Queuing Theory 2014 - Exercises

Ioannis Glaropoulos

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1 Probability Theory and Transforms

1.1 Exercise 1.2

X is a random variable chosen from X_1 with probability a and from X_2 with probability b. Calculate $E[X]$ and σ_X for $\alpha = 0.2$ and $b = 0.8$. X_1 is an exponentially distributed r.v. with parameter $\lambda_1 = 0.1$ and X_2 is an exponentially distributed r.v. with parameter $\lambda_2 = 0.02$. Let the r.v. Y be chosen from D_1 with probability α and from D_2 with probability b, where D_1 and D_2 are deterministic r.v.s. Calculate the values D_1 and D_2 so that $E[X] = E[Y]$ and $\sigma_X = \sigma_Y$.
Solution: a) We directly apply the **conditional expectation** formula:

$$E[X] = \alpha E[X_1] + bE[X_2].$$

We can do this since the expectation is a raw moment – not central. The proof is straightforward: we have

$$\begin{aligned} &f_X(x) = \alpha f_{X_1}(x) + b f_{X_2}(x) \rightarrow \\ &\rightarrow E[X] = \textstyle\int_0^\infty x f_X(x)dx = \alpha \int_0^\infty x f_{X_1}(x)dx + b \int_0^\infty x f_{X_2}(x)dx = \\ &\quad = \alpha E[X_1] + bE[X_2]. \end{aligned}$$

We then replace the given data

$$E[X] = \alpha \frac{1}{\lambda_1} + b\frac{1}{\lambda_2} = 0.2\frac{1}{0.1} + 0.8\frac{1}{0.02} = 42. \tag{1}$$

We can not calculate the variance (or the standard deviation) in the same way, since this is a central moment. Instead, we proceed with calculating the expected square of the r.v. X, which is a **raw** moment:

$$\begin{aligned} &E[X^2] = \textstyle\int_0^\infty x^2 f_X(x)dx = \alpha \int_0^\infty x^2 f_{X_1}(x)dx + b \int_0^\infty x^2 f_{X_2}(x)dx = \\ &\quad = \alpha E[X_1^2] + bE[X_2^2]. \end{aligned}$$

Replacing the data we get

$$E[X] = \alpha \frac{2}{\lambda_1^2} + b\frac{2}{\lambda_2^2} = 0.2\frac{2}{0.1^2} + 0.8\frac{2}{0.02^2} = 4040. \tag{2}$$

Finally, we use the relation between the expectation, square mean and variance

$$\sigma_X^2 = E[X^2] - [E[X]]^2 = 4040 - 42^2 \rightarrow \sigma_X = 47.70. \tag{3}$$

b) We have $E[Y] = \alpha d_1 + bd_2$ and $E[Y^2] = \alpha d_1^2 + bd_2^2$. So the system of equations becomes

$$\begin{aligned} &0.2d_1 + 0.8d_2 = 42, \\ &0.2d_1^2 + 0.8d_2^2 = 4040 \end{aligned} \tag{4}$$

Solving this 2 by 2 non-linear system we obtain the solution. Notice that because of the second order of the equation we may **in general** have more than one solutions.

1.2 Exercise 1.3

X is a discrete stochastic variable, $p_k = P(X = k) = \frac{a^k}{k!}e^{-a}, k = 0, 1, 2, ...$ and a is a positive constant.

a) Prove that $\sum_{k=0}^{\infty} p_k = 1$.

b) Determine the z-transform (generating function) $P(z) = \sum_{k=0}^{\infty} z^k p_k$.

c) Calculate $E[X], \mathrm{Var}[X]$ and $E[X(X-1)...(X-r+1)], r = 1, 2, ...$ with and without using z-transforms.

Solution a) We have

$$\sum_{k=0}^{\infty} p_k = \sum_{k=0}^{\infty} \frac{a^k}{k!}e^{-a} = e^{-a}\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^{-a}e^{a} = 1.$$

Notice this useful and well-known infinite series summation.

b) We replace the definition of the mass function and gradually have:

$$P(z) = \sum_{k=0}^{\infty} z^k \frac{a^k}{k!}e^{-a} = e^{-a}\sum_{k=0}^{\infty} z^k \frac{a^k}{k!} = e^{-a}\sum_{k=0}^{\infty} \frac{(za)^k}{k!} = e^{-a}e^{az} = e^{-a(1-z)}.$$

c) First, we try without the z-transform, i.e. using the definitions in the probability domain. We start from the third sentence, using the **definition** of expectation:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx \tag{5}$$

$$\begin{aligned} E[X(X-1)...(X-r+1)] &= \textstyle\sum_{k=0}^{\infty} k(k-1)...(k-r+1)p_k = \\ &= \textstyle\sum_{k=0}^{\infty} k(k-1)...(k-r+1)\frac{a^k}{k!}e^{-a} = \sum_{k=0}^{\infty} \frac{a^k}{(k-r)!}e^{-a} = \\ &= e^{-a}a^r \textstyle\sum_{k=0}^{\infty} \frac{a^{(k-r)}}{(k-r)!} = a^r e^{-a} e^{a} = a^r. \end{aligned}$$

Then clearly, we have (by setting $r = 1$) $E[X] = a^1 = a$. And, finally,

$$\mathrm{Var}[X] = E[X^2] - [E[X]]^2 = E[X^2] - a^2 = E[X(X-1)] + E[X] - a^2 = a^2 + a - a^2.$$

We try, now, with the z-transform. We differentiate r times the definition of the z-transform:

$$\frac{d^r}{dz^r}P(z) = \frac{d^r}{dz^r}\sum_{k=0}^{\infty} z^k p_k = \sum_{k=0}^{\infty} k(k-1)...(k-r+1)z^{k-r}p_k$$

If we replace $z = 1$ we get

$$\left.\frac{d^r}{dz^r}P(z)\right\}_{z=1} = E[X(X-1)...(X-r+1)].$$

We, then, calculate,

$$\left.\frac{d^r}{dz^r}P(z)\right\}_{z=1} = a^r e^{-a(1-1)} = a^r.$$

1.3 Exercise 1.4

X_i's are independent Poisson distributed random variables, thus, $p_k = \frac{a_i^k}{k!}e^{-a_i}$, $k = 0, 1, 2, ...$, and each $a_i, i = 1, 2, ..., n$ is a positive constant. Give the probability distribution function of $X = \sum_{i=1}^{n}$.

Solution: This problem indicates the usefulness of the z-transform in the calculation of the distribution of the sum of variables. We have proven that **the ZT of the sum of independent random variables is the product of their individual z-transforms**. Thus,

$$P(z) = \prod_{i=1}^{n} P_i(z) = \prod_{i=1}^{n} e^{-a_i(1-z)} = e^{\sum_{i=1}^{n} -a_i(1-z)} = e^{-\alpha(1-z)},$$

where $\alpha = \sum_{i=1}^{n} -a_i$. This proves that the distribution is also Poisson with parameter α, i.e. the sum of parameters. The proof is based on the uniqueness of z-transform[1]. As a result, the distribution function will be

$$p_X(k) = \frac{\alpha^k}{k!}e^{-\alpha}$$

1.4 Exercise 1.5

X is a positive stochastic continuous variable with probability distribution function (PDF)

$$F(x) = P(X \leq x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-ax}, x \geq 0. \end{cases}$$

a) Give the probability density function $f(x) = dF(x)/dx$.

b) Give $\overline{F}(x) = P(X > x)$.

c) Calculate the Laplace Transform $f^*(s) = E[e^{-sX}] = \int_0^\infty e^{-sx} f(x)dx$.

d) Calculate the expected values $m = E[X], E[X^k]$, $k = 0, 1, 2, ...$, the variance σ_X^2, the standard deviation σ_X and the coefficient of variation $c = \sigma/m$, with and without the transform $F^*(s)$.

Solution: a) For the calculation of $f(x)$ we just need to differentiate:

$$f(x) = dF(x)/dx = d(1 - e^{-ax})/dx = ae^{-ax}.$$

b) The complementary PDF is simply given as

$$\overline{F}_X(x) = P(X > x) = 1 - P(X \leq x) = 1 - F_X(x) = e^{-ax}.$$

c) Calculation of the Laplace Transform with simple integration

$$f^*(s) = \int_0^\infty e^{-sx} f(x)dx = \int_0^\infty e^{-sx} ae^{-ax}dx = a\int_0^\infty e^{-x(s+a)}dx = \frac{a}{s+a}.$$

d) We proceed first, without the help of Laplace transforms, using the definition of the expectation

$$E[X^0] = \int_0^\infty x^0 f(x)dx = \int_0^\infty f(x)dx = 1.$$

[1]or the 1-1 correspondence between the mass function and the ZT

$$E[X^k] = \int_0^\infty x^k f(x)dx = \int_0^\infty x^k a e^{-ax}dx = a\frac{-1}{a}\int_0^\infty x^k (e^{-ax})'dx =$$
$$= k\int_0^\infty x^{k-1}e^{-ax}dx = \frac{k}{a}\int_0^\infty x^{k-1}ae^{-ax}dx = \int_0^\infty x^{k-1}f(x)dx =$$
$$= \frac{k}{a}E[X^{k-1}].$$

This is a recursive formula that enables the calculation of any moment. We have:

$$E[X^k] = \frac{k}{a}E[X^{k-1}] = \frac{k}{a}\frac{k-1}{a}E[X^{k-2}] = \frac{k}{a}\frac{k-1}{a}..\frac{1}{a}E[X^0] = \frac{k}{a}\frac{k-1}{a}..\frac{1}{a} = \frac{k!}{a^k}$$

which gives, simply, $E[X] = 1/a$, for $k = 1$. The variance is calculated through the usual formula, and the raw moments are taken from above:

$$\sigma^2 = E[X^2] - [E[X]]^2 = \frac{2}{a^2} - \left(\frac{1}{a}\right)^2 = 1/a^2.$$

so the standard deviation is simply the square root of the variance, $1/a$, and the coefficient of variation is 1. Notice that this is special for the exponential distribution.

We try, now, with the help of the Laplace transforms.

$$E[X^k] = (-1)^k \frac{d^k}{ds^k} f^*(s) = (-1)^k \frac{d^k}{ds^k}\frac{a}{s+a} = \frac{(-1)^k ak!}{(s+a)^{k+1}}.$$

We find this formula by differentiating k times the Laplace transform and replacing $s = 0$. The rest follows with simple replacement $k = 1, 2, ...$

1.5 Exercise 1.6

X_i's are independent, exponentially distributed random variables with a mean value of $1/a, a > 0, i = 1, 2, ..., n$. Calculate $P(X \le x)$ and $P(X \ge x)$ where

a) $X = \min(X_1, X_2, ..., X_n)$,

b) $X = \max(X_1, X_2, ..., X_n)$.

Solution: a) The key point in this exercise is the fact that the random variables are independent (mutually independent). We gradually have:

$$P(X \le x) = P(\min(X_1, X_2, ..., X_n) \le x) = 1 - P(\min(X_1, X_2, ..., X_n) > x)$$
$$= 1 - P(X_1 > x, X_2 > x, ..., X_n > x) = 1 - \prod_{i=1}^n P(X_i > x)$$
$$= 1 - \prod_{i=1}^n e^{-ax} = 1 - e^{-\sum_{i=1}^n ax} = 1 - e^{-nax}$$

This shows that the minimum of exponentially distributed random variables is also an exponential variable and its rate is the sum of the individual rates.

b) Similar calculations:

$$P(X \le x) = P(\max(X_1, X_2, ..., X_n) \le x) = P(X_1 \le x, X_2 \le x, ..., X_n \le x)$$
$$= \prod_{i=1}^n P(X_i \le x) = \prod_{i=1}^n (1 - e^{-ax}) = (1 - e^{-ax})^n.$$

Cleary, the variable X is, now, not exponential.

2 Balance equations, birth-death processes, continuous Markov Chains

2.1 Exercise 3.2

Consider a birth-death process with 3 states, where the transition rate from state 2 to state 1 is $q_{21} = \mu$ and $q_{23} = \lambda$. Show that the mean time spent in state 2 is exponentially distributed with mean $1/(\lambda + \mu)$.[2]

Solution: Suppose that the system has just arrived at state 2. The time until next "birth" – denoted here as T_B – is exponentially distributed with cumulative distribution function $F_{T_B}(t) = 1 - e^{-\lambda t}$. Similarly, the time until next "death" – denoted here as T_D – is exponentially distributed with cumulative distribution function $F_{T_D}(t) = 1 - e^{-\mu t}$. The random variables T_B and T_D are independent.

Denote by T_2 the time the system spends in state 2. The system will depart from state 2 when the first of the two events (birth or death) occurs. Consequently we have $T_2 = \min\{T_B, T_D\}$. We, then, apply the result from exercise 1.6, that is the minimum of independent exponential random variables is an exponential random variable. We can actually show this:

$$\begin{aligned}
F_{T_2}(t) &= \Pr\{T_2 \le t\} = \\
&= \Pr\{\min\{T_B, T_D\} \le t\} = \\
&= 1 - \Pr\{\min\{T_B, T_D\} > t\} = \\
&= 1 - \Pr\{T_B > t, T_D > t\} = \\
&= 1 - \Pr\{T_B > t\} \cdot \Pr\{T_D > t\} = \\
&= 1 - e^{-\lambda t} \cdot e^{-\mu t} = \\
&= 1 - e^{-(\lambda+\mu)t}
\end{aligned}$$

so T_2 is exponentially distributed with parameter $\lambda + \mu$.

Notice that we can generalize to the case with more than two transition branches. This exercise reveals the property of continuous time Markov chains, that is, the time spent on a state is exponentially distributed.

2.2 Exercise 3.3

Assume that the number of call arrivals between two locations has Poisson distribution with intensity λ. Also, assume that the holding times of the conversations are exponentially distributed with a mean of $1/\mu$. Calculate the average number of call arrivals for a period of a conversation.

Solution: Denote by N_C the number of arriving calls during the period of one conversation. Denote by T the duration of this conversation. Given that $T = t$, $N_C | T = t$ is Poisson distributed with parameter $\lambda \cdot t$ so the probability mass function of the number of calls will be

$$\Pr\{\text{arriving calls within } t = k\} = P_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

[2] This exercise is similar to Exercise 6 from Chapter 1: "The minimum of independent exponential variables is exponential."

with an average number of calls: $E[N_C|T = t] = \lambda t$.

Moreover T is exponentially distributed, with parameter μ so the density function will be:

$$f_T(t) = \mu e^{-\mu t}.$$

We apply the conditional expectation formula:

$$E[N_C] = \int_0^\infty E[N_C|T = t] \cdot f_T(t)dt = \int_0^\infty \lambda t \mu e^{-\mu t} dt = \lambda \int_0^\infty t \mu e^{-\mu t} dt = \frac{\lambda}{\mu}.$$

2.3 Exercise 3.4

Consider a communication link with a constant rate of $4.8kbit/sec$. *Over the link we transmit two types of messages, both of exponentially distributed size. Messages arrive in a Poisson fashion with* $\lambda = 10$ *messages/second. With probability 0.5 (independent from previous arrivals) the arriving message is of type 1 and has a mean length of 300 bits. Otherwise a message of type 2 arrives with a mean length of 150 bits. The buffer at the link can at most hold one message of type 1 or two messages of type 2. A message being transmitted still takes a place in the buffer.*

a) Determine the mean and the coefficient of variation of the service time of a randomly chosen arriving message.

b) Determine the average times in the system for accepted messages of type 1 and 2.

c) Determine the message loss probabilities for messages of type 1 and 2.

Solution:

a) We have a link with a constant transmission rate. So the service time distributions follow the packet length distributions. Consequently, the service times of both packet types are exponential with mean values of

- Type 1: $E[T_1] = \frac{300}{4800} = \frac{1}{16}$ sec,
- Type 2: $E[T_2] = \frac{150}{4800} = \frac{1}{32}$ sec.

As a result the parameters of the exponential distributions are $\mu_1 = 16$ and $\mu_2 = 32$, respectively. A random arriving packet is of Type 1 or Type 2 with probability 0.5. We apply the conditional expectation [3]:

$$E[T] = \frac{1}{2}E[T_1] + \frac{1}{2}E[T_2] = \frac{3}{64}$$

Similarly, we calculate the mean square:

$$E[T^2] = \frac{1}{2}E[T_1^2] + \frac{1}{2}E[T_2^2] = \frac{1}{2}\frac{2}{\mu_1^2} + \frac{1}{2}\frac{2}{\mu_2^2} = 16^{-2} + 32^{-2} = \frac{5}{4} \cdot 16^{-2}.$$

The variance of T is derived from $Var[T] = E[T^2] - (E[T])^2$. Then we compute the standard deviation σ_T as $\sigma_T = \sqrt{Var[T]}$, and finally the coefficient of variation is given as: $c_T = \frac{\sigma_T}{E[T]}$.

[3] This is similar to exercise 1.2

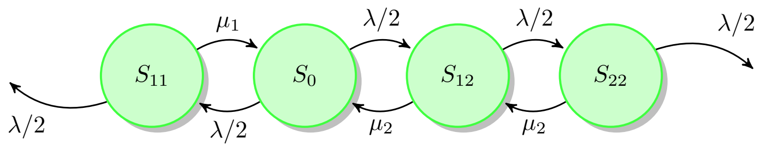

Figure 1: State Diagram for Exercise 3.4

b) For this part of the exercise, we need to draw the Markov Chain (Fig. 2) and solve it in the **steady state**. The state space must be defined in such a way that we can guarantee that all transitions – from state to state – have an exponential rate. We choose here to define such a Markov chain with 4 states:
State 0; Empty buffer.
State 11; 1 packet of Type 1.
State 21; 1 packet of Type 2.
State 22; 2 packets of Type 2.
Then we solve the balance equations in the local form:

$$\mu_2 P_{22} = \lambda/2 P_{21}$$
$$\mu_2 P_{21} = \lambda/2 P_0$$
$$\mu_1 P_{11} = \lambda/2 P_0$$
$$P_0 + P_{11} + P_{21} + P_{22} = 1 (\text{norm. equation})$$

Solution:

$$P_0 = 0.670, P_{21} = 0.105, P_{22} = 0.016, P_{11} = 0.209.$$

An accepted message of Type 1 can only arrive at state 0, otherwise it is rejected. So its the average service time will be $E[T_1]$.
An accepted message of Type 2 can arrive at states 0 and 21, otherwise it is rejected. Then, the average service time will be $(E[T_2]P_0 + 2E[T_2]P_{(21)})/(P_0 + P_{21})$. [4]

c) The loss probabilities are equal to the probabilities of the system being in BLOCKING states, for each of the two packet types. We underline that this is always true for *homogeneous* Markov chains, that is, Markov chains where the arrival rates do not depend on the system state.

2.4 Exercise 3.5

Consider a Markovian system with discouraged job arrivals. Jobs arrive to a server in a Poisson fashion, with an intensity of one job per 7 seconds. The jobs observe the queue. They do NOT join the queue with probability l_k if they observe k jobs in the queue. $l_k = k/4$ if $k < 4$, or 0, otherwise. The service time is exponentially distributed with mean time of 6 seconds.

a) Determine the mean number of customers in the system, and

b) the number of jobs served in 100 seconds.

Solution:

[4] The occurrence of acceptance reduces the sample space to two states only. Then the probabilities are normalized.

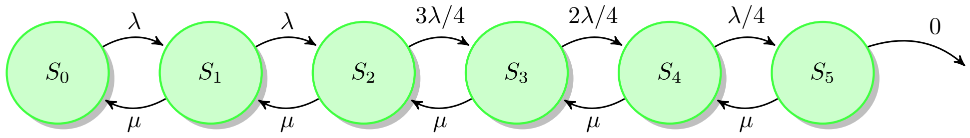

Figure 2: State diagram for Exercise 3.5

a) This is a simple model but requires careful design. After building the correct state diagram, the solution is found, based on the LOCAL balance equations.

We have a system with 6 states. *State space*: $S_k : k$ jobs in the system. The system diagram is shown in Fig. 3).

Balance Equation System:

$$\lambda P_0 = \mu P_1$$
$$\lambda P_1 = \mu P_2$$
$$3\lambda/4 P_2 = \mu P_3$$
$$\lambda/2 P_2 = \mu P_4$$
$$\lambda/4 P_2 = \mu P_5$$
$$\sum_{k=1}^{5} P_k = 1$$

Solution: $P_0 \approx 0.3$, and the remaining probabilities are computed based on P_0 and the equations above. After determining the state probabilities, we derive the average number of customers in the system through

$$E[N] = \sum_{k=0}^{5} k \cdot P_k$$

We find $E[N] \approx 1.43$.

b) We have, here, a system with different arrival rates in each state. These systems are defined as *non-homogeneous*. However, the service rate is constant. The server is busy with probability $(1 - P_0)$. When it is busy, it serves jobs. The service rate is $\mu = 1/6 sec^{-1}$. As a result, the server can serve $100 \cdot \mu \cdot (1 - P_0)$ jobs in 100 seconds on AVERAGE!

2.5 Exercise 3.6

Consider a network node that can serve 1 and store 2 packets altogether. Packets arrive to the node according to a Poisson process. Serving a packet involves two independent sequentially performed tasks: the ERROR CHECK and the packet TRANSMISSION to the output link. Each task requires an exponentially distributed time with an average of 30msec. Give, that we observe that the node is empty in 60% of the time, what is the average time spend in the node for one packet?

Solution: As always, we need to construct the state diagram is such a way

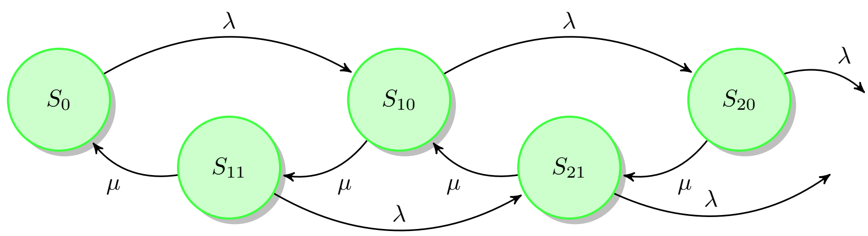

Figure 3: State Diagram for Exercise 3.6

that all transitions rates are guaranteed to be exponential. The selected state space:

S_0: Empty network node,
S_{11}: One packet under transmission,
S_{10}: One packet under error-check,
S_{20}: One packet under error-check and one buffered,
S_{21}: One packet under transmission and one buffered,

The state diagram is shown in Fig. 4. We can form the *global* balance equations parameterized by λ. Then we apply information that is given: $P_0 = 0.6$; This extra information enables the solution of the system of equations, and leads to the calculation of λ:

$$\lambda P_0 = \mu P_{11}$$
$$(\lambda + \mu) P_{11} = \mu P_{10}$$
$$(\lambda + \mu) P_{10} = \lambda P_0 + \mu P_{21}$$
$$\mu P_{21} = \lambda P_{11} + \mu P_{22}$$
$$P_{11} + P_{10} + P_{21} + P_{22} = 1 - P_0 = 0.4$$

Solution: $P_{10} \approx 0.1636, P_{11} \approx 0.1337, P_{20} \approx 0.0365, P_{21} \approx 0.0663, \lambda \approx 7.63$. For the calculation of the total average *system* time for a packet, we apply Little's formula.

$$\overline{N} = \lambda_{eff} \cdot E[T_{sys}] \rightarrow E[T_{sys}] = \frac{\overline{N}}{\lambda_{eff}} = \frac{1 \cdot (P_{10} + P_{11}) + 2 \cdot (P_{21} + P_{22})}{\lambda_{eff}}.$$

We always apply the effective arrival rate at Little's formula, because the formula needs the actual average arrival rate at the system, excluding possible drops. Here, the effective rate is not equal to λ, since we have packet drops. However, since the arrival rate for this system does not change with time, the effective arrival rate is simply:

$$\lambda_{eff} = \lambda \cdot (P_0 + P_{10} + P_{11}).$$