MATRIX ALGEBRA
MAGNUS JANSSON
Deadline: 2014–04–09, 10.00

Homework # 2

Numbers below refer to problems in Horn, Johnson "Matrix analysis." A number 1.1.P.2 refers to Problem 2 in Section 1.1.

1. (1.0.P.2) Assume that $A^T = A \in M_n(R)$ is symmetric. Show that $\max\{x^T Ax \;:\; x^T x = 1,\; x \in \mathbb{R}^n\}$ is the *largest* real eigenvalue of A.

2. (1.1.P.1) Suppose $A \in M_n$ is nonsingular. For each $\lambda \in \sigma(A)$, show that $\lambda^{-1} \in \sigma(A^{-1})$. If $Ax = \lambda x$ and $x \neq 0$, show that $A^{-1}x = \lambda^{-1}x$.

3. (1.1.P.5) Let $A \in M_n$ be idempotent, that is, $A^2 = A$. Show that each eigenvalue of A is either 0 or 1. Explain why I is the only nonsingular idempotent matrix.

4. (1.1.P.6) Show that all eigenvalues of a nilpotent matrix are 0. Give an example of a nonzero nilpotent matrix. Explain why 0 is the only nilpotent idempotent matrix.

5. (1.2.P.2) For matrices $A \in M_{m,n}$ and $B \in M_{m,n}$, show by direct calculation that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. For any $A \in M_n$ and nonsingular $S \in M_n$, deduce that $\mathrm{tr}(S^{-1}AS) = \mathrm{tr}(A)$. Use multiplicativity of the determinant function to show that $\det(S^{-1}AS) = \det(A)$.

   The matrix $S^{-1}AS$ is called a similarity of A, and these results say that both the trace and the determinant are similarity invariant on M_n.

6. (1.3.P.4, approx. 1.3.P.5 in old book) If $A \in M_n$ has distinct eigenvalues $\alpha_1, \ldots, \alpha_n$ and commutes with a given matrix $B \in M_n$, show that B is diagonalizable and that there is a polynomial $p(t)$ of degree at most $n-1$, such that $B = p(A)$.

7. (1.3.P.7) A matrix $A \in M_n$ is a square root of $B \in M_n$ if $A^2 = B$. Show that every diagonalizable $B \in M_n$ has a square root. Does $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ have a square root? Why?

8. (1.4.P.1) Let nonzero vectors $x, y \in M_n$ be given, let $A = xy^*$ and let $\lambda = y^*x$. Show that

   (a) λ is an eigenvalue of A;

   (b) x is a right and y is a left eigenvector of A associated with λ;

   (c) if $\lambda \neq 0$, then it is the *only* nonzero eigenvalue of A (algebraic multiplicity=1).

   Explain why any vector that is orthogonal to y is in the null space of A. What is the geometric multiplicity of the eigenvalue 0? Explain why A is diagonalizable if and only if $y^*x \neq 0$.

9. (1.4.P.7) In this problem we outline a simple version of the *power method* for finding the largest modulus eigenvalue and an associated eigenvector of $A \in M_n$. Suppose that $A \in M_n$ has distinct eigenvalues $\lambda_1, \dots, \lambda_n$ and that there is exactly one eigenvalue λ_n of maximum modulus $\rho(A)$. If $x^{(0)} \in \mathbb{C}^n$ is *not* orthogonal to a left eigenvector associated with λ_n, show that the sequence

$$x^{(k+1)} = \frac{1}{\sqrt{x^{(k)*}x^{(k)}}} A x^{(k)}, \quad k = 0, 1, 2, \dots$$

   converges to an eigenvector of A, and the ratios of a given nonzero entry in the vectors $Ax^{(k)}$ and $x^{(k)}$ converge to λ_n.

10. (1.4.P.8) As a continuation of the previous exercise, further eigenvalues (and eigenvectors) of A can be calculated by combining the power method with a *deflation* that delivers a square matrix of size one smaller, whose spectrum (with multiplicities) contains all but one eigenvalue of A. Let $S \in M_n$ be nonsingular and have as its first column an eigenvector $y^{(n)}$ associated with eigenvalue λ_n. Show that $S^{-1}AS = \begin{bmatrix} \lambda_n & * \\ 0 & B \end{bmatrix}$ and the eigenvalues of $B \in M_{n-1}$ are $\lambda_1, \dots, \lambda_{n-1}$. Another eigenvalue may be calculated from B and the deflation can be repeated until all eigenvalues have been found.