PROBLEM 1

Use the Gauss theorem to calculate the flux of the vector field:

$$\overline{A} = (x^3+2y,\ y^3,\ z^3-3z^2+3z)$$

through the surface $x^2+y^2+(z-1)^2=1$ $\Rightarrow$ sphere centered in $(0,0,1)$

SOLUTION

1) $\displaystyle \iint_S \bar{A}\cdot d\bar{S} = \iiint_V \operatorname{div}\bar{A}\ dV = \iiint_V 3x^2+3y^2+3(z^2-2z+1)\ dV =$

$\displaystyle = 3\iiint_V x^2+y^2+(z-1)^2\ dV$

2) Introduce spherical coord.
But spherical coord. correspond to a sphere centered in $(0,0,0)$
we define new variables.

$$\left.\begin{array}{l} x' = x \\ y' = y \\ z' = z-1 \end{array}\right\} \Rightarrow 3\iiint_V x'^2+y'^2+z'^2\ dV$$

And the "new" surface will be: $\underbrace{x'^2+y'^2+z'^2=1}_{\text{Sphere centered in }(0,0,0)}$

$\Rightarrow$ we can use.

$$\left.\begin{array}{l} x' = r\cos\varphi\sin\theta \\ y' = r\sin\varphi\sin\theta \\ z' = r\cos\theta \end{array}\right\} \Rightarrow x'^2+y'^2+z'^2 = r^2$$

$$3\iiint_V x'^2+y'^2+z'^2\ dV = 3\int_0^{\pi}\int_0^{2\pi}\int_0^1 r^2\ \overbrace{r^2\sin\theta\, dr\, d\varphi\, d\theta}^{dV} = \frac{3}{5}\int_0^{\pi}\int_0^{2\pi}[r^5]_0^1 \sin\theta\, d\varphi\, d\theta =$$

$$= \frac{3}{5}\int_0^{\pi}[\varphi]_0^{2\pi}\sin\theta\, d\theta = \frac{6\pi}{5}\left[-\cos\theta\right]_0^{2\pi} = \frac{12\pi}{5}$$

PROBLEM 2

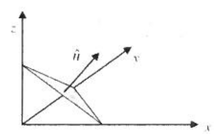
Calculate the integral $\oint_L \overline{A}\cdot d\overline{r}$ using the Stokes' theorem for:

(a) $\overline{A} = (x+2y,\ y-3z,\ z-x)$

L: unit circle in the xy plane oriented anti-clockwise.

(b) $\overline{A} = (0, xy, 0)$

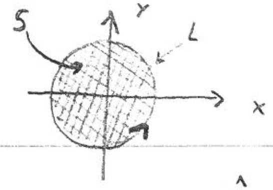
L: triangle defined by points (1,0,0), (0,1,0) and (0,0,1)



(c) $\overline{A} = (0, x^2, z^2)$

L: the boundary curve of the part of the surface $x^2+y^3+z^4=1$ that lies in the first octant $(x>0, y>0, z>0)$, oriented anti-clockwise ~~from~~ to the origin.

(a) $\displaystyle \int_L \bar{A}\cdot d\bar{r} = \iint_S \operatorname{rot}\bar{A}\cdot d\bar{S}$



$$\operatorname{rot}\bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y & y-3z & z-x \end{vmatrix} = (3, 1, -2)$$

$\hat{n} = \hat{e}_z$ From the right-hand rule!

$\displaystyle \iint_S \operatorname{rot}\bar{A}\cdot d\bar{S} = \iint_S \operatorname{rot}\bar{A}\cdot\hat{n}\ dS$

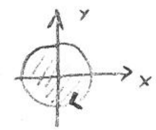
Cylindrical coordinates

$dx\,dy = \rho\, d\rho\, d\varphi$

$$= \iint_S (3,1,-2)\,\overbrace{(0,0,1)}^{\hat{e}_z}\ \rho\, d\rho\, d\varphi = \int_0^1\int_0^{2\pi} -2\rho\, d\rho\, d\varphi =$$

$$= -2\int_0^{2\pi}\left[\frac{\rho^2}{2}\right]_0^1 d\varphi = -\int_0^{2\pi} d\varphi = -2\pi$$

OBSERVATION
If orientation were "clockwise":



then $\hat{n} = -\hat{e}_z$

CONTINUE $\rightarrow$

PROBLEM 3

$\operatorname{grad}(fg) \Rightarrow$ vector	$\operatorname{rot}(f\cdot\bar{A}) \Rightarrow$ WRONG	$\operatorname{div}(\bar{A}\times\bar{B}) \Rightarrow$ scalar	$\operatorname{rot}(fg\,\bar{A}\times\bar{B}) \Rightarrow$ vector
$\operatorname{div}(f\bar{A}) \Rightarrow$ scalar	$\operatorname{rot}(f\bar{A}) \Rightarrow$ vector	$f\bar{A}(\bar{A}\cdot\bar{B}) \Rightarrow$ vector	$(\operatorname{rot}(\operatorname{rot}\bar{A}))\times\bar{B} \Rightarrow$ vector
	$\operatorname{div}(\bar{A}\cdot\bar{B}) \Rightarrow$ WRONG	$\operatorname{grad}(f\bar{A}\cdot\bar{B}) \Rightarrow$ vector	

(b) $\text{rot}\,\bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & xy & 0 \end{vmatrix} = (0, 0, y)$

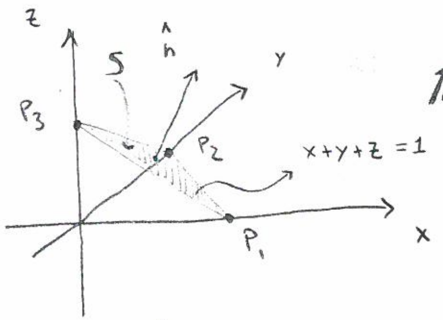
$$\int_L \bar{A}\, d\bar{r} = \iint_S \text{rot}\,\bar{A} \cdot d\bar{S} =$$

[$dx\,dy$ is the projection on xy-plane of $d\bar{S}$

$\Rightarrow dx\,dy = d\bar{S}\cdot\hat{e}_z = dS\, \hat{n}\cdot\hat{e}_z \Rightarrow dS = \dfrac{dx\,dy}{\hat{n}\cdot\hat{e}_z}$

$\Rightarrow d\bar{S} = \hat{n}\, dS = \dfrac{\hat{n}}{\hat{n}\cdot\hat{e}_z}\, dx\,dy$]

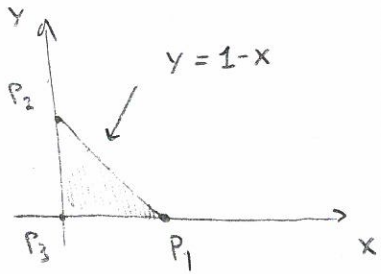
$$= \iint_S \text{rot}\,\bar{A} \cdot \frac{\hat{n}}{\hat{n}\cdot\hat{e}_z}\, dx\,dy$$



$\Uparrow$

$\hat{n} = \dfrac{(1,1,1)}{\sqrt{3}} \Rightarrow \dfrac{\hat{n}}{\hat{n}\cdot\hat{e}_z} = \dfrac{(1,1,1)}{(1,1,1)(0,0,1)} = (1,1,1)$

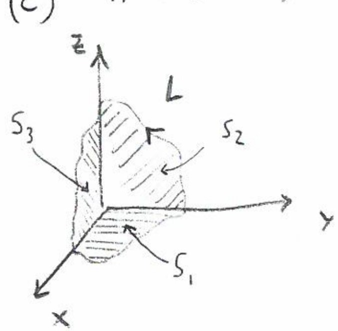
From the "top" (i.e. looking along the z-axis), the figure looks like;



$\Rightarrow \displaystyle\int_0^1 \int_0^{1-x} (0,0,y)(1,1,1)\, dx\,dy =$

$$= \int_0^1 \int_0^{1-x} y\, dx\,dy = \int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx = \int_0^1 \frac{1-2x+x^2}{2}\, dx = \left[\frac{1}{2}x - \frac{x^2}{2} + \frac{x^3}{6}\right]_0^1 = \frac{1}{6}$$

(c) $\bar{A} = (\cancel{0}, x^2, z^2) \Rightarrow \text{rot}\,\bar{A} = (0,0,2x)$ ($\Rightarrow$ parallel to $\hat{e}$)



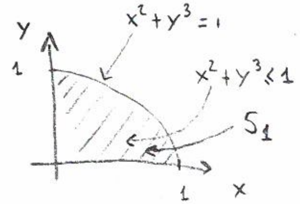
$$\int_L \bar{A}\cdot d\bar{r} = \iint_S \text{rot}\,\bar{A}\cdot d\bar{S}$$

Smart choice of S: $S = S_1 + S_2 + S_3$

with S_1 in xy plane
S_2 in yz plane
S_3 in xz plane } // to $\hat{e}_z \Rightarrow \iint_{S_2} = \iint_{S_3} = 0$

$$\iint_S \text{rot}\,\bar{A}\cdot d\bar{S} = \iint_{S_1} (0,0,2x)\cdot \hat{e}_z\, dx\,dy = \iint_{S_1} 2x\, dx\,dy$$

[S_1 is on xy plane $\Rightarrow z=0 \Rightarrow x^2+y^3=1$



$\Rightarrow y < (1-x^2)^{1/3}$]

$$= \int_0^1 \int_0^{(1-x^2)^{1/3}} 2x\, dx\,dy = \int_0^1 \left[2xy\right]_0^{(1-x^2)^{1/3}} dx = \int_0^1 2x(1-x^2)^{1/3}\, dx = ?$$

DIFFICULT !!!

Let's change variable! $w = x^2 \Rightarrow dw = 2x\,dx$

$$= \iint_{S_1} dw\,dy = \int_{y=0}^{y=1} \int_{w=0}^{w=1-y^3} dw\,dy = \int_0^1 1-y^3\, dy = \left[y - \frac{1}{4}y^4\right]_0^1 = \frac{3}{4}$$

ANOTHER WAY

S: $x^2+y^3+z^4=1 \Rightarrow$ If $\phi = x^2+y^3+z^4-1$ then $\bar{n} = \text{grad}\,\phi = (2x, 3y^2, 4z^3)$ is perpendicular to S

But $d\bar{S} = \dfrac{\hat{n}}{\hat{n}\cdot\hat{e}_z}\, dx\,dy \Rightarrow d\bar{S} = \dfrac{(2x, 3y^2, 4z^3)}{4z^3}\, dx\,dy$

$\Rightarrow \displaystyle\iint_S \text{rot}\,\bar{A}\cdot d\bar{S} = \iint (0,0,2x)\cdot\left(\frac{2x}{4z^3}, \frac{3y^2}{4z^3}, 1\right) dx\,dy$

$= \iint 2x\, dx\,dy$

This can be solved like above