Queuing Theory 2014 - Exercises

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1 Probability Theory and Transforms

1.1 Exercise 1.2

X is a random variable chosen from X_1 with probability a and from X_2 with probability b. Calculate $E[X]$ and σ_X for $\alpha = 0.2$ and $b = 0.8$. X_1 is an exponentially distributed r.v. with parameter $\lambda_1 = 0.1$ and X_2 is an exponentially distributed r.v. with parameter $\lambda_2 = 0.02$. Let the r.v. Y be chosen from D_1 with probability α and from D_2 with probability b, where D_1 and D_2 are deterministic r.v.s. Calculate the values D_1 and D_2 so that $E[X] = E[Y]$ and $\sigma_X = \sigma_Y$.
Solution: a) We directly apply the **conditional expectation** formula:

$$E[X] = \alpha E[X_1] + bE[X_2].$$

We can do this since the expectation is a raw moment – not central. The proof is straightforward: we have

$$\begin{aligned} & f_X(x) = \alpha f_{X_1}(x) + b f_{X_2}(x) \rightarrow \\ & \rightarrow E[X] = \int_0^\infty x f_X(x)dx = \alpha \int_0^\infty x f_{X_1}(x)dx + b \int_0^\infty x f_{X_2}(x)dx = \\ & \quad = \alpha E[X_1] + bE[X_2]. \end{aligned}$$

We then replace the given data

$$E[X] = \alpha\frac{1}{\lambda_1} + b\frac{1}{\lambda_2} = 0.2\frac{1}{0.1} + 0.8\frac{1}{0.02} = 42. \tag{1}$$

We can not calculate the variance (or the standard deviation) in the same way, since this is a central moment. Instead, we proceed with calculating the expected square of the r.v. X, which is a **raw** moment:

$$\begin{aligned} E[X^2] &= \int_0^\infty x^2 f_X(x)dx = \alpha \int_0^\infty x^2 f_{X_1}(x)dx + b \int_0^\infty x^2 f_{X_2}(x)dx = \\ &= \alpha E[X_1^2] + bE[X_2^2]. \end{aligned}$$

Replacing the data we get

$$E[X] = \alpha\frac{2}{\lambda_1^2} + b\frac{2}{\lambda_2^2} = 0.2\frac{2}{0.1^2} + 0.8\frac{2}{0.02^2} = 4040. \tag{2}$$

Finally, we use the relation between the expectation, square mean and variance

$$\sigma_X^2 = E[X^2] - [E[X]]^2 = 4040 - 42^2 \rightarrow \sigma_X = 47.70. \tag{3}$$

b) We have $E[Y] = \alpha d_1 + bd_2$ and $E[Y^2] = \alpha d_1^2 + bd_2^2$. So the system of equations becomes

$$\begin{aligned} & 0.2d_1 + 0.8d_2 = 42, \\ & 0.2d_1^2 + 0.8d_2^2 = 4040 \end{aligned} \tag{4}$$

Solving this 2 by 2 non-linear system we obtain the solution. Notice that because of the second order of the equation we may **in general** have more than one solutions.

1.2 Exercise 1.3

X is a discrete stochastic variable, $p_k = P(X = k) = \frac{a^k}{k!}e^{-a}, k = 0, 1, 2, ...$ and a is a positive constant.

a) Prove that $\sum_{k=0}^{\infty} p_k = 1$.

b) Determine the z-transform (generating function) $P(z) = \sum_{k=0}^{\infty} z^k p_k$.

c) Calculate $E[X], \text{Var}[X]$ and $E[X(X-1)...(X-r+1)], r = 1, 2, ...$ with and without using z-transforms.

Solution a) We have

$$\sum_{k=0}^{\infty} p_k = \sum_{k=0}^{\infty} \frac{a^k}{k!}e^{-a} = e^{-a}\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^{-a}e^{a} = 1.$$

Notice this useful and well-known infinite series summation.

b) We replace the definition of the mass function and gradually have:

$$P(z) = \sum_{k=0}^{\infty} z^k \frac{a^k}{k!}e^{-a} = e^{-a}\sum_{k=0}^{\infty} z^k \frac{a^k}{k!} = e^{-a}\sum_{k=0}^{\infty} \frac{(za)^k}{k!} = e^{-a}e^{az} = e^{-a(1-z)}.$$

c) First, we try without the z-transform, i.e. using the definitions in the probability domain. We start from the third sentence, using the **definition** of expectation:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx \tag{5}$$

$$\begin{aligned} E[X(X-1)...(X-r+1)] &= \textstyle\sum_{k=0}^{\infty} k(k-1)...(k-r+1)p_k = \\ &= \textstyle\sum_{k=0}^{\infty} k(k-1)...(k-r+1)\frac{a^k}{k!}e^{-a} = \sum_{k=0}^{\infty} \frac{a^k}{(k-r)!}e^{-a} = \\ &= e^{-a}a^r \textstyle\sum_{k=0}^{\infty} \frac{a^{(k-r)}}{(k-r)!} = a^r e^{-a} e^{a} = a^r. \end{aligned}$$

Then clearly, we have (by setting $r = 1$) $E[X] = a^1 = a$. And, finally,

$$\text{Var}[X] = E[X^2] - [E[X]]^2 = E[X^2] - a^2 = E[X(X-1)] + E[X] - a^2 = a^2 + a - a^2.$$

We try, now, with the z-transform. We differentiate r times the definition of the z-transform:

$$\frac{d^r}{dz^r}P(z) = \frac{d^r}{dz^r}\sum_{k=0}^{\infty} z^k p_k = \sum_{k=0}^{\infty} k(k-1)...(k-r+1)z^{k-r}p_k$$

If we replace $z = 1$ we get

$$\left.\frac{d^r}{dz^r}P(z)\right\}_{z=1} = E[X(X-1)...(X-r+1)].$$

We, then, calculate,

$$\left.\frac{d^r}{dz^r}P(z)\right\}_{z=1} = a^r e^{-a(1-1)} = a^r.$$

1.3 Exercise 1.4

X_i's are independent Poisson distributed random variables, thus, $p_k = \frac{a_i^k}{k!}e^{-a_i}$, $k = 0, 1, 2, ...$, and each $a_i, i = 1, 2, ..., n$ is a positive constant. Give the probability distribution function of $X = \sum_{i=1}^{n}$.

Solution: This problem indicates the usefulness of the z-transform in the calculation of the distribution of the sum of variables. We have proven that **the ZT of the sum of independent random variables is the product of their individual z-transforms**. Thus,

$$P(z) = \prod_{i=1}^{n} P_i(z) = \prod_{i=1}^{n} e^{-a_i(1-z)} = e^{\sum_{i=1}^{n} -a_i(1-z)} = e^{-\alpha(1-z)},$$

where $\alpha = \sum_{i=1}^{n} -a_i$. This proves that the distribution is also Poisson with parameter α, i.e. the sum of parameters. The proof is based on the uniqueness of z-transform[1]. As a result, the distribution function will be

$$p_X(k) = \frac{\alpha^k}{k!}e^{-\alpha}$$

1.4 Exercise 1.5

X is a positive stochastic continuous variable with probability distribution function (PDF)

$$F(x) = P(X \leq x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-ax}, & x \geq 0. \end{cases}$$

a) Give the probability density function $f(x) = dF(x)/dx$.

b) Give $\overline{F}(x) = P(X > x)$.

c) Calculate the Laplace Transform $f^*(s) = E[e^{-sX}] = \int_0^\infty e^{-sx} f(x)dx$.

d) Calculate the expected values $m = E[X], E[X^k]$, $k = 0, 1, 2, ...$, the variance σ_X^2, the standard deviation σ_X and the coefficient of variation $c = \sigma/m$, with and without the transform $F^*(s)$.

Solution: a) For the calculation of $f(x)$ we just need to differentiate:

$$f(x) = dF(x)/dx = d(1 - e^{-ax})/dx = ae^{-ax}.$$

b) The complementary PDF is simply given as

$$\overline{F}_X(x) = P(X > x) = 1 - P(X \leq x) = 1 - F_X(x) = e^{-ax}.$$

c) Calculation of the Laplace Transform with simple integration

$$f^*(s) = \int_0^\infty e^{-sx} f(x)dx = \int_0^\infty e^{-sx} ae^{-ax}dx = a\int_0^\infty e^{-x(s+a)}dx = \frac{a}{s+a}.$$

d) We proceed first, without the help of Laplace transforms, using the definition of the expectation

$$E[X^0] = \int_0^\infty x^0 f(x)dx = \int_0^\infty f(x)dx = 1.$$

[1]or the 1-1 correspondence between the mass function and the ZT

$E[X^k] = \int_0^\infty x^k f(x)dx = \int_0^\infty x^k ae^{-ax}dx = a\frac{-1}{a}\int_0^\infty x^k (e^{-ax})'dx =$
$= k\int_0^\infty x^{k-1}e^{-ax}dx = \frac{k}{a}\int_0^\infty x^{k-1}ae^{-ax}dx = \int_0^\infty x^{k-1}f(x)dx =$
$= \frac{k}{a}E[X^{k-1}].$

This is a recursive formula that enables the calculation of any moment. We have:

$$E[X^k] = \frac{k}{a}E[X^{k-1}] = \frac{k}{a}\frac{k-1}{a}E[X^{k-2}] = \frac{k}{a}\frac{k-1}{a}..\frac{1}{a}E[X^0] = \frac{k}{a}\frac{k-1}{a}..\frac{1}{a} = \frac{k!}{a^k}$$

which gives, simply, $E[X] = 1/a$, for $k = 1$. The variance is calculated through the usual formula, and the raw moments are taken from above:

$$\sigma^2 = E[X^2] - [E[X]]^2 = \frac{2}{a^2} - \left(\frac{1}{a}\right)^2 = 1/a^2.$$

so the standard deviation is simply the square root of the variance, $1/a$, and the coefficient of variation is 1. Notice that this is special for the exponential distribution.

We try, now, with the help of the Laplace transforms.

$$E[X^k] = (-1)^k \frac{d^k}{ds^k} f^*(s) = (-1)^k \frac{d^k}{ds^k}\frac{a}{s+a} = \frac{(-1)^k ak!}{(s+a)^{k+1}}.$$

We find this formula by differentiating k times the Laplace transform and replacing $s = 0$. The rest follows with simple replacement $k = 1, 2, ...$

1.5 Exercise 1.6

X_i's are independent, exponentially distributed random variables with a mean value of $1/a, a > 0, i = 1, 2, ..., n$. Calculate $P(X \leq x)$ and $P(X \geq x)$ where

a) $X = \min(X_1, X_2, ..., X_n)$,

b) $X = \max(X_1, X_2, ..., X_n)$.

Solution: a) The key point in this exercise is the fact that the random variables are independent (mutually independent). We gradually have:

$P(X \leq x) = P(\min(X_1, X_2, ..., X_n) \leq x) = 1 - P(\min(X_1, X_2, ..., X_n) > x)$
$= 1 - P(X_1 > x, X_2 > x, ..., X_n > x) = 1 - \prod_{i=1}^n P(X_i > x)$
$= 1 - \prod_{i=1}^n e^{-ax} = 1 - e^{-\sum_{i=1}^n ax} = 1 - e^{-nax}$

This shows that the minimum of exponentially distributed random variables is also an exponential variable and its rate is the sum of the individual rates.

b) Similar calculations:

$P(X \leq x) = P(\max(X_1, X_2, ..., X_n) \leq x) = P(X_1 \leq x, X_2 \leq x, ..., X_n \leq x)$
$= \prod_{i=1}^n P(X_i \leq x) = \prod_{i=1}^n (1 - e^{-ax}) = (1 - e^{-ax})^n.$

Cleary, the variable X is, now, not exponential.

2 Balance equations, birth-death processes, continuous Markov Chains

2.1 Exercise 3.2

Consider a birth-death process with 3 states, where the transition rate from state 2 to state 1 is $q_{21} = \mu$ and $q_{23} = \lambda$. Show that the mean time spent in state 2 is exponentially distributed with mean $1/(\lambda + \mu)$.[2]

Solution: Suppose that the system has just arrived at state 2. The time until next "birth" – denoted here as T_B – is exponentially distributed with cumulative distribution function $F_{T_B}(t) = 1 - e^{-\lambda t}$. Similarly, the time until next "death" – denoted here as T_D – is exponentially distributed with cumulative distribution function $F_{T_D}(t) = 1 - e^{-\mu t}$. The random variables T_B and T_D are independent.

Denote by T_2 the time the system spends in state 2. The system will depart from state 2 when the first of the two events (birth or death) occurs. Consequently we have $T_2 = \min\{T_B, T_D\}$. We, then, apply the result from exercise 1.6, that is the minimum of independent exponential random variables is an exponential random variable. We can actually show this:

$$\begin{aligned}
F_{T_2}(t) &= \Pr\{T_2 \leq t\} = \\
&= \Pr\{\min\{T_B, T_D\} \leq t\} = \\
&= 1 - \Pr\{\min\{T_B, T_D\} > t\} = \\
&= 1 - \Pr\{T_B > t, T_D > t\} = \\
&= 1 - \Pr\{T_B > t\} \cdot \Pr\{T_D > t\} = \\
&= 1 - e^{-\lambda t} \cdot e^{-\mu t} = \\
&= 1 - e^{-(\lambda+\mu)t}
\end{aligned}$$

so T_2 is exponentially distributed with parameter $\lambda + \mu$.

Notice that we can generalize to the case with more than two transition branches. This exercise reveals the property of continuous time Markov chains, that is, the time spent on a state is exponentially distributed.

2.2 Exercise 3.3

Assume that the number of call arrivals between two locations has Poisson distribution with intensity λ. Also, assume that the holding times of the conversations are exponentially distributed with a mean of $1/\mu$. Calculate the average number of call arrivals for a period of a conversation.

Solution: Denote by N_C the number of arriving calls during the period of one conversation. Denote by T the duration of this conversation. Given that $T = t$, $N_C | T = t$ is Poisson distributed with parameter $\lambda \cdot t$ so the probability mass function of the number of calls will be

$$\Pr\{\text{arriving calls within } t = k\} = P_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

[2]This exercise is similar to Exercise 6 from Chapter 1: "The minimum of independent exponential variables is exponential."

with an average number of calls: $E[N_C|T=t] = \lambda t$.

Moreover T is exponentially distributed, with parameter μ so the density function will be:

$$f_T(t) = \mu e^{-\mu t}.$$

We apply the conditional expectation formula:

$$E[N_C] = \int_0^\infty E[N_C|T=t] \cdot f_T(t)dt = \int_0^\infty \lambda t \mu e^{-\mu t} dt = \lambda \int_0^\infty t \mu e^{-\mu t} dt = \frac{\lambda}{\mu}.$$

2.3 Exercise 3.4

Consider a communication link with a constant rate of $4.8 kbit/sec$. *Over the link we transmit two types of messages, both of exponentially distributed size. Messages arrive in a Poisson fashion with* $\lambda = 10$ *messages/second. With probability 0.5 (independent from previous arrivals) the arriving message is of type 1 and has a mean length of 300 bits. Otherwise a message of type 2 arrives with a mean length of 150 bits. The buffer at the link can at most hold one message of type 1 or two messages of type 2. A message being transmitted still takes a place in the buffer.*

a) Determine the mean and the coefficient of variation of the service time of a randomly chosen arriving message.

b) Determine the average times in the system for accepted messages of type 1 and 2.

c) Determine the message loss probabilities for messages of type 1 and 2.

Solution:

a) We have a link with a constant transmission rate. So the service time distributions follow the packet length distributions. Consequently, the service times of both packet types are exponential with mean values of

- Type 1: $E[T_1] = \frac{300}{4800} = \frac{1}{16}$ sec,
- Type 2: $E[T_2] = \frac{150}{4800} = \frac{1}{32}$ sec.

As a result the parameters of the exponential distributions are $\mu_1 = 16$ and $\mu_2 = 32$, respectively. A random arriving packet is of Type 1 or Type 2 with probability 0.5. We apply the conditional expectation [3]:

$$E[T] = \frac{1}{2}E[T_1] + \frac{1}{2}E[T_2] = \frac{3}{64}$$

Similarly, we calculate the mean square:

$$E[T^2] = \frac{1}{2}E[T_1^2] + \frac{1}{2}E[T_2^2] = \frac{1}{2}\frac{2}{\mu_1^2} + \frac{1}{2}\frac{2}{\mu_2^2} = 16^{-2} + 32^{-2} = \frac{5}{4} \cdot 16^{-2}.$$

The variance of T is derived from $Var[T] = E[T^2] - (E[T])^2$. Then we compute the standard deviation σ_T as $\sigma_T = \sqrt{Var[T]}$, and finally the coefficient of variation is given as: $c_T = \frac{\sigma_T}{E[T]}$.

[3] This is similar to exercise 1.2

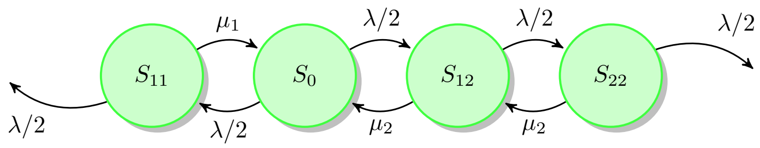

Figure 1: State Diagram for Exercise 3.4

b) For this part of the exercise, we need to draw the Markov Chain (Fig. 1) and solve it in the **steady state**. The state space must be defined in such a way that we can guarantee that all transitions – from state to state – have an exponential rate. We choose here to define such a Markov chain with 4 states:
State 0; Empty buffer.
State 11; 1 packet of Type 1.
State 21; 1 packet of Type 2.
State 22; 2 packets of Type 2.
Then we solve the balance equations in the local form:

$$\mu_2 P_{22} = \lambda/2 P_{21}$$
$$\mu_2 P_{21} = \lambda/2 P_0$$
$$\mu_1 P_{11} = \lambda/2 P_0$$
$$P_0 + P_{11} + P_{21} + P_{22} = 1 (\text{norm. equation})$$

Solution:

$$P_0 = 0.670, P_{21} = 0.105, P_{22} = 0.016, P_{11} = 0.209.$$

An accepted message of Type 1 can only arrive at state 0, otherwise it is rejected. So its the average service time will be $E[T_1]$.
An accepted message of Type 2 can arrive at states 0 and 21, otherwise it is rejected. Then, the average service time will be $(E[T_2]P_0 + 2E[T_2]P_{(21)})/(P_0 + P_{21})$. [4]

c) The loss probabilities are equal to the probabilities of the system being in BLOCKING states, for each of the two packet types. We underline that this is always true for *homogeneous* Markov chains, that is, Markov chains where the arrival rates do not depend on the system state.

2.4 Exercise 3.5

Consider a Markovian system with discouraged job arrivals. Jobs arrive to a server in a Poisson fashion, with an intensity of one job per 7 seconds. The jobs observe the queue. They do NOT join the queue with probability l_k if they observe k jobs in the queue. $l_k = k/4$ if $k < 4$, or 0, otherwise. The service time is exponentially distributed with mean time of 6 seconds.

a) Determine the mean number of customers in the system, and

b) the number of jobs served in 100 seconds.

Solution:

[4]The occurrence of acceptance reduces the sample space to two states only. Then the probabilities are normalized.

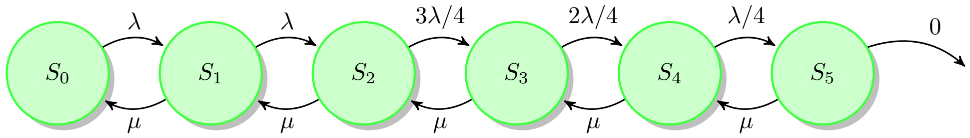

Figure 2: State diagram for Exercise 3.5

a) This is a simple model but requires careful design. After building the correct state diagram, the solution is found, based on the LOCAL balance equations.

We have a system with 6 states. *State space*: S_k : k jobs in the system. The system diagram is shown in Fig. 2).

Balance Equation System:

$$\lambda P_0 = \mu P_1$$
$$\lambda P_1 = \mu P_2$$
$$3\lambda/4 P_2 = \mu P_3$$
$$\lambda/2 P_2 = \mu P_4$$
$$\lambda/4 P_2 = \mu P_5$$
$$\sum_{k=1}^{5} P_k = 1$$

Solution: $P_0 \approx 0.3$, and the remaining probabilities are computed based on P_0 and the equations above. After determining the state probabilities, we derive the average number of customers in the system through

$$E[N] = \sum_{k=0}^{5} k \cdot P_k$$

We find $E[N] \approx 1.43$.

b) We have, here, a system with different arrival rates in each state. These systems are defined as *non-homogeneous*. However, the service rate is constant. The server is busy with probability $(1-P_0)$. When it is busy, it serves jobs. The service rate is $\mu = 1/6sec^{-1}$. As a result, the server can serve $100 \cdot \mu \cdot (1 - P_0)$ jobs in 100 seconds on AVERAGE!

2.5 Exercise 3.6

Consider a network node that can serve 1 and store 2 packets altogether. Packets arrive to the node according to a Poisson process. Serving a packet involves two independent sequentially performed tasks: the ERROR CHECK and the packet TRANSMISSION to the output link. Each task requires an exponentially distributed time with an average of 30msec. Give, that we observe that the node is empty in 60% of the time, what is the average time spend in the node for one packet?

Solution: As always, we need to construct the state diagram is such a way

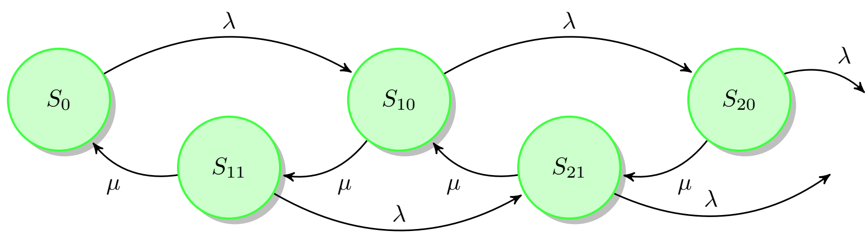

Figure 3: State Diagram for Exercise 3.6

that all transitions rates are guaranteed to be exponential. The selected state space:

S_0: Empty network node,
S_{11}: One packet under transmission,
S_{10}: One packet under error-check,
S_{20}: One packet under error-check and one buffered,
S_{21}: One packet under transmission and one buffered,

The state diagram is shown in Fig. 3. We can form the *global* balance equations parameterized by λ. Then we apply information that is given: $P_0 = 0.6$; This extra information enables the solution of the system of equations, and leads to the calculation of λ:

$$\lambda P_0 = \mu P_{11}$$
$$(\lambda + \mu)P_{11} = \mu P_{10}$$
$$(\lambda + \mu)P_{10} = \lambda P_0 + \mu P_{21}$$
$$\mu P_{21} = \lambda P_{11} + \mu P_{22}$$
$$P_{11} + P_{10} + P_{21} + P_{22} = 1 - P_0 = 0.4$$

Solution: $P_{10} \approx 0.1636, P_{11} \approx 0.1337, P_{20} \approx 0.0365, P_{21} \approx 0.0663, \lambda \approx 7.63$. For the calculation of the total average *system* time for a packet, we apply Little's formula.

$$\overline{N} = \lambda_{eff} \cdot E[T_{sys}] \rightarrow E[T_{sys}] = \frac{\overline{N}}{\lambda_{eff}} = \frac{1 \cdot (P_{10} + P_{11}) + 2 \cdot (P_{21} + P_{22})}{\lambda_{eff}}.$$

We always apply the effective arrival rate at Little's formula, because the formula needs the actual average arrival rate at the system, excluding possible drops. Here, the effective rate is not equal to λ, since we have packet drops. However, since the arrival rate for this system does not change with time, the effective arrival rate is simply:

$$\lambda_{eff} = \lambda \cdot (P_0 + P_{10} + P_{11}).$$

3 Chapter 4 – Queuing Systems

3.1 Exercise 4.1

Packets arrive to a communication node with a single output link according to a Poisson Process. Give the Kendall notation for the following cases:

1. *the packet lengths are exponentially distributed, the buffer capacity at the node is infinite*
2. *the packet length is fixed, the buffer can store n packets*
3. *the packet length is L with probability p_L amd l with probability p_l and there is no buffer in the node*

Solution: Kendall Notation

1. Arrival Process
2. Service Time
3. Number of Servers
4. Number of Total Positions (servers and queues)
5. Population

The Poisson arrivals (M) and the Single server (1) are fixed: M/?/1/?/?

1. M/M/1, as the buffer is infinite
2. M/D/1/n+1, as the service is deterministic and the buffer is n
3. M/G/1/1, as the service is general and there is no buffer

3.2 Exercise 4.2

Give the Kendall notation for the following systems. Telephone calls arrive to a PBX with C output links. The calls arrive as Poisson process and the call holding times are exponentially distributed.

1. *Calls arriving when all the output links are busy are blocked*
2. *Up to c calls can wait when all the output links are blocked*

Solution:

1. M/M/C/C
2. M/M/C/C+c

3.3 Exercise 4.3

Why is it not a good idea to have a G/G/10/12/5 System? Solution: 10 servers for 5 users!

4 Exercise 4.4

Which system provides the best performance, an M/M/3/300/100 or an M/M/3/100/100?

Solution: They have the same performance, since the users fit to both queues. Of course, the first system wastes buffer positions!

4.1 Exercise 4.5

A PBX was installed to handle the voice traffic generated by 300 employees in an office. Each employee on average makes 2 calls per hour with an average call duration of 4.5 minutes The PBX has 90 outgoing links.

1. *What is the offered load to the PBX?*
2. *What is the utilization of the outgoing links? Assume that calls arriving when all the links are busy are queued up.*

Solution: Offered Load $\rho \to \lambda\overline{T} = 300 \cdot \frac{2}{60} \cdot 4.5 = 45$ Erlang. Generally, the actual load is not the offered load.

Link Utilization:

$$\frac{\text{actual load}}{\#\text{ servers}}$$

The existence of an infinite queue here means that no load is dropped, or that the offered load is the actual load. So,

$$\text{utilization} = \frac{\text{offered load}}{90} = \frac{45}{90} = 0.5$$

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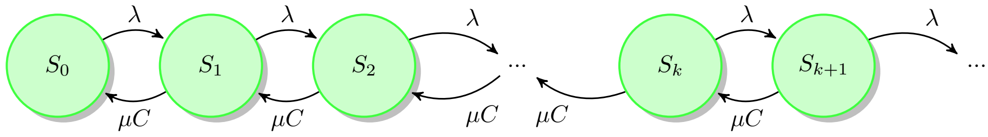

Figure 4: System diagram for the M/M/1 chain of exercise 5.1

5 Chapter 5 – M/M/1 Systems

5.1 Exercise 5.1

In a computer network a link has a transmission rate of C bit/s. Messages arrive to this link in a Poisson fashion with rate λ messages per second. Assume that the messages have exponentially distributed length with a mean of $1/\mu$ bits and the messages are queued in a FCFS fashion if the link is busy.
a) Determine the minimum required C for given λ and μ such that the average system time (service time + waiting time) is less than a given time T_0.

Solution: System Description

- Single communication link: C bits per second
- Poisson arrivals: λ messages per second
- Exponential Service times: $E[T] = E[X]/C = 1/(\mu C)$, so the *exponential rate* is μC.
- First Come First Served policy
- Infinite Queue[5]

This is a typical M/M/1 System. We see the system diagram in Fig. 4. We first derive the state distribution (steady-state) of this system through the solution of the balance equations. We *define* $\rho = \lambda/(\mu C)$. For a no-loss system, ρ is the OFFERED and, at the same time, the ACTUAL load.

$$\lambda P_0 = (\mu C)P_1 \rightarrow P_1 = \rho P_0$$
$$\lambda P_1 = (\mu C)P_2 \rightarrow P_2 = \rho P_1 = \rho^2 P_0$$
$$\lambda P_2 = (\mu C)P_3 \rightarrow P_3 = \rho P_2 = \rho^3 P_0$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\lambda P_k = (\mu C)P_{k+1} \rightarrow P_{k+1} = \rho P_k = \rho^k P_0$$
$$\dots\dots\dots\dots$$

Then, we calculate the P_0 through the normalization equation:

$$\sum_{k=0}^{\infty} P_k = 1 \rightarrow \sum_{k=0}^{\infty} \rho^k P_0 = 1 \rightarrow P_0 \sum_{k=0}^{\infty} \rho^k = 1 \rightarrow P_0 \cdot \frac{1}{1-\rho} = 1 \rightarrow P_0 = 1 - \rho.$$

[5] If no buffer capacity is mentioned, we always assume that this is infinite.

Finally, the state distribution is given as

$$P_k = (1-\rho)\rho^k.$$

We, now, derive the average number of messages in the system, using the state distribution:

$$\overline{N} = \sum_{k=0}^{\infty} kP_k = \sum_{k=0}^{\infty} k(1-\rho)\rho^k = (1-\rho)\rho \sum_{k=0}^{\infty} k\rho^{k-1} =$$
$$= (1-\rho)\rho \sum_{k=0}^{\infty} \frac{d\rho^k}{d\rho} = (1-\rho)\rho \frac{d\left(\sum_{k=0}^{\infty}\rho^k\right)}{d\rho} = (1-\rho)\rho \frac{d(1/(1-\rho))}{d\rho} = \frac{\rho}{1-\rho}.$$

In order to solve the first question we can use the LITTLE's formula:

$$\overline{N} = \lambda_{\text{eff}} E[T_{total}] \to E[T_{\text{total}}] = \frac{\overline{N}}{\lambda} = \frac{\rho/(1-\rho)}{\lambda} = \frac{\lambda/(\mu C)/(1-\lambda/(\mu C))}{\lambda},$$

since $\lambda_{eff} = \lambda$, so, finally,

$$E[T_{total}] = \frac{1}{(\mu C) - \lambda}.$$

The minimum required C is determined by:

$$\frac{1}{\mu C - \lambda} \le T_0 \to \mu C - \lambda \ge T_0^{-1} \to C \ge \frac{\lambda + T_0^{-1}}{\mu}.$$

6 Exercise 5.5

Consider a queuing system with a single server. The arrival events can be modeled with Poisson distribution, but two customers arrive at the system at each arrival event. Each customer requires an exponentially distributed service time.

1. Draw the state diagram
2. Determine p_k using local balance equations
3. Let $P(z) = \sum_{k=0}^{\infty} z^k p_k$. Calculate $P(z)$ for the system. Note, that $P(z)$ must be finite for $|z| < 1$, and we know $P(1) = 1$.
4. Calculate the mean number of customers in the system with the help of $P(z)$ and compare it with the one of the M/M/1 system.

Solution: The system can be described by an M/M/1 model, since there is a single server, the service times are exponential service and the arrival process is Poisson. We must notice, however, that this Poisson Process models arrival events, but the events consist of two customer arrivals. (The departure events are still one-by-one, though.)

As always, for a Markovian System we must guarantee that all transitions are exponential. We define the usual state space: S_k : k customers in the

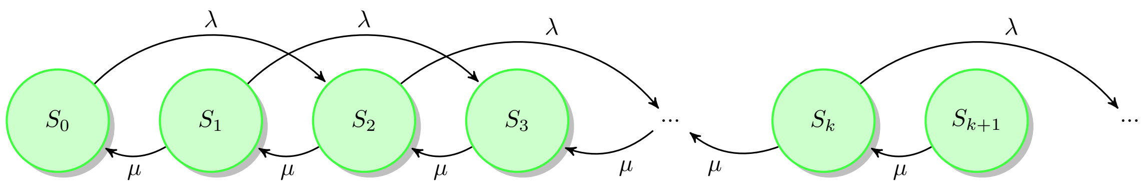

Figure 5: System diagram for the M/M/1 chain of exercise 5.5

system. Then, the state diagram is straightforward. Special care must be taken on determining the transitions and rates from state to state.

$$\textbf{Departure rate} = \mu$$
$$\textbf{Arrival Event rate} = \lambda$$

Clearly, the average customer arrival rate is 2λ and is NOT Poisson! What IS Poisson is the group arrival rate. We also DEFINE $\rho = \frac{\lambda}{\mu}$. This is neither the offered nor the actual load. We just use ρ to define this fraction.
The system diagram is given in Fig. 5.
Local Balance Equations:

$$\lambda P_0 = \mu P_1$$
$$\lambda P_{k-2} + \lambda P_{k-1} = \mu P_k, \quad k \geq 2$$

We can go ahead and solve them numerically. Alternatively, we can use the ZT methodology, since we only want to compute the average number of customers.

We consider the parametric local balance equation:

$$\mu P_k = \lambda P_{k-1} + \lambda P_{k-2} \rightarrow$$

$$\rightarrow \sum_{k=2}^{\infty} z^k \mu P_k = \sum_{k=2}^{\infty} z^k (\lambda P_{k-1} + \lambda P_{k-2})$$

$$\rightarrow \mu(P(z) - zP_1 - P_0) = \sum_{k=2}^{\infty} \lambda z^k P_{k-1} + \sum_{k=2}^{\infty} \lambda z^k P_{k-2}$$

$$\rightarrow \mu(P(z) - zP_1 - P_0) = \lambda z \sum_{k=2}^{\infty} \lambda z^{k-1} P_{k-1} + \lambda z^2 \sum_{k=2}^{\infty} \lambda z^{k-2} P_{k-2}$$

$$\rightarrow \mu(P(z) - zP_1 - P_0) = \lambda z(P(z) - P_0) + \lambda z^2 P(z)$$

We solve the equation with respect to $P(z)$

$$P(z) = \frac{\mu P_0 + \mu z P_1 - \lambda z P_0}{\mu - \lambda z - \lambda z^2} = \frac{P_0 + zP_1 - \rho z P_0}{1 - \rho z - \rho z^2}. \tag{6}$$

We need to apply two conditions that HOLD, in order to determine the unknown terms above. The first condition comes from the balance equation that we did not consider. We replace $P_1 = \rho P_0$ in (6), and obtain:

$$P(z) = \frac{P_0}{1 - \rho z - \rho z^2}. \tag{7}$$

The second condition comes from the NORMALIZATION in the probability or in the Z-domain:

$$\sum_{k=0}^{\infty} P_k = 1, \quad \text{or,} \quad P(z=1) = 1.$$

Replacing that in (7) we obtain $P_0 = 1 - 2\rho$, so finally

$$P(z) = \frac{1-2\rho}{1-\rho z - \rho z^2} \tag{8}$$

Finally, we need to compute the mean number of customers. We have

$$\overline{N} = \left[\frac{dP(z)}{dz}\right]_{z=1}.$$

Proof:

$$\left[\frac{dP(z)}{dz}\right]_{z=1} = \left[\frac{d\sum_{k=0}^{\infty} z^k P_k}{dz}\right]_{z=1} = \left[\sum_{k=0}^{\infty} k z^{k-1} P_k\right]_{z=1} = \sum_{k=0}^{\infty} k P_k = \overline{N}.$$

So, this is what we will do. We differentiate the derived ZT in (8):

$$\frac{dP(z)}{dz} = \frac{(-1)(1-2\rho)(-\rho - 2\rho z)}{(1-\rho z - \rho z^2)^2}$$

Replacing $z = 1$ we obtain

$$\overline{N} = \frac{3\rho}{1-2\rho} = \frac{3\lambda}{\mu - 2\lambda}.$$

The typical M/M/1 system with the same average customer arrival rate (2λ) and service rate (μ) has $\overline{N}_{M/M/1} = \frac{\rho}{1-\rho}$, where ρ is its offered load, and is equal to $\rho = 2\lambda/\mu$. So, finally,

$$\overline{N}_{M/M/1} = \frac{2\lambda}{\mu - 2\lambda}$$

so it is different, and, actually, less. Why?

7 Exercise 5.6

A queuing system has one server and infinite queuing capacity. The number of customers in the system can be modeled as a birth-death process with $\lambda_k = \lambda$ and $\mu_k = k\mu$, $k = 0, 1, 2, \ldots$ thus, the server increases the speed of the service with the number of customers in the queue. Calculate the average number of customers in the system as a function of $\rho = \lambda/\mu$.

Solution: The system is an M/M/1 queue, since it has infinite buffer, 1 server, and Markovian arrival and departure process. However, as we can see, it is not a typical M/M/1 case, as the service rates depend on the current system state. The system diagram is shown in Fig. 6. We need to solve the system of balance equations:

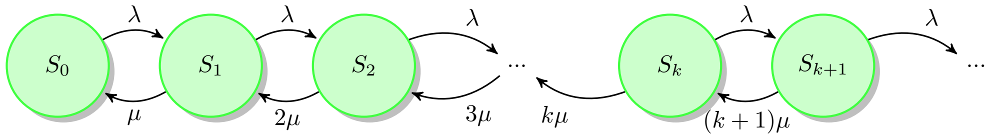

Figure 6: System diagram for the M/M/1 chain of exercise 5.6

$\lambda P_0 = \mu P_1 \rightarrow P_1 = \rho P_0$
$\lambda P_1 = 2\mu P_2 \rightarrow P_2 = \frac{1}{2}\rho P_1 = \frac{1}{2}\rho^2 P_0$
$\lambda P_2 = 3\mu P_3 \rightarrow P_3 = \frac{1}{3}\rho P_2 = \frac{1}{2\cdot 3}\rho^3 P_0$
....
$\lambda P_{k-1} = k\mu P_k \rightarrow P_k = \frac{1}{k}\rho P_{k-1} = ... = \frac{1}{k!}\rho^k P_0$
....
$\sum_{k=0}^{\infty} P_k = 1$ (normalization)

From the last general equation and the normalization equation we obtain the state distribution:

$$\sum_{k=0}^{\infty} \frac{\rho^k}{k!} P_0 = 1 \rightarrow P_0 e^{\rho} = 1 \rightarrow P_0 = e^{-\rho}.$$

so finally, for each k

$$P_k = \frac{\rho^k}{k!} e^{-\rho}$$

so the state distribution is POISSON! Then, we can calculate the average number of customers from the state distribution

$$\overline{N} = \sum_{k=0}^{\infty} k P_k = \rho$$

or simply say that the average is ρ, from the Poisson distribution.
From LITTLE we can, also, calculate the average system time

$$E[T_{total}] = \frac{\overline{N}}{\lambda} = \frac{1}{\mu}.$$

This means that the arriving customers only stay in the system for an average time equal to the service time![6]

8 Exercise 5.7

Customers arrive to a single server system in groups of 1,2,3 and 4 customers. The number of customers per group is i.i.d. There are in total 4 places in the

[6]This is equivalent to the case where there is no queue and each customer is served in parallel with the others, so actually this system is equivalent to an M/M/∞ system!

system. If a group of customers does not fit into the system, none of the members of the group joins the queue. 10% of the customers arrive in a group of 1, 20% of the customers arrive in groups of 2, 30% in a group of 3 and 40% in a group of 4 customers. The average number of arriving customers is 75 customers per hour, the interarrival time between groups is exponentially distributed. The service time is exponentially distributed with a mean of 0.5 minutes.

1. *Give the Kendall notation of the system and draw the state transition diagram.*
2. *Calculate the average number of customers in the queue and the mean waiting time per customer.*
3. *Calculate the probability that the system is full and the probability that a customer arriving in a group of k customers can not join the queue.*
4. *Calculate the probability that an arriving customer in general can not join the queue and the probability that an arriving group of customers can not join the queue.*
5. *What is the average waiting time for a customer arriving in a group of 3 customers?*

Solution: This is a very interesting problem that reveals the problems when the arriving process is complex and not straightforward so it must be derived.

First, we give the Kendall notation of the system. We have:

- Exponential GROUP inter-arrival times, so the arrival time will be Markovian.[7]
- The service times are exponentially distributed
- The system has a single server
- The total capacity is 4

Consequently, the Kendall notation is M/M/1/4.
We must draw the state transition diagram. We consider the typical state space where S_k means "k customers in the system". As a result the system has 5 states in total. The service rates are always the same, with

$$\mu = \frac{1}{E[T_s]} = \frac{1}{0.5/60} = 120h^{-1}.$$

The difficulty lies in deriving the arrival transition rates. We are given that the number of customers per group is i.i.d. We assume, naturally, that the GROUP arrival process is HOMOGENEOUS, that is, groups arrive in each state with the same rate! Also, since the inter-arrival times between GROUP arrivals are exponential we conclude that the GROUP arrivals is a Poisson process.

[7] It is our task to find an appropriate state space where the event arrival process is Poisson.

Figure 7: System diagram for the M/M/1 chain of exercise 5.6

We are given that the number of customers per group is an i.i.d. process, but we are NOT given the distribution. Let

$$q_1, \quad q_2, \quad q_3, \quad q_4$$

denote the probabilities that a random arriving group contains 1,2,3,4 customers respectively.

Let λ_G denote the Poisson **group** arrival rate. Then, the individual rates for each groups is ALSO a Poisson process, based on the *Poisson split* property, with rates

$$\lambda_G q_1, \quad \lambda_G q_2, \quad \lambda_G q_3, \quad \lambda_G q_4.$$

For any $i = 1, 2, 3, 4,$

$$\lambda_G \cdot q_i$$

defines the (average) rate of arrivals for group's of type i, and, consequently,

$$\lambda_G \cdot q_i \cdot i$$

defines the (average) rate of arrivals of customers belonging to group of type i.

Based on the given data from the exercise regarding the ratio of customers arriving in any of the groups, we obtain the following equations:

$$\lambda_G q_1 \cdot 1 = 10\% \cdot 75$$
$$\lambda_G q_2 \cdot 2 = 20\% \cdot 75$$
$$\lambda_G q_3 \cdot 3 = 30\% \cdot 75$$
$$\lambda_G q_4 \cdot 4 = 40\% \cdot 75$$

From the above it is clear that $q_1 = q_2 = q_3 = q_4 \rightarrow q_i = \frac{1}{4}, \quad \forall i = 1, 2, 3, 4.$ Finally, using any of the above equations we compute the group arrival rate:

$$\lambda_G = 30 \text{ groups / hour}$$

We can now complete the state diagram (Fig. 7). Then, we solve the LOCAL balance equations, to define the state probabilities.

We can compute BOTH the average number of customers in the system, and the average number of customers in the queue:

$$\overline{N}_{queue} = 1 \cdot P_2 + 2 \cdot P_3 + 3 \cdot P_4$$
$$\overline{N} = 1 \cdot P_1 + 2 \cdot P_2 + 3 \cdot P_3 + 4 \cdot P_4$$

For the average waiting time, we could apply the LITTLE result. For that we need the effective customer arrival rate, which is different from the nominal, since there are losses in the system.

It is important to notice again that the system is HOMOGENEOUS in group arrivals but not in customer arrivals.

We have

$$\lambda_{eff} = P_0\lambda_G\cdot(q_1+2q_2+3q_3+4q_4)+P_1\lambda_G\cdot(q_1+2q_2+3q_3)+P_2\lambda_G\cdot(q_1+2q_2)+P_3\lambda_G\cdot(q_1).$$

Then, from LITTLE we compute first the system time, $\overline{N} = \lambda_{eff}E[T]$, and then the average waiting time will be $\overline{W} = E[T] - E[T_s]$.

The probability that the system is full (as seen by an independent observer) is simply P_4.

The probability that a customer of a group k does not join the queue is, actually, the probability that the whole particular group does not join the queue. Since the system is homogeneous in group arrivals (a random group SEES state distribution),

$$\begin{aligned}
\Pr(\text{a random group 1 is blocked}) &= P_4\\
\Pr(\text{a random group 2 is blocked}) &= P_4 + P_3\\
\Pr(\text{a random group 3 is blocked}) &= P_4 + P_3 + P_2\\
\Pr(\text{a random group 4 is blocked}) &= P_4 + P_3 + P_2 + P_1
\end{aligned}$$

An arriving customer in general, belongs to groups 1,2,3,4 with probabilities 10%, 20%, 30% and 40%. So given these probabilities, he follows the group blocking probabilities:

$$\Pr(\text{a random customer is blocked}) =$$
$$= 40\% \cdot (P_1 + P_2 + P_3 + P_4) + 30\% \cdot (P_2 + P_3 + P_4) + 20\%2 \cdot (P_3 + P_4) + 10\% \cdot P_4$$

An arriving group of customers is blocked with probability

$\Pr(\text{a random group is blocked}) =$
$= \sum_{i=1}^{4} \Pr\{\text{a random group has } i \text{ customers}\} \cdot \Pr\{\text{a random group } i \text{ is blocked}\} =$
$= P_4 + P_3 \cdot 3/4 + P_2 \cdot 1/2 + P_1 \cdot 1/4.$

A customer that arrives in group of 3 customers MEANS that the arrived group sees either state 0 or state 1, otherwise there is no mean of WAITING time, since the group is rejected. The arrivals are homogeneous in groups, so the groups see state 0, 1 with probabilities P_0, P_1, respectively. So

$$\overline{W}_3 = \frac{P_0}{P_0 + P_1} \cdot W_3^0 + \frac{P_1}{P_0 + P_1} \cdot W_3^1$$

where the two waiting times are

$$W_3^0 = \frac{1}{3}(0.5 + 1 + 0), \qquad W_3^1 = \frac{1}{3}(0.5 + 1 + 1.5)$$