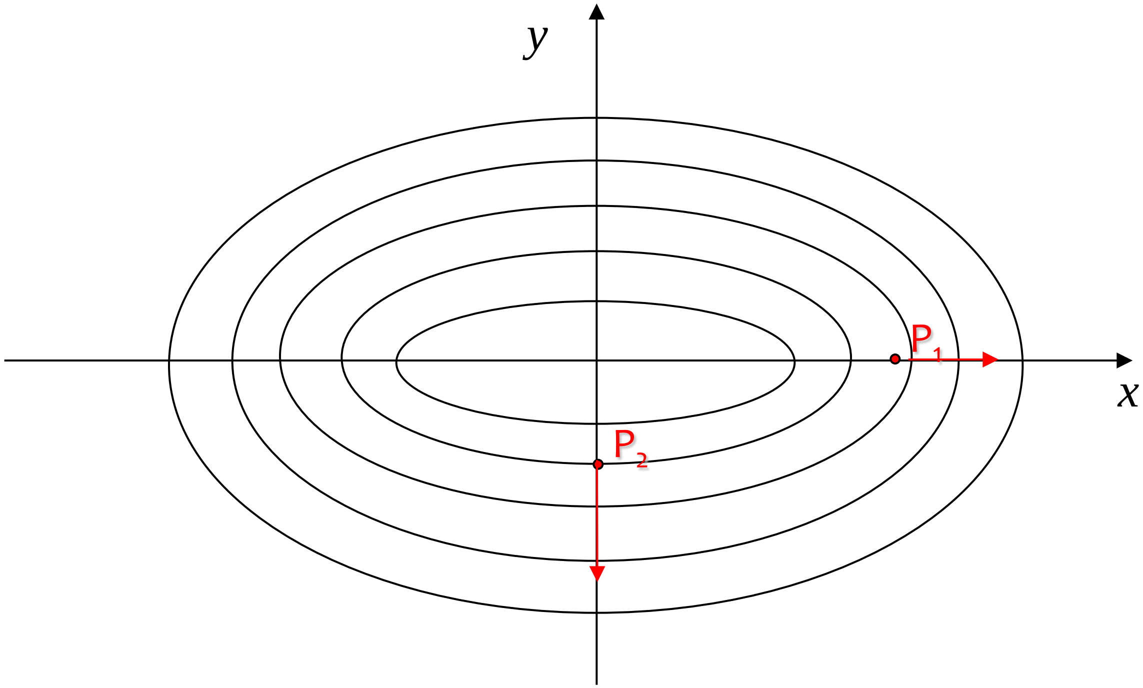
2D-EXAMPLE

- Theorems 1, 2 and 3 are valid also in two dimensions.
- $grad\phi$ is a vector field that:
  - in each point is orthogonal to the level curve in that point and
  - always points along the direction in which the height grows faster

Plot the level curves of the scalar field: $\phi = \frac{x^2}{4} + y^2$

and calculate the gradient in the points $P_1=(2,0)$ and $P_2=(0,-1)$

$$grad\phi = \left(\frac{x}{2}, 2y\right)$$



$$grad\phi|_{P1} = (1,0)$$

$$grad\phi|_{P2} = (0,-2)$$

ELECTROSTATIC POTENTIAL AND ELECTRIC FIELD

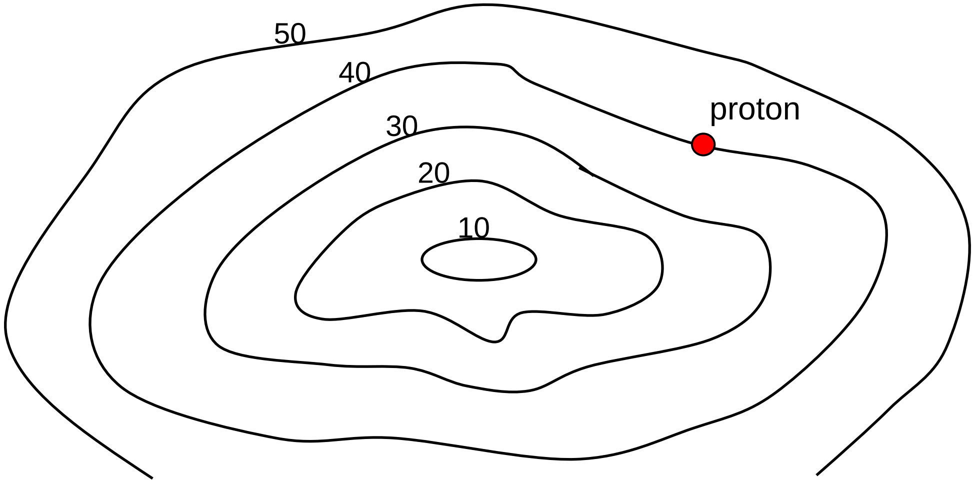
- Consider an electrostatic potential $V(\bar{r})$
- The electric field produced by $V(\bar{r})$ is: $\bar{E}(\bar{r}) = -gradV(\bar{r})$

  (see the TET course for details)

- The force produced by $\bar{E}(\bar{r})$ on an electric charge q is: $\bar{F}(\bar{r}) = q\bar{E} = -q\,gradV(\bar{r})$

EXERCISE:

- Consider a proton in an electrostatic potential:



In which direction will the proton move?

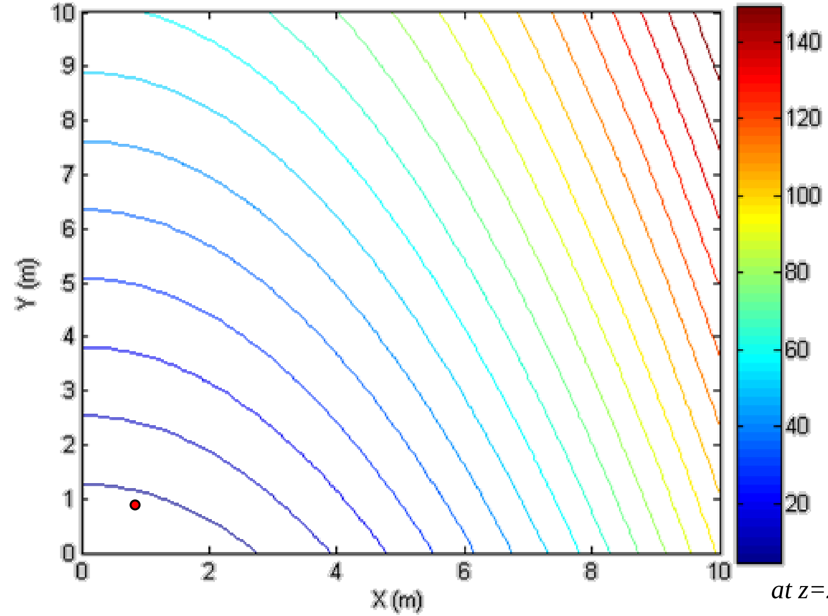
TARGET PROBLEM

A mosquito is flying around in the room.

The temperature is described by the scalar field:

$$T(x,y,z)=x^2+2yz-z \quad [^\circ C]$$

The mosquito is in the point P=(1,1,2)



at z=2

(a) In which direction the mosquito will fly to be in a warmer place as quick as possible?

(b) How much the temperature changes in time if the mosquito flies with velocity 3m/s in direction (-2,2,1)?

TARGET PROBLEM

(a) In which direction the mosquito will fly to be warm as quick as possible?

We use theorem 1: The gradient in the point P is a vector that points to the direction in which the scalar field in P has the highest growth.

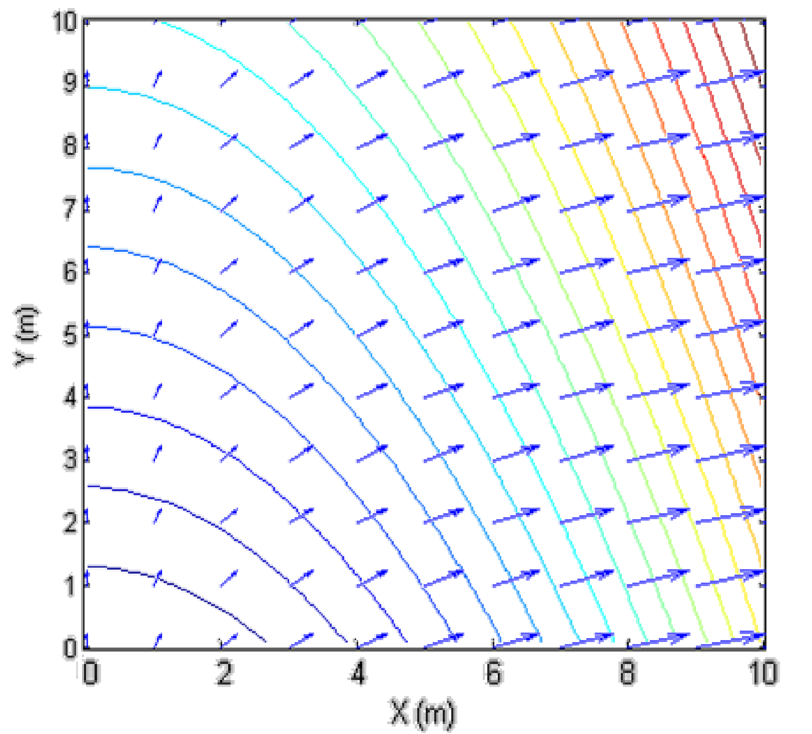
From definition (5):

$$gradT = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right)$$

$$T(x, y, z) = x^2 + 2yz - z$$

$$\frac{\partial T}{\partial x} = 2x, \qquad \frac{\partial T}{\partial y} = 2z, \qquad \frac{\partial T}{\partial z} = 2y - 1$$

$$gradT = (2x, 2z, 2y - 1)$$



The mosquito is in P=(1,1,2)

$$(gradT)_{P=(1,1,2)} = (2\cdot 1, 2\cdot 2, 2\cdot 1 - 1) = (2,4,1)$$

The mosquito will fly in direction (2,4,1)

TARGET PROBLEM

(b) How fast the temperature changes if the mosquito flies with velocity 3m/s in direction (-2,2,1)?

We must calculate $\frac{dT}{dt}$ where t is the time

Using equation (6):

$$\frac{dT}{dt} = gradT \cdot \frac{d\bar{r}}{dt} = gradT \cdot \hat{e}\frac{ds}{dt}$$

where
$$\left\{ \begin{aligned} &\frac{ds}{dt} = |\bar{v}| = 3m/s \\ &\hat{e} = \frac{\bar{v}}{|\bar{v}|} = \frac{(-2,2,1)}{|(-2,2,1)|} = \frac{(-2,2,1)}{\sqrt{(-2)^2+2^2+1^2}} = \frac{(-2,2,1)}{3} \end{aligned} \right\} \Rightarrow \frac{d\bar{r}}{dt} = \frac{(-2,2,1)}{3}\cdot 3 = (-2,2,1)$$

$$\frac{dT}{dt} = gradT \cdot \frac{d\bar{r}}{dt} = (2,4,1)\cdot(-2,2,1) = 5 \quad [\text{C/s}]$$

WHICH STATEMENT IS WRONG?

1- A scalar field associates a real number to a point in space (yellow)

2- The increase of a scalar field in a given direction is given by the directional derivative:

$$\frac{d\phi}{ds} = grad\phi \cdot \hat{e}$$

(red)

3- If ϕ is a scalar then $grad\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$ in R^3 (green)

4- A scalar field can be written as $\overline{A} = \overline{A}(x, y, z)$ (blue)