EP2200 Queueing theory and teletraffic systems

Queueing networks

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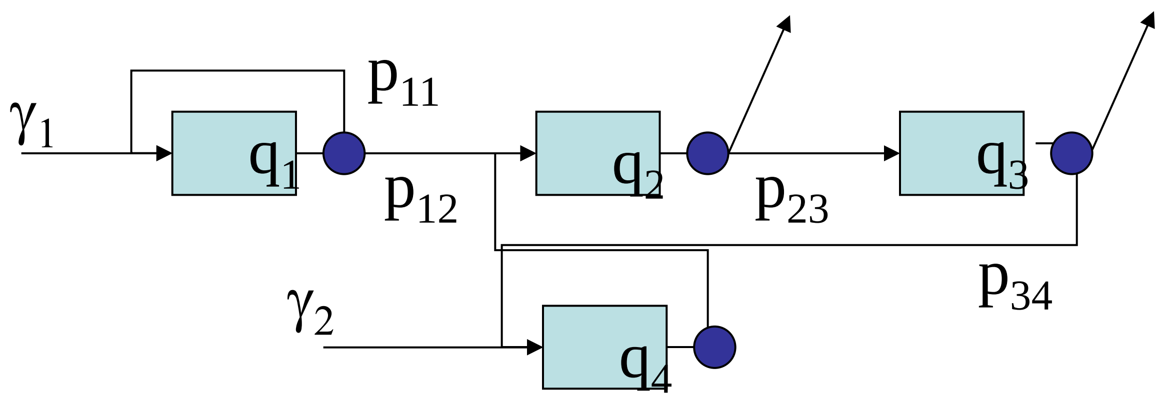
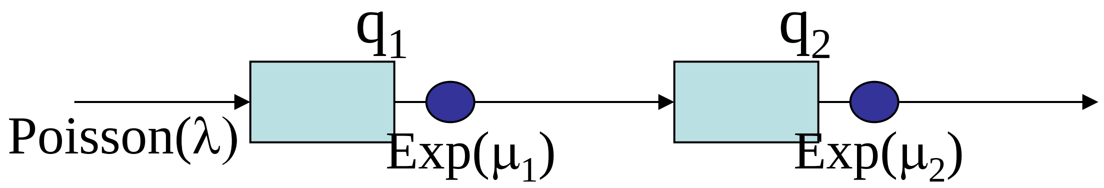
Open and closed queuing networks

- Queuing network: network of queuing systems
  - E.g., data packets traversing the network from router to router
- Open and closed networks
  - Open queuing network: customers arrive and leave the network (typical application: data communication)
  - Closed queueing networks: in and out flows are missing – constant number of customers circulate in the network (application: computer systems)



Open queuing networks- A tandem system



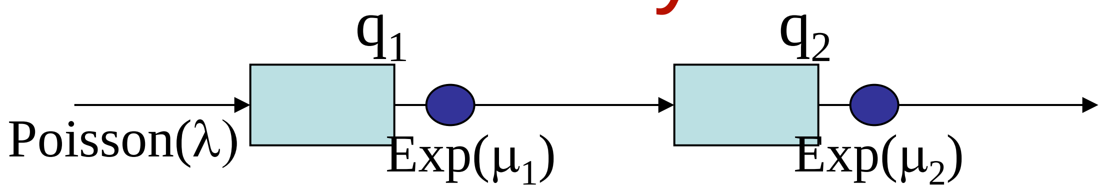
- The most simple open queuing network
- Assume a Poisson arrival process and independent, exponentially distributed service times
- What is the departure process from queue 1?
  - Interdeparture time:
    - Customer leaves queue behind: time of service of next customer
    - Customer leaves empty system behind: time to next arrival + time of service

$$L(f_\tau(t)) = \rho \frac{\mu}{s+\mu} + (1-\rho)\frac{\lambda}{s+\lambda}\frac{\mu}{s+\mu} =$$

$$\frac{\rho\mu(s+\lambda)+\lambda\mu-\rho\lambda\mu}{(s+\lambda)(s+\mu)} = \frac{\lambda s+\lambda^2+\lambda\mu-\lambda^2}{(s+\lambda)(s+\mu)} = \frac{\lambda}{s+\lambda}$$

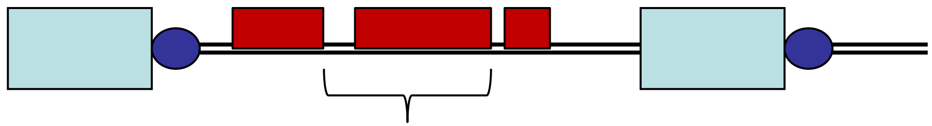
  - Departure process: Poisson (λ)!
- Same for M/M/m, but not for systems with losses and not for M/G/m systems

A tandem system



- Tandem system
  - Queue 1 is an M/M/1 queue
  - Departure process from Queue 1 is Poisson
  - Thus Queue 2 is also an M/M/1 queue
- State of the tandem queue: $N=(n_1,n_2)$, $p(\underline{n})=p(n_1,n_2)$
- Jackson theorem: the network behaves as if set of independent queues, that is:
  - $p(n_1,n_2) = p(n_1)p(n_2)$
  - Proof: see Virtamo notes

Modeling communication networks - note on the independence assumption

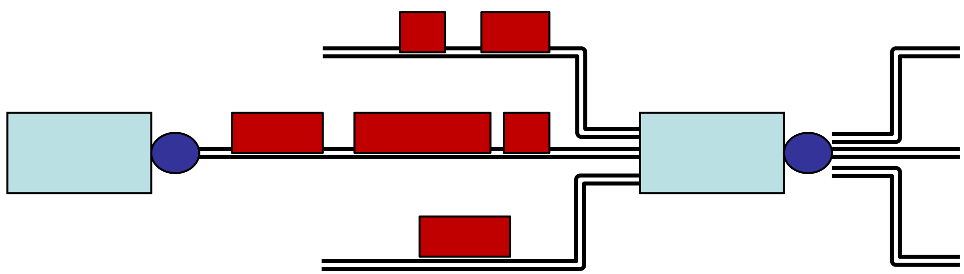


interarrival time □ packet size □ correlation!

- The product form $p(n_1, n_2) = p(n_1)p(n_2)$ applies only if the arrival and service processes are independent
- For two transmission links in series, queue 2 is not a M/M/1-queue
  - Correlation between service times of a customer in the two queues – determined by the packet length and the link transmission rate
  - Correlation between arrival and service times
    - For two consecutive packets, the interarrival time at the second queue can not be smaller than the service (that is, transmission) time of the first packet at the first queue
    - E.g., there will not be any queuing in queue 2 if the transmission rate at queue 2 is larger
  - Product form solution does not apply

Modeling communication networks - note on the indepdence assumption

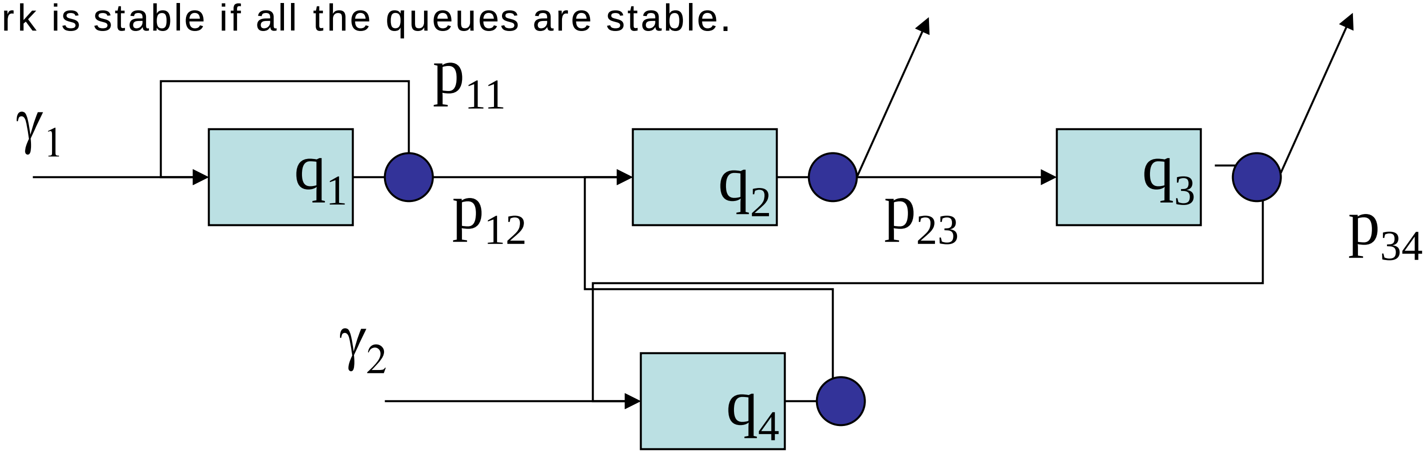
- Kleinrock's assumption on independence
  - Traffic to a queue comes from several upstream queues
    - Superposition of Poisson processes give a Poisson process
  - Traffic from a queue is spread randomly to several downstream queues
    - Partial processes are Poisson with intensity $p_i \lambda$ ($\Sigma p_i = 1$)
  - It is assumed to create independent arrival and service processes
  - Product form solution applies
  - E.g., network of large routers



Open Jackson's queuing networks – where the product form works

- Open queuing network
  - arrivals to the network
  - from all arrival point a departure point is reachable
- M queues with infinite storage and m exponential servers
  - Even finite storage if "last queue" in the networks
- Customers from outside of the network arrive to node *i* as a Poisson process with intensity $\gamma_i \geq 0$
- The service times are independent of the arrival process (and service times in other queues)
- A customer comes from node *i* to node *j* after service with the probability p_{ij} or leaves the network with the probability $p_{i0}=1-\Sigma p_{ij}$.
- Note, it allows feedback (e.g, p_{11}). The arrival process in not Poisson anymore, but the queue behaves as if the arrival would be Poissonian.

- Network is stable if all the queues are stable.



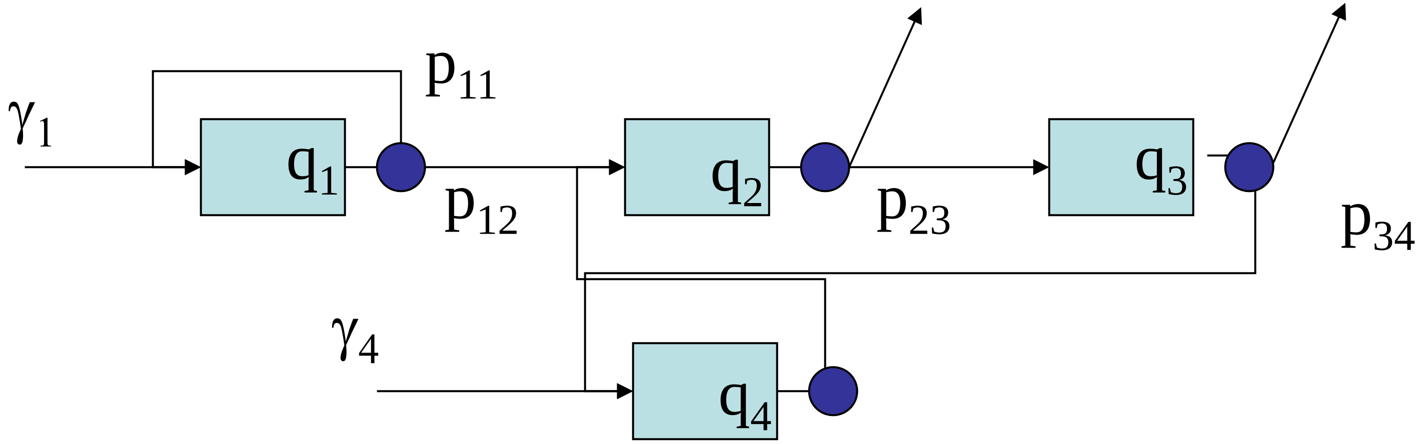
Open Jackson's queuing networks

- Flow conservation: arrival intensity to node *j* is

$$\lambda_j = \gamma_j + \sum_{i=1}^{M} \lambda_i p_{ij}$$

- Jacksons theorem: The distribution of number of customers in the network has *product form* – queues behave as independent M/M/m queues! (we do not prove – same as for tandem queues)

$$p(n_1, n_2, \ldots, n_M) = p_1(n_1) \cdots p_M(n_M)$$

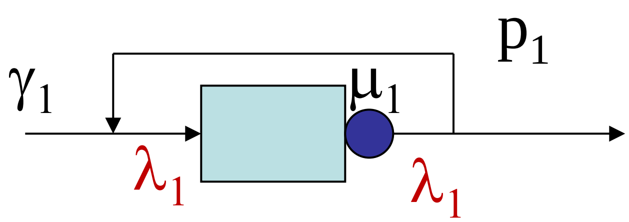


Open Jackson's queuing networks

- Flow conservation: arrival intensity to node *j*:

$$\lambda_j = \gamma_j + \sum_{i=1}^{M} \lambda_i p_{ij}$$

- Example 1: single feedback queue



$$\lambda_1 = \gamma_1 + \lambda_1 p_1 \qquad \rho = \frac{\lambda_1}{\mu_1}$$

$$\lambda_1 = \frac{\gamma_1}{1 - p_1} \qquad p(k) = (1-\rho)\rho^k$$

  - Performance measures as if it would be M/M/1
  - Though the arrival process is not Poisson
  - Stability: $\lambda_1 / \mu_1 < 1$

Open Jackson's queuing networks

- Arrival intensity and state probability

$$\lambda_j = \gamma_j + \sum_{i=1}^{M} \lambda_i p_{ij}$$

$$P(n_1, n_2, \ldots, n_M) = P_1(n_1) \cdots P_M(n_M)$$

- For the M/M/1 case:

$$P(n_i) = (1 - \rho_i)\rho_i^{n_i} \text{ and } \rho_i = \lambda_i / \mu_i < 1$$

- Example 2
  - calculate arrival intensities, give the "stability region", the possible arrival rates, when the network is stable
  - calculate the probability that the network is empty
  - calculate the probability that there is one customer in the network

Open Jackson's queuing networks
Mean performance measures

- Little's theorem applies to the entire network! – Good, because T is hard to calculate if there are feedback loops.
- The mean number of customers in the network and the average time spent in the network are (e.g., M/M/1 case)

$$N = \sum_{j=1}^{M} N_j = \sum_{j=1}^{M} \frac{\rho_j}{1-\rho_j}$$

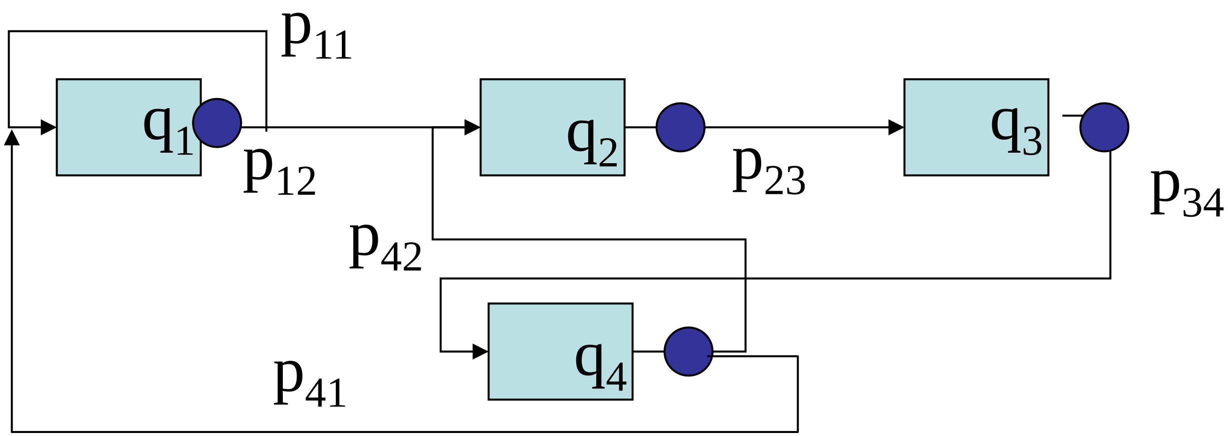
$$T = N / \sum_{j=1}^{M} \gamma_j$$

- The mean number of nodes a customer visits before leaving:
  - {Sum arrival intensity to the queues} / {arrival intensity to the network}

$$V = \sum_{j=1}^{M} \lambda_j / \sum_{j=1}^{M} \gamma_j, \quad \lambda_j = \gamma_j + \sum_{i=1}^{M} \lambda_i p_{ij}$$

Closed Jackson's queuing networks

- Not exam material this year
- Closed queuing network
- M queues with infinite storage and m exponential servers
- K customers circulating in the network, no arrivals and departures
- The service times are independent of the arrival process (and service times in other queues)
- A customer comes from node *i* to node *j* after service with the probability p_{ij}
- Queues can not be independent, since there is a fixed number of customers



Closed Jackson's queuing networks

- Flow conservation: arrival intensity to node *j*, the problem is that none of the -s are known*:*

$$\lambda_j = \sum_{i=1}^{M} \lambda_i p_{ij} \qquad (*)$$

- Limited set of states, since the sum of the customers is constant K:

$$S = \{(n_1, n_2, \ldots, n_M), \quad n_i \geq 0, \sum_{i=1}^{M} n_i = K\}$$

- MC based solution: state: vector of number of customers per queue - complex
- Algorithmic solution – e.g., M/M/1
  - (*) gives a set of dependent equations, with solution of e.g.:

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots \lambda_M\} = \alpha\{1, e_2, e_3, e_4, \ldots e_M\}$$

  - we have to select the one that gives sum of network state probabilities equal to one
  - Gordon-Newell: state probabilities, without calculating arrival intensities (without proof)

$$P(\underline{n}) = \frac{1}{G_M^K} \prod_{i=1}^{M} \left(\frac{e_i}{\mu_i}\right)^{n_i}, \quad G_M^K = \sum_{\underline{n} \in S} \prod_{i=1}^{M} \left(\frac{e_i}{\mu_i}\right)^{n_i}$$

Summary

- Queuing networks:
  - set of queuing systems
  - customers move from queue to queue
- Applied to networking problems: independence of queues have to be ensured
- Open queuing networks
  - Burke: Output process of an M/M/m queue is Poissonian
  - Jackson theorem: network state probability has product form if M/M/m queues
- Closed queuing networks – not exam material
  - Number of customers constant
  - State of queues is dependent – Gordon-Newell normalization