Exercise 1.6: Conductivity and work

Unsolved problem from exercise class 1.

Consider a current $\mathbf{J}(\mathbf{x},t)$ driven linearly by the electric field $\mathbf{E}(\mathbf{x},t)$ such

$$\mathbf{J}(\mathbf{k},\omega) = \sigma \bullet \mathbf{E}(\mathbf{k},\omega)$$

where σ is the conductivity tensor (2-tensor, i.e. has matrix representations with components σ_{ij}).

How does the work performed by electric field $\mathbf{E}$ on a current $\mathbf{J}$ depend on the hermitian and anti-hermitian parts of the conductivity tensor?

Hint: Use Plancherel's theorem

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}^*(\omega)\hat{g}(\omega)d\omega$$

Exercise 3.1: Birefringence

Consider a laser that emits primarily at a frequency, f, but also emits small components at the harmonic frequencies $2f$, $3f$, etc.

We will now study how the polarisation of this light changes when it passes though a quarter wave plate.

a) Derive the dispersion and eigenvectors for the O- and X-modes propagating in the quarter wave plate when $\mathbf{k}$ is in the x-direction and the dielectric tensor:

$$\mathbf{K} = \begin{bmatrix} K_\perp & 0 & 0 \\ 0 & K_\perp & 0 \\ 0 & 0 & K_\parallel \end{bmatrix}$$

b) When entering the plate the polarisation is $\mathbf{e}_{in}=[0,1,1]$
Express this polarisation using the eigenvectors of the O-mode and the X-mode.

c) Place this polarisation on the Poincare sphere when $\mathbf{e}^1=\mathbf{e}_y$ amd $\mathbf{e}^2=\mathbf{e}_z$.

d) If the plate is a quarter wave plate at a frequency f, what is it's length?

e) What is the polarisation of the three first harmonic components when leaving the crystal?

f) Construct the Muller matrix for the quarter wave plate at the frequency f.

Refresh: Stokes vectors and Muller matrixes

The Stokes vector and the Muller calculus can be used to describe both polarised and unpolarised waves.

The intensity tensor: $I^{\alpha\beta} = < E^{\alpha}(t,\mathbf{x})\, E^{\beta}(t,\mathbf{x}) >$ (unpolarised light is stochastic)

Define the Stokes vector S=[I, Q, U, V] such that

$$I^{\alpha\beta} = \frac{1}{2}\left[I\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + Q\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + U\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + V\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right]$$

- The Stokes parameter I is the intensity – all waves have an intensity!
- Q is the amplitude of a linearly polarised wave $\mathbf{E} = [0,1,0] \Rightarrow [I,Q,U,V] = [1,1,0,0]$
- Similarly for U : $\mathbf{E} = [0,1,1]/\sqrt{2} \Rightarrow [I,Q,U,V] = [1,0,1,0]$
- And for $-U$: $\mathbf{E} = [0,1,-1]/\sqrt{2} \Rightarrow [I,Q,U,V] = [1,0,-1,0]$
- V is the amplitude of cirular polarised wave: $\mathbf{E} = [0,1,i]/\sqrt{2} \Rightarrow [I,Q,U,V] = [1,0,0,1]$
- Unpolarised wave, let ξ_1 , ξ_1 be random: $\mathbf{E} = [0,\xi_1,\xi_2]/\sqrt{2} \Rightarrow [I,Q,U,V] = [1,0,0,0]$

The Muller matrix, $\mathbf{M}$:

$$S_{out} = \mathbf{M} S_{in} \iff \begin{bmatrix} I_{out} \\ Q_{out} \\ U_{out} \\ V_{out} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \begin{bmatrix} I_{in} \\ Q_{in} \\ U_{in} \\ V_{in} \end{bmatrix}$$

Exercise 3.2: Faraday rotation

In magnetoactive media a type of birefringence can be found called Faraday rotation. Unlike birefringence in a crystal this effect appear when the wave propagates **along** the axis of the media (axis of the magnetic field).

The effect originates from the Lorentz-force. When $\mathbf{B}$ is in the z-direction, then

$$q\mathbf{v}\times\mathbf{B} = q\mathbf{e}_i\varepsilon_{ijk}v_jB_k = q\left[\varepsilon_{ijk}B_k\right]\left[v_j\right] = q\begin{bmatrix}0 & B & 0\\ -B & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_x\\ v_y\\ v_z\end{bmatrix}$$

This property gives rise to off-diagonal terms in the dielectric tensor $\mathbf{K} = \begin{bmatrix}S & -iD & 0\\ iD & S & 0\\ 0 & 0 & P\end{bmatrix}$

a) Find the transverse modes when $\mathbf{k}=k\mathbf{e}_z$.

b) Show that the eigenvectors are orthogonal, i.e. that they form an orthogonal basis for all polarised transverse waves.

c) Express the $\mathbf{e}_x$ and the $\mathbf{e}_y$ unit vectors in terms of the eigenvectors.

d) With initial condition $\mathbf{E}=E\mathbf{e}_x$, how does the Faraday rotation change the polarisation after travelling a distance L?

e) Assume $D=B$ and $S=const$. How does the Faraday rotation depend on B?