LECTURE 3: OUTLINE

- Ch. 2: Unitary equiv, QR factorization, Schur's thm, Cayley-H., Normal matrices, Spectral thm, Singular value decomp.
- Ch. 3: Canonical forms: Jordan/Matrix factorizations



KTH - Signal Processing

Magnus Jansson / Bhavani Shankar / Joakim Jaldén / Mats Bengtsson

UNITARY MATRICES

- ullet A set of vectors $\{x_i\}\in {f C}^n$ are called
 - orthogonal if $x_i^* x_i = 0, \forall i \neq j$ and
- *orthonormal* if they are orthogonal and $x_i^*x_i = 1, \forall i$.
- A matrix $U \in M_n$ is unitary if $U^*U = I$.



- A matrix $U \in M_n(\mathbf{R})$ is real orthogonal if $U^T U = I$.
- (A matrix $U \in M_n$ is orthogonal if $UU^T = I$.)
- If U, V are unitary then UV is unitary.
 - Unitary matrices form a group under multiplication.

UNITARY MATRICES CONT'D

The following are equiv.

- 1. U is unitary
- 2. U is nonsingular and $U^{-1}=U^{st}$
- 3. $UU^*=I$



- 4. U^{*} is unitary
- 5. the columns of U are orthonormal
- 6. the rows of U are orthonormal
- 7. for all $x \in \mathbb{C}^n$, the Euclidean length of y = Ux equals that of x.

Def: A linear transformation $T: \mathbf{C}^n \to \mathbf{C}^m$ is a *Euclidean isometry* if $x^*x = (Tx)^*(Tx)$ for all $x \in \mathbf{C}^n$

Unitary U is an Euclidean isometry.

KTH - Signal Processing

Magnus Jansson / Bhavani Shankar / Joakim Jaldén / Mats Bengtssor

EUCLIDEAN ISOMETRY AND PARSEVAL'S THEOREM

1. F_N be the FFT (Fast Fourier Transform matrix) of dimension $N \times N$, i. e.

$$F_N(m,n) = \frac{1}{\sqrt{N}} e^{\frac{-2\pi(m-1)(n-1)}{N}}$$

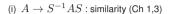
2. F is a unitary matrix.



- 3. Let $y = F_N x$ i.e, y is the N point FFT of x.
- (a) Length of x = Length of y
- (b) $\sum_{j=1}^N |x(j)|^2 = \sum_{j=1}^N |y(j)|^2$: This is energy conservation or Parseval's Theorem in DSP.

UNITARY EQUIVALENCE

Def: A matrix $B\in M_n$ is unitarily equivalent (or similar) to $A\in M_n$ if $B=U^*AU$ for some unitary matrix U.





(ii) $A o S^*AS$: *congruence (Ch 4)

(iii) $A \to S^*AS$ with S unitary : unitary similarity (Ch 2)

Since in (iii) $S^* = S^{-1}$, we have that (iii) is "included" in both (i) and (ii).

Theorem: If A and B are unitarily equivalent then

$$||A||_F^2 \triangleq \sum_{i,j} |a_{ij}|^2 = \sum_{i,j} |b_{ij}|^2 = ||B||_F^2$$

KTH – Signal Processing 5 Magnus Jansson / Bhavani Shankar / Joakim Jaldén / Mats Bengtssor

UNITARY MATRICES AND PLANE ROTATIONS: 2-D CASE

- Consider rotating the 2-D Euclidean plane counter-clockwise by an angle θ .
- · Resulting coordinates,

$$x' = x\cos\theta - y\sin\theta$$

$$y' = x \sin \theta + y \cos \theta$$



Equivalently

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\bullet \ \, \text{Note that} \, U = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \text{ is unitary}.$$

Unitary matrices and Plane Rotations : General Case



$$U(\theta, 2, 4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 0 & 1 & 0 \\ 0 & \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

- $U(\theta, 2, 4)$ rotates the *second* and *fourth* axes in \mathbb{R}^4 counter clock-wise by θ .
- The other axes are not changed.
- ullet Left multiplication by $U(\theta,2,4)$ affects only rows 2 and 4.
- Note that $U(\theta, 2, 4)$ is unitary.
- Such $U(\theta, m, n)$ are called Givens rotations.

KTH - Signal Processing

Magnus Jansson / Bhavani Shankar / Joakim Jaldén / Mats Bengtssor

PRODUCT OF GIVENS ROTATIONS

- $U = U(\theta_1, 1, 3)U(\theta_2, 2, 4)$ rotates
 - second and fourth axes in \mathbb{R}^4 counter clock-wise by θ_2 .
 - first and third axes in \mathbb{R}^4 counter clock-wise by θ_1 .
- U is unitary \Rightarrow product of Givens rotations is unitary.



• Such matrices are used in Least-Squares and eigenvalue computations.

SPECIAL UNITARY MATRICES: HOUSEHOLDER MATRICES

Let $w \in \mathbf{C}^n$ be a normalized ($w^*w = 1$) vector and define

$$U_w = I - 2ww^*$$

Properties:



- 1. U_w is unitary and Hermitian.
- 2. $U_w x = x, \forall x \perp w$.
- 3. $U_w w = -w$
- 4. There is a Householder matrix such that

$$y = U_w x$$

for any given real vectors x and y of the same length.

KTH - Signal Processing

Magnus Jansson / Bhavani Shankar / Joakim Jaldén / Mats Bengtssor

QR-FACTORIZATION

Thm: If $A \in M_{n,m}$ and $n \ge m$, then

$$A = QR$$



with $Q\in M_{n,m}$ such that $Q^*Q=I$ and $R\in M_m$ is upper triangular.

- If m=n then Q is unitary.
- If A is nonsingular, then the diagonal elements of R can be taken to be positive (Q and R are in this case unique).
- Gram Schmidt orthogonalization followed by book keeping.
- Useful in Least squares solutions, eigenvalue computations etc.

THE QR-ALGORITHM FOR EIGENVALUE COMPUTATION

Initialization: Given $A \in M_n$, set k = 0, $A_k = A$,

- 1. $A_k = Q_k R_k$ (QR decomposition)
- 2. Obtain $A_{k+1} = R_k Q_k$.
- 3. Set k = k + 1 and go to Step 1.



Prove : A_k and A are unitarily equivalent for all k.

Under certain conditions A_k converges to an upper triangular matrix (as $k\to\infty$) whose diagonal elements correspond to the eigenvalues of A (see p. 114-115 in Horn and Johnson).

KTH - Signal Processing

Magnus Jansson / Bhavani Shankar / Joakim Jaldén / Mats Bengtsson

SCHUR'S UNITARY TRIANGULARIZATION THM

Theorem: Given $A\in M_n$ with eigenvalues $\lambda_1,\ldots,\lambda_n$, there is a unitary matrix $U\in M_n$ such that

$$U^*AU = T = [t_{ij}]$$

is upper triangular with $t_{ii}=\lambda_i$ $(i=1,\ldots,n)$ in any prescribed order. If $A\in M_n(\mathbf{R})$ and all λ_i are real, U may be chosen real and orthogonal.



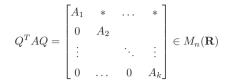
Consequence: Any matrix in M_n is unitarily similar to an upper (or lower) triangular matrix. Note that $A=UTU^*$.

Uniqueness:

- 1. Neither U nor T is unique.
- 2. Eigenvalues can appear in any order
- 3. Two triangular matrices can be unitarily similar

SCHUR: THE GENERAL REAL CASE

Given $A \in M_n(\mathbf{R})$, there is a real orthogonal matrix $Q \in M_n(\mathbf{R})$ such that





where $A_i\ (i=1,\ldots,k)$ are real scalars or 2 by 2 blocks with a non-real pair of complex conjugate eigenvalues.

KTH - Signal Processing

Magnus Jansson / Bhavani Shankar / Joakim Jaldén / Mats Bengtssor

IMPLICATIONS OF THE SCHUR THEOREM

- 1. $\operatorname{tr} A = \sum_{i} \lambda_{j}(A)$
- 2. $\det A = \prod_i \lambda_i(A)$
- 3. Cayley-Hamilton theorem.





CAYLEY-HAMILTON THEOREM

Let $p_A(t) = \det(tI - A)$ be the characteristic polynomial of $A \in M_n$. Then

$$p_A(A) = 0$$



- $\bullet \ A^{n+k} = q_k(A) \ (k \geq 0) \ \text{for some polynomials} \ q_k(t) \ \text{of degrees} \leq n-1.$
- If A is nonsingular: $A^{-1} = q(A)$ for some polynomial q(t) of degree < n-1.

Important : Note $p_A(C)$ is a matrix valued function.

KTH - Signal Processing

Magnus Jansson / Bhavani Shankar / Joakim Jaldén / Mats Bengtsson

NORMAL MATRICES

Def: A matrix $A \in M_n$ is normal if $A^*A = AA^*$.

Examples:

All unitary matrices are normal.

All Hermitian matrices are normal.

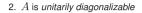


Def: $A \in M_n$ is unitarily diagonalizable if A is unitarily equivalent to a diagonal

FACTS FOR NORMAL MATRICES

The following are equivalent:

1. A is normal





3.
$$||A||_F^2 \triangleq \sum_{i,j} |a_{ij}|^2 = \sum_{i=1}^n |\lambda_i|^2$$

4. there is an orthonormal set of n eigenvectors of A

The equivalence of 1 and 2 is called "the Spectral Theorem for Normal matrices."

KTH - Signal Processing

17

Magnus Jansson / Bhavani Shankar / Joakim Jaldén / Mats Bengtsson

IMPORTANT SPECIAL CASE: HERMITIAN (SYM) MATRICES

Spectral theorem for Hermitian matrices:

If $A \in M_n$ is Hermitian, then,

- all eigenvalues are real
- ullet A is unitarily diagonalizable.



If $A \in M_n(\mathbf{R})$ is symmetric, then A is real orthogonally diagonalizable.

SVD: SINGULAR VALUE DECOMPOSITION

Theorem: Any $A \in M_{m,n}$ can be decomposed as

$$A = V\Sigma W^*$$

• $V \in M_m$: Unitary with columns being eigenvectors of AA^* .



• $W \in M_n$: Unitary with columns being eigenvectors of A^*A .

•
$$\Sigma = [\sigma_{ij}] \in M_{m,n}$$
 has $\sigma_{ij} = 0, \ \forall \ i \neq j$

Suppose rank(A) = k and $q = min\{m, n\}$, then

- $\sigma_{11} \ge \cdots \ge \sigma_{kk} > \sigma_{k+1,k+1} = \cdots = \sigma_{qq} = 0$
- $\sigma_{ii} \equiv \sigma_i$ square roots of non-zero eigenvalues of AA^* (or A^*A)
- Unique : σ_i , Non-unique : V, W

KTH - Signal Processing

Magnus Jansson / Bhavani Shankar / Joakim Jaldén / Mats Bengtssor

CANONICAL FORMS

- An equivalence relation partitions the domain.
- Simple to study equivalence if two objects in an equivalence class can be related to one representative object.
- Requirements of the representatives



- Belong to the equivalence class.
- One per class.
- Set of such representatives is a Canonical form
- We are interested in a canonical form for equivalence relation defined by similarity.

CANONICAL FORMS: JORDAN FORM

Every equivalence class under similarity contains *essentially* only one, so called, Jordan matrix:

$$J = \begin{bmatrix} J_{n_1}(\lambda_1) & 0 \\ & \ddots & \\ 0 & J_{n_k}(\lambda_k) \end{bmatrix}$$



where each block $J_k(\lambda) \in M_k$ has the structure

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & \lambda & 1 \\ 0 & & & & \lambda \end{bmatrix}$$

KTH - Signal Processing

1

Magnus Jansson / Bhavani Shankar / Joakim Jaldén / Mats Bengtsson

THE JORDAN FORM THEOREM

Note that the orders n_i or λ_i are generally not distinct.

Theorem: For a given matrix $A\in M_n$, there is a nonsingular matrix $S\in M_n$ such that $A=SJS^{-1}$ and $\sum_i n_i=n$. The Jordan matrix is unique up to permutations of the Jordan blocks.



The Jordan form may be numerically unstable to compute but it is of theoretical interest.

JORDAN FORM CONT'D

- The number k of Jordan blocks is the number of linearly independent eigenvectors. (Each block is associated with an eigenvector from the standard basis.)
- J is diagonalizable iff k=n.



- The number of blocks corresponding to the same eigenvalue is the geometric multiplicity of that eigenvalue.
- The sum of the orders (dimensions) of all blocks corresponding to the same eigenvalue equals the algebraic multiplicity of that eigenvalue.

KTH - Signal Processing

23

Magnus Jansson / Bhavani Shankar / Joakim Jaldén / Mats Bengtsson

APPLICATIONS OF THE JORDAN FORM

Linear systems:

$$\dot{x}(t) = Ax(t); \ x(0) = x_0$$

The solution may be "easily" obtained by changing state variables to the Jordan form.



Convergent matrices: If all elements of A^m tend to zero as $m\to\infty$, then A is a convergent matrix. Fact: A is convergent iff $\rho(A)<1$ (that is, iff $|\lambda_i|<1,\ \forall i$). This may be proved, e.g., by using the Jordan canonical form.

TRIANGULAR FACTORIZATIONS

Linear systems of equations are easy to solve if we can factorize the system matrix as A=LU where $L\left(U\right)$ is lower (upper) triangular.

Theorem: If $A \in M_n$, then there exist permutation matrices $P,Q \in M_n$ such that





(in some cases we can take Q=I and/or P=I).

KTH - Signal Processing

25

Magnus Jansson / Bhavani Shankar / Joakim Jaldén / Mats Bengtsson

WHEN TO USE WHAT?



	Theoretical	Practical
	derivations	implem.
Schur triangularization	<u> </u>	(
QR factorization	<u> </u>	<u> </u>
Spectral dec.	<u> </u>	€ (?)
SVD	©	©
Jordan form	<u> </u>	∷‼