EP2200 Queuing theory and teletraffic systems

3rd lecture

Markov chains
Birth-death process
- Poisson process

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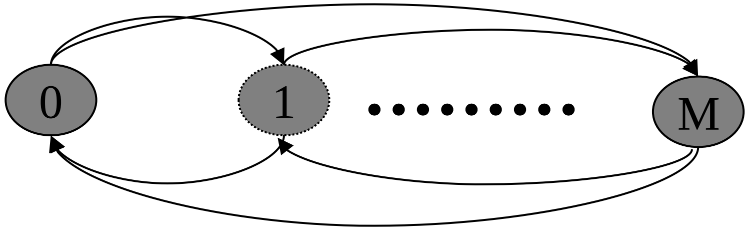
Outline for today

- Markov processes
  - Continuous-time Markov-chains
  - Graph and matrix representation
- Transient and steady state solutions
- Balance equations – local and global
- Pure Birth process – Poisson process as special case
- Birth-death process as special case

Markov processes

- Stochastic process
  - $p_i(t)=P(X(t)=i)$
- The process is a Markov process if *the future of the process depends on the current state only* - Markov property
  - $P(X(t_{n+1})=j \mid X(t_n)=i, X(t_{n-1})=l, \ldots, X(t_0)=m) = P(X(t_{n+1})=j \mid X(t_n)=i)$
  - Homogeneous Markov process: the probability of state change is unchanged by time shift, depends only on the time interval

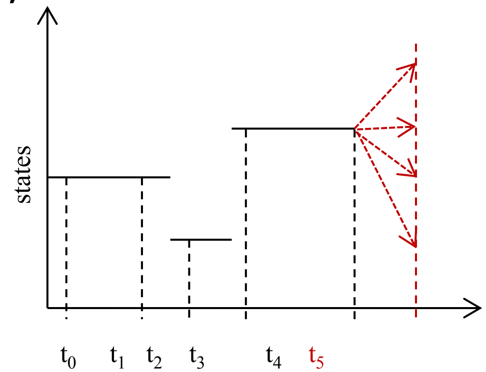
    $P(X(t_{n+1})=j \mid X(t_n)=i) = p_{ij}(t_{n+1}-t_n)$
- Markov chain: if the state space is discrete
  - A homogeneous Markov chain can be represented by a graph:
    - States: nodes
    - State changes: edges



Continuous-time Markov chains (homogeneous case)

- Continuous time, discrete space stochastic process, with Markov property, that is:

$$P(X(t_{n+1}) = j \mid X(t_n) = i, X(t_{n-1}) = l, \dots X(t_0) = m) =$$
$$P(X(t_{n+1}) = j \mid X(t_n) = i), \quad t_0 < t_1 < \dots < t_n < t_{n+1}$$



- State transition can happen in any point of time
- Example:
  - number of packets waiting at the output buffer of a router
  - number of customers waiting in a bank

- The time spent in a state has to be exponential to ensure Markov property:
  - the probability of moving from state *i* to state *j* sometime between t_n and t_{n+1} does not depend on the time the process already spent in state *i* before t_n.

Continuous-time Markov chains (homogeneous case)

- State change probability depends on the time interval:
  $P(X(t_{n+1})=j \mid X(t_n)=i) = p_{ij}(t_{n+1}-t_n)$

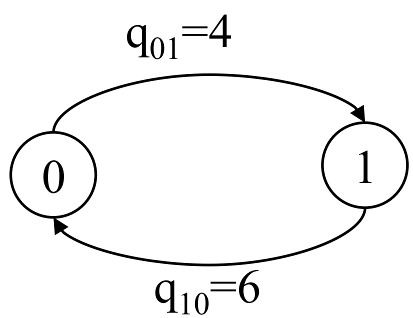
- Characterize the Markov chain with the state transition rates instead:

$$q_{ij} = \lim_{\Delta t \to 0} \frac{P(X(t+\Delta t)=j \mid X(t)=i)}{\Delta t}, \quad i \neq j$$ - rate (intensity) of state change

$$q_{ii} = -\sum_{j \neq i} q_{ij}$$ - defined to easy calculation later on

Transition rate matrix **Q**:

$$\mathbf{Q} = \begin{bmatrix} q_{00} & q_{01} & \cdots & q_{0M} \\ \vdots & \ddots & & \\ & & & q_{(M-1)M} \\ q_{M0} & \cdots & q_{M(M-1)} & q_{MM} \end{bmatrix}$$



$$\mathbf{Q} = \begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix}$$

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Transient solution

- The transient - time dependent – state probability distribution
- $\underline{p}(t)=\{p_0(t), p_1(t), p_2(t),...\}$ – probability of being in state *i* at time t, given $\underline{p}(0)$.

$$q_{ij} = \lim_{\Delta t \to 0} \frac{P(X(t+\Delta t)=j \mid X(t)=i)}{\Delta t} \quad \Rightarrow \quad P(X(t+\Delta t)=j \mid X(t)=i) = q_{ij}\Delta t + o(\Delta t)$$

$$p_i(t+\Delta t) = p_i(t) - \underbrace{p_i(t)\sum_{j\neq i} q_{ij}\Delta t}_{\text{leaves the state}} + \underbrace{\sum_{j\neq i} p_j(t) q_{ji}\Delta t}_{\text{arrives to the state}} + o(\Delta t), \quad \lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t} = 0$$

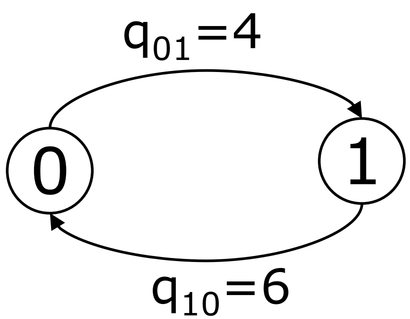
$$p_i(t+\Delta t) - p_i(t) = p_i(t) q_{ii}\Delta t + \sum_{j\neq i} p_j(t) q_{ji}\Delta t + o(\Delta t) = \sum_j p_j(t) q_{ji}\Delta t + o(\Delta t) \quad (-\sum_{j\neq i} q_{ij} = q_{ii})$$

$$\frac{p_i(t+\Delta t) - p_i(t)}{\Delta t} = \sum_j p_j(t) q_{ji} + \frac{o(\Delta t)}{\Delta t} \quad \Rightarrow \quad \frac{dp_i(t)}{dt} = \sum_j p_j(t) q_{ji}$$

$$\frac{d\underline{p}(t)}{dt} = \underline{p}(t)\mathbf{Q}, \quad \underline{p}(t) = \underline{p}(0)\cdot e^{\mathbf{Q}t}$$

Transient solution

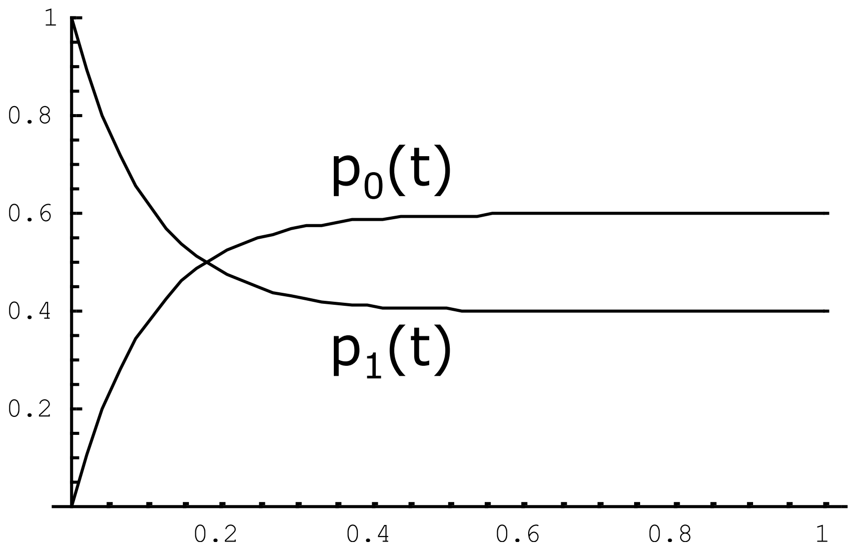
Example – transient solution



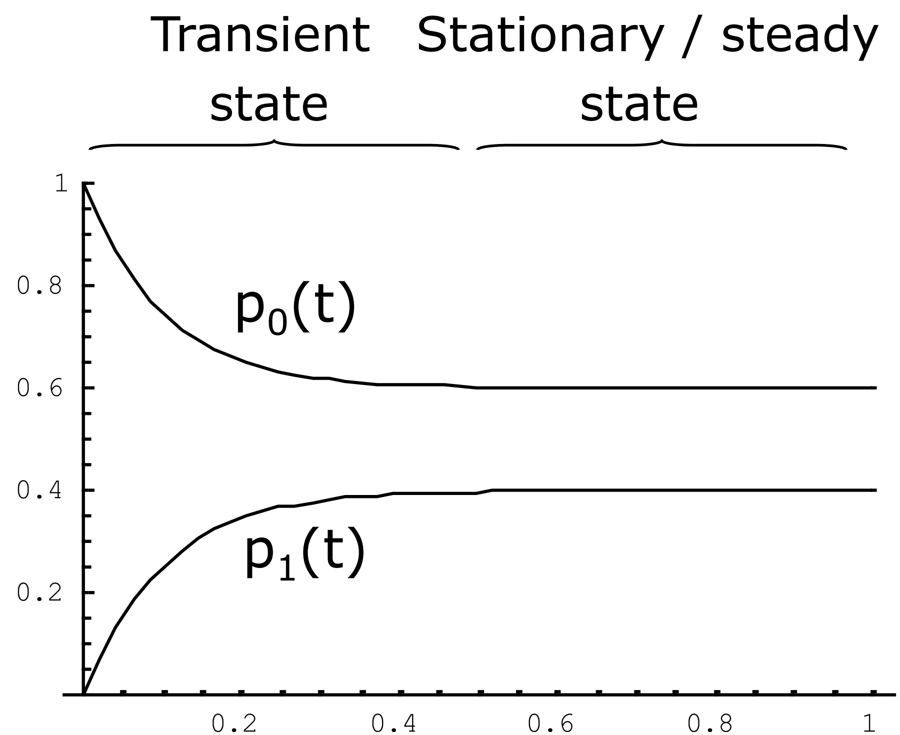
$$\mathbf{Q} = \begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix}$$

$$\underline{p}(t)\mathbf{Q} = \frac{d\underline{p}(t)}{dt}$$

$$\underline{p}(t) = \underline{p}(0) \cdot e^{\mathbf{Q}t}$$



A: $p_0(0)=0$, $p_1(0)=1$



B: $p_0(0)=1$, $p_1(0)=0$

Transient solution - comment

- Matrix exponential
- Matrix exponentials are defined as:

$$e^{\mathbf{X}} = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{X}^k$$

And are difficult to calculate.

- Therefore, for the small case on the 2-state MC I solved the original set of differential equations instead:

$$\frac{dp(t)}{dt} = p(t)\mathbf{Q}$$

$$p_0'(t) = p_0(t)q_{00} + p_1(t)q_{10}$$

$$p_1'(t) = p_0(t)q_{01} + p_1(t)q_{11}$$

$$p_0(t) + p_1(t) = 1$$

- To solve linear differential equations is still tough. See relevant course books
- Fortunately, math programs can do it for you (I used Mathematica)

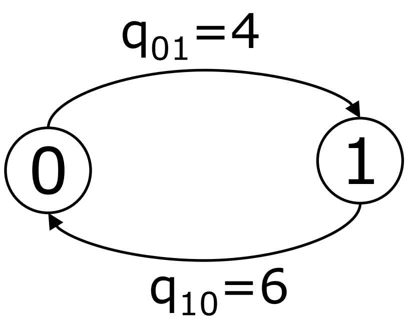
Stationary solution (steady state)

- Def: stationary state probability distribution (stationary solution)
  - $\underline{p} = \lim_{t\to\infty} \underline{p}(t)$ exists
  - $\underline{p}$ is independent from $\underline{p}(0)$
- The stationary solution $\underline{p}$ has to satisfy:

$$\underline{p}(t)\mathbf{Q} = \frac{d\underline{p}(t)}{dt} = 0, \quad \sum p_i(t) = 1$$

$$\mathbf{Q} = \begin{bmatrix} q_{00} & q_{01} & \cdots & q_{0M} \\ \vdots & \ddots & & \\ & & & q_{(M-1)M} \\ q_{M0} & \cdots & q_{M(M-1)} & q_{MM} \end{bmatrix}$$

Note: the rank of Q_{MM} is M-1!



$$[p_0, p_1]\begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix} = [0,0], \quad p_0 + p_1 = 1$$

$$p_0 = 0.6, \quad p_1 = 0.4$$

Stationary solution (steady state)

Important theorems – without the proof

- Stationary solution exists, if
  - The Markov chain is irreducible (there is a path between any two states) and
  - $p\mathbf{Q}=0, \quad p\times\mathbf{1}=1$ has positive solution

- Equivalently, stationary solution exists, if
  - The Markov chain is irreducible
  - For all states: the mean time to return to the state is finite

- Finite state, irreducible Markov chains always have stationary solution.

- Markov chains with stationary solution are also ergodic:
  - p_i gives the portion of time a single realization spends in state i, and
  - the probability that one out of many realizations are in state i at arbitrary point of time

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Balance equations

- How can we find the stationary solution? $\underline{p}\mathbf{Q}=\underline{0}$

$$0=\underline{p}\mathbf{Q} \quad \Rightarrow$$

State 1:

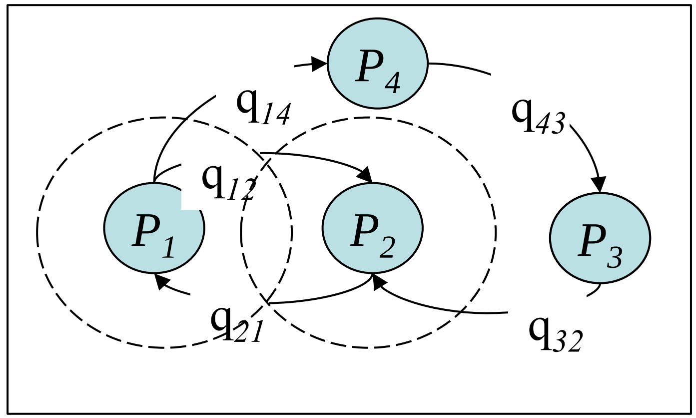
$$0=-(q_{12}+q_{14})p_1+q_{21}p_2$$

$$q_{21}p_2=(q_{12}+q_{14})p_1$$

State 2:

$$0=q_{12}p_1-q_{21}p_2+q_{32}p_3$$

$$\underbrace{q_{12}p_1+q_{32}p_3}_{\text{flow in}}=\underbrace{q_{21}p_2}_{\text{flow out}}$$



$$\mathbf{Q}=\begin{bmatrix}-(q_{12}+q_{14}) & q_{12} & 0 & q_{14}\\ q_{21} & -q_{21} & 0 & 0\\ 0 & q_{32} & -q_{32} & 0\\ 0 & 0 & q_{43} & -q_{43}\end{bmatrix}$$

- Global balance conditions
  - In equilibrium (for the stationary solution)
  - the transition rate out of a state – or a group of states - must equal the transition rate into the state (or states)
    - flow in = flow out
  - defines a global balance equation

Group work

- Global balance equation for state 1 and 2:

$$0 = \underline{p}\mathbf{Q} \quad \Rightarrow$$

State 1:

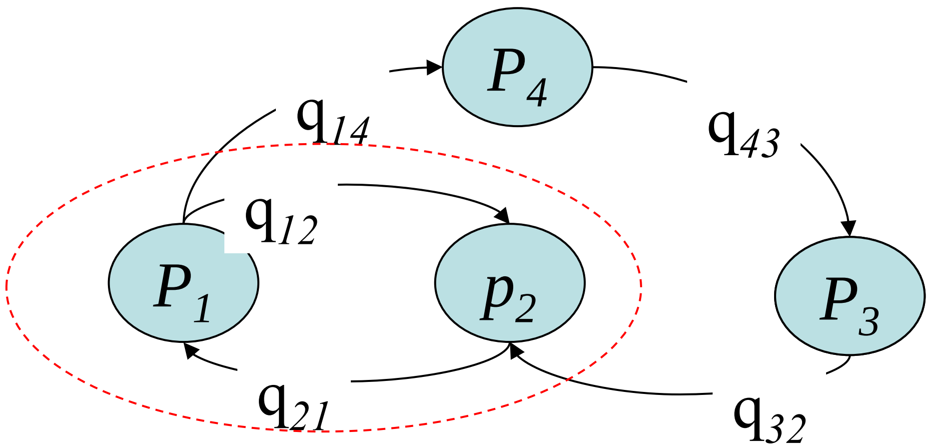
$$0 = -(q_{12} + q_{14})p_1 + q_{21}p_2$$

$$q_{21}p_2 = (q_{12} + q_{14})p_1$$

State 2:

$$0 = q_{12}p_1 - q_{21}p_2 + q_{32}p_3$$

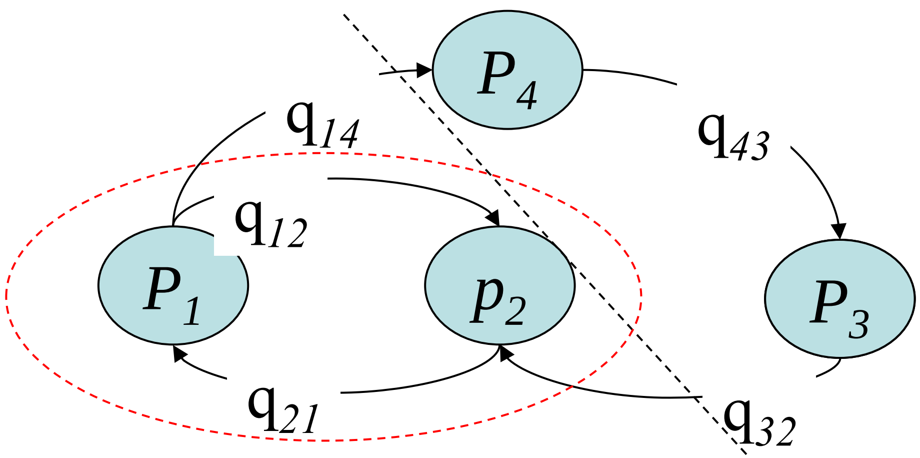
$$q_{12}p_1 + q_{32}p_3 = q_{21}p_2$$



- Is there a global balance equation for the circle around states 1 and 2?

Balance equations

- Local balance conditions in equilibrium
  - the local balance means that the total flow from one part of the chain must be equal to the flow back from the other part
  - for all possible cuts
  - defines a local balance equation
- The local balance equation is the same as a global balance equation around a set of states!



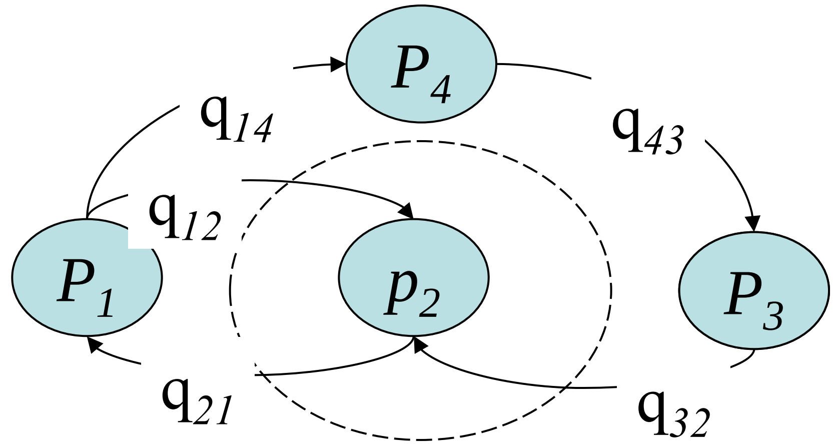
Balance equations

- Set of linear equations instead of a matrix equation

$$\mathbf{0} = \underline{p}Q \quad \Rightarrow$$

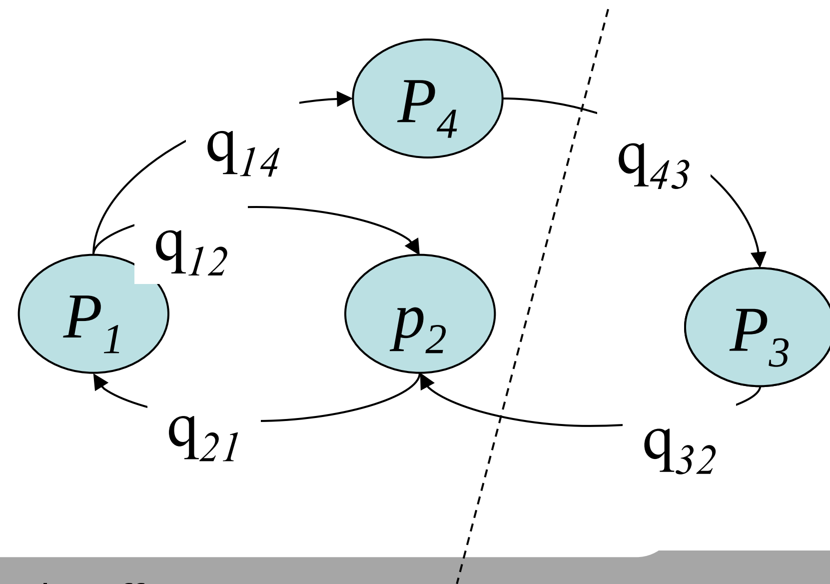
$$0 = q_{12}p_1 - q_{21}p_2 + q_{32}p_3$$

$$\underbrace{q_{12}p_1 + q_{32}p_3}_{\text{flow in}} = \underbrace{q_{21}p_2}_{\text{flow out}}$$



- Global balance :
  - flow in = flow out around a state
  - or around many states
- Local balance equation:
  - flow in = flow out across a cut

$$q_{43}p_4 = q_{32}p_3$$



- M states
  - M-1 independent equations
  - $\Sigma p_i = 1$

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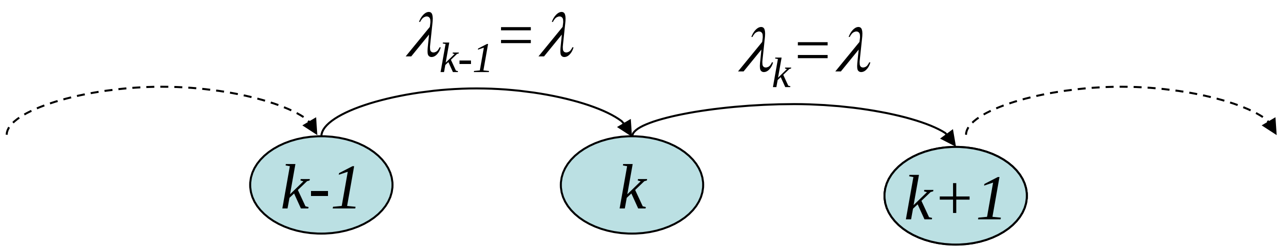
Pure birth process

- Continuous time Markov-chain, infinite state space
- Transitions occur only between neighboring states
  - State independent birth intensity: $\lambda_i = \lambda, \quad \forall i$

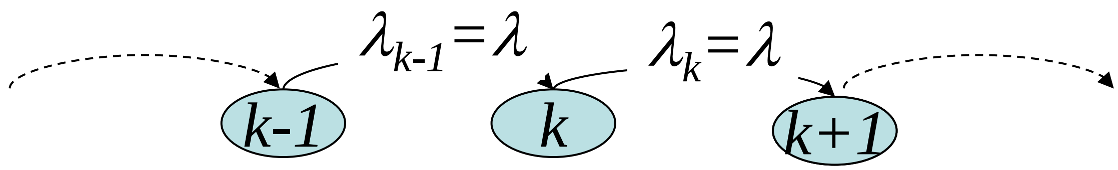


$$\mathbf{Q} = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots \\ 0 & -\lambda & \lambda & 0 \\ 0 & 0 & -\lambda & \lambda \\ 0 & 0 & 0 & \ddots \end{bmatrix}$$

- No stationary solution
- Transient solution:
  - $p_k(t)$=P(system in state k at time t)
  - number of events (births) in an interval t

Pure birth process

- Transient solution – number of events (births) in an interval (0,t]



$$\mathbf{Q} = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots \\ 0 & -\lambda & \lambda & 0 \\ 0 & 0 & -\lambda & \lambda \\ 0 & 0 & 0 & \ddots \end{bmatrix}$$

$$\underline{p}'(t) = \underline{p}(t)\mathbf{Q}, \quad p_0(0) = 1, \quad p_k(0) = 0 \quad \mathit{for} \quad k \neq 0$$

$$p'_0(t) = -\lambda p_0(t) \qquad \rightarrow \boxed{p_0(t) = e^{-\lambda t}}$$

$$p'_1(t) = \lambda p_0(t) - \lambda p_1(t) \qquad \rightarrow p'_1(t) = \lambda e^{-\lambda t} - \lambda p_1(t) \qquad \rightarrow \boxed{p_1(t) = \lambda t e^{-\lambda t}}$$

$$\vdots$$

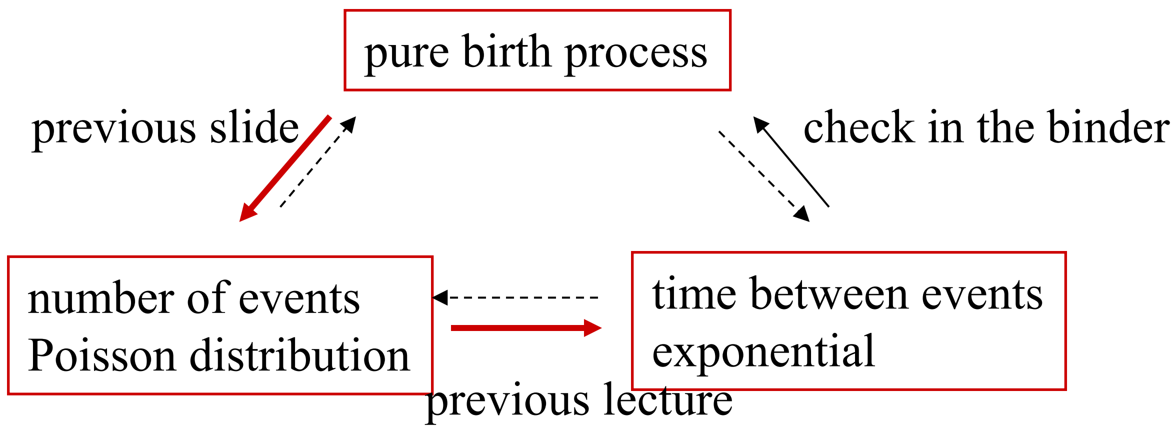
$$p'_k(t) = \lambda p_{k-1}(t) - \lambda p_k(t) \qquad \Rightarrow \boxed{p_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}}$$

- Pure birth process gives Poisson process! – time between state transitions is Exp(λ)

Equivalent definitions of Poisson process

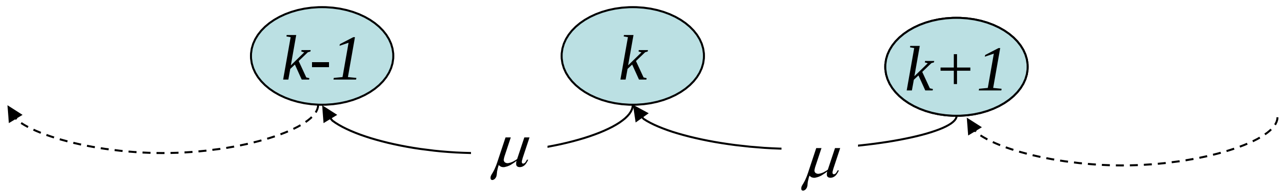
1. Pure birth process with intensity λ
2. The number of events in period (0,t] has Poisson distribution with parameter λ
3. The time between events is exponentially distributed with parameter λ

$$P(X < t) = 1 - e^{-\lambda t}$$



Pure death process

- Continuous time Markov-chain, infinite state space
- Transitions occur only between neighboring states
  - State independent death intensity: $\mu_i = \mu, \quad \forall i \neq 0$



- No stationary solution
- Pure death process gives Poisson process until reaching state 0
- Time between state transitions is Exp(μ)

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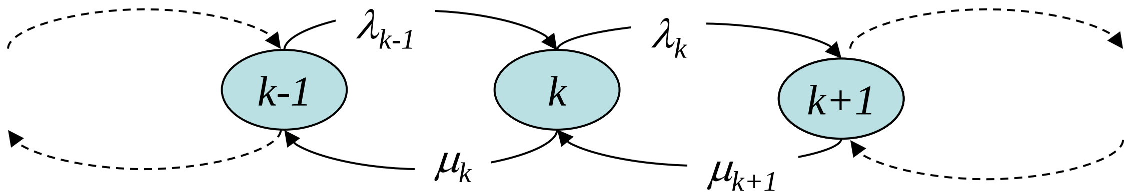
Birth-death process

- Continuous time Markov-chain
- Transitions occur only between neighboring states

$i \to i+1$ birth with intensity λ_i

$i \to i-1$ death with intensity μ_i (for $i>0$)

} modells population



$$\mathbf{Q} = \begin{bmatrix} -\sum q_{0j} & q_{01} & q_{02} & \cdots \\ q_{10} & -\sum q_{1j} & q_{12} & \cdots \\ q_{20} & q_{21} & -\sum q_{2j} & \cdots \\ \vdots & & & \ddots \end{bmatrix} = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 \\ & & & \ddots \end{bmatrix}$$

- State holding time – length of time spent in a state *k*
  - Until transition to states *k-1* or *k+1*
  - Minimum of the times to the first birth or first deaths $\to$ minimum of two Exponentially distributed random variables: Exp($\lambda_k+\mu_k$)

B-D process - stationary solution

- Local balance equations, like for general Markov-chains
- Stability: positive solution for <u>p</u> (since the MC is irreducible)

Cut 1: $\lambda_{k-1} p_{k-1} = \mu_k p_k \quad \Rightarrow \quad p_k = \frac{\lambda_{k-1}}{\mu_k} p_{k-1}$

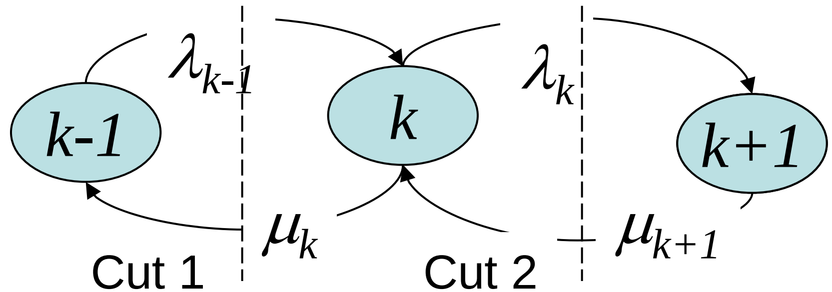
Cut 2: $\lambda_k p_k = \mu_{k+1} p_{k+1} \quad \Rightarrow \quad p_{k+1} = \frac{\lambda_k}{\mu_{k+1}} p_k = \frac{\lambda_k \lambda_{k-1}}{\mu_{k+1}\mu_k} p_{k-1}$

$\vdots$

$$\Rightarrow p_k = \frac{\lambda_0 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k} p_0 = \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}} p_0,$$

$$\sum p_k = 1 \quad \Rightarrow$$

$$p_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}},$$

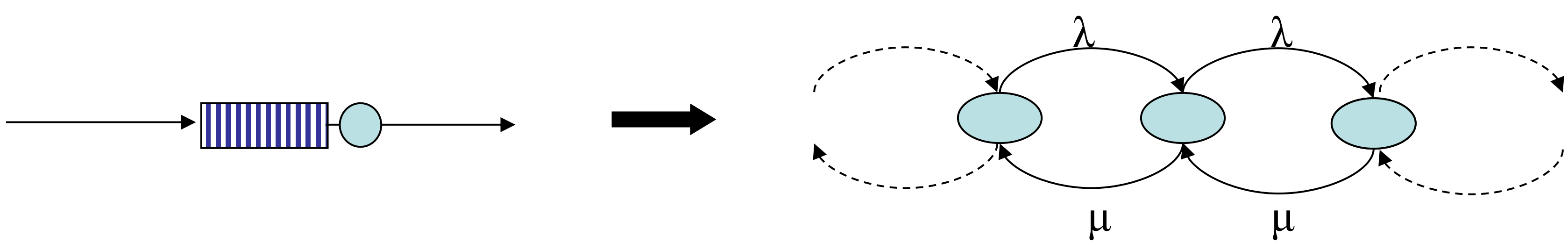


Group work: stationary solution for state independent transition rates:

$$\lambda_i = \lambda, \ \mu_i = \mu.$$

Markov-chains and queuing systems

- Why do we like Poisson and B-D processes?
  How are they related to queuing systems?
  - If arrivals in a queuing system can be modeled as Poisson process $\rightarrow$ also as a pure birth process
  - If services in a queuing systems can be modeled with exponential service times $\rightarrow$ also as a (pure) death process
  - Then the queuing system can be modeled as a birth-death process



Summary – Continuous time Markov-chains

- Markovian property: next state depends on the present state only
- State lifetime: exponential
- State transition intensity matrix **Q**
- Stationary solution: $\underline{p}\mathbf{Q}=\underline{0}$, or balance equations

- Poisson process
  - pure birth process (λ)
  - number of events has Poisson distribution, $E[X]=\lambda t$
  - interarrival times are exponential $E(\tau)=1/\lambda$

- Birth-death process: transition between neighboring states

- B-D process may model queuing systems!

Summary

- Continuous-time Markov chains
- Balance equations (global, local)
- Pure birth process and Poisson process
- Birth-death process