EP2200 Queueing theory and teletraffic systems

Lectures 5-7

Summary of M/M/*/* systems

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M/M/*/* systems

- Poisson arrival, Exponential service time

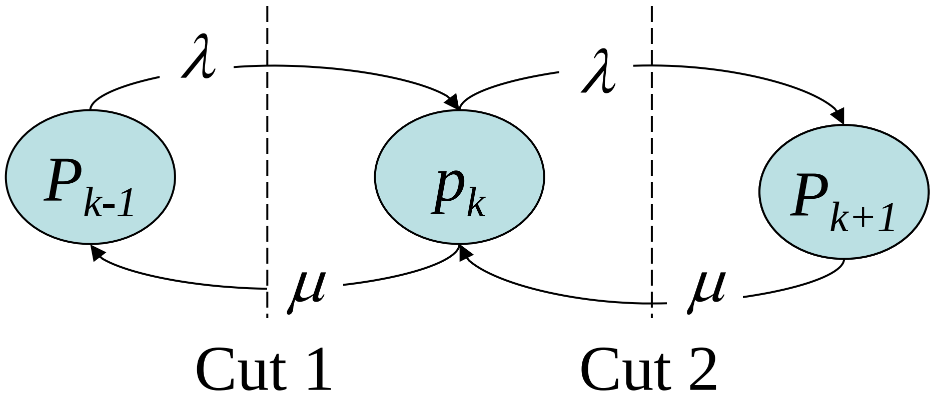
- M/M/1
- M/M/1/K
- M/M/m/m (Erlang loss system)
- M/M/m (Erlang wait system)
- M/M/m/m/C (Engset loss system)

M/M/1

- Single server, infinite waiting room
- Service times are exponentially distributed
- Arrival process Poisson
  - Models a large population of independent customers
  - Each customer generates requests with low rate
  - The total arrival process tends towards a Poisson process
- The queueing system can be modelled by a homogeneous birth-death process

Local balance equations

- Total transition rates from one part of the chain must balance the transition rates from the other part in stationarity



$$\text{Cut 1:}\ \lambda p_{k-1} = \mu p_k \Rightarrow p_k = \frac{\lambda}{\mu} p_{k-1}$$

$$\text{Cut 2:}\ \lambda p_k = \mu p_{k+1} \Rightarrow p_{k+1} = \frac{\lambda}{\mu} p_k = \left(\frac{\lambda}{\mu}\right)^2 p_{k-1}$$

Performance results

- The system is in state k with probability $p_k=(1-\rho)\rho^k$
- An arriving customer finds k customers in the system with probability p_k
  - –PASTA: Poisson Arrivals See Time Averages
- Expected number of customers in the system is $N=\rho/(1-\rho)$
  - –Time measures by Little's law
- Waiting time distribution
  - – In transform domain and in time domain

$$L(f_w(t)) = \sum_{k=0}^{\infty} \left(\frac{\mu}{s+\mu} \right)^k (1-\rho)\rho^k$$

$$P(W < t) = W(t) = 1 - \rho e^{-(\mu-\lambda)t}, t \geq 0$$

M/M/1/K systems

- Poisson arrival, exponential service time, 1 server, finite buffer capacity
- State transition diagram:
  - K+1 states
  - $\lambda_i=\lambda$, for $i \leq K$
  - $\mu_i=\mu$, for $i > 0$
- State probabilities in equilibrium and blocking probability from the local balance equations

$$p_k = \frac{\rho^k(1-\rho)}{1-\rho^{K+1}}, \quad P(block) = p_K$$

$$\lambda_{eff} = \lambda(1-p_K)$$

$$\overline{N} = \sum_{k=0}^{K} k p_k = \frac{\rho}{1-\rho}\left(1-(K+1)p_K\right), \quad \overline{T} = \overline{N}/\lambda_{eff}$$

M/M/m/m - loss systems (Erlang loss systems)

- Poisson arrival, exponential service time, m identical servers, no buffer,
- Offered load: $a=\lambda/\mu$
- State transition diagram:
  - m+1 states
  - $\lambda_i=\lambda$
  - $\mu_i=i\mu$
- State probabilities and performance measures

$$p_k = \frac{a^k / k!}{\sum_{i=0}^{m} a^i / i!}, \quad P(block) = E_m(a) = B(m,a) = p_m$$ (Erlang-B form)

$$N = \sum_{k=0}^{m} k p_k = a(1-p_m)$$

M/M/m systems (Erlang wait systems)

- Poisson arrival, exponential service time, m identical and independent servers, infinite buffer capacity
- Offered load a=λ/μ [Erlang], a<m for stability
- State transition diagram:
  - infinite states
  - $\lambda_i=\lambda$
  - $\mu_i=i\mu$, for $0 < i \leq m$
  - $\mu_i=m\mu$, for $i > m$
- Probability of waiting and waiting time distribution

$$P(wait) = D_m(a) = C(m,a) = \sum_{k=m}^{\infty} p_k$$

$$D_m(a) = \frac{mE_m(a)}{m - a(1 - E_m(a))}$$ (Erlang-C form)

$$P(W < t) = W(t) = 1 - D_m(a)e^{-(m\mu-\lambda)t}, t \geq 0$$

M/M/m/m/C – finite population Engset loss system

- Exponential service time, m identical servers, infinite buffer capacity
- BUT: finite population – can not be modeled with state independent arrivals
- Modeling a single user:
  - thinking time Exp(λ)
  - holding time (or service time) Exp(μ)
  - after blocked call new thinking time
- Markov-chain model:
  - $\lambda_i=(C-i)\lambda$
  - $\mu_i=i\mu$
  - p_i from the balance equations

M/M/m/m/C – finite population

- Time blocking: the (%) proportion of time in blocking state = p_m
- Call blocking: the probability that an arriving call gets blocked $=a_m$
- Call blocking ≠ time blocking

$$a_k = \frac{\lambda_k p_k}{\sum_{i=0}^{m} \lambda_i p_i}$$

- Effective load and average number of active users

$$\lambda_{eff} = \sum_{i=0}^{m-1} (C-i)\lambda p_i$$

$$\overline{N} = \lambda_{eff} / \mu$$