Why tensors in this course?

- A key concept in this couse is "the response of a media to electromagnetic fields"
- Given an electric field $\mathbf{E}$ this response can be described in term of the relation with the induced current $\mathbf{J}$.
- For linear media the Fourier transformed relation can often be described by a conductivity tensor $\boldsymbol{\sigma}$:

$$\mathbf{J} = \sigma \bullet \mathbf{E} \quad \text{or} \quad J_i = \sigma_{ij} E_j$$

- Example: In isotropic media the tensor I diagonal (no tensor description needed)

$$\sigma_{ij} = \sigma_0 \delta_{ij} \quad \Rightarrow \quad \mathbf{J} = \sigma_0 \mathbf{E} \quad \text{or} \quad J_i = \sigma_0 E_i$$

- Example: Uniaxial crystal conducts differently along the perpendicular to the crystal-plane. If the normal to the crystal-plane is in the z-directions then

$$\sigma_{ij} = \begin{bmatrix} \sigma_{\parallel} & 0 & 0 \\ 0 & \sigma_{\parallel} & 0 \\ 0 & 0 & \sigma_{\perp} \end{bmatrix} = \sigma_{\parallel}\delta_{ij} + \left(\sigma_{\perp} - \sigma_{\parallel}\right)\delta_{i3}\delta_{j3} \quad \Rightarrow \quad J_{1,2} = \sigma_{\parallel} E_{1,2} \;\&\; J_3 = \sigma_{\perp} E_3$$

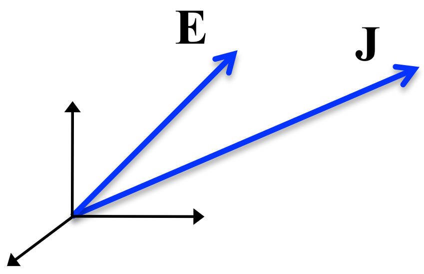
- In magnetised media the response often has off-diagonal components

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

Why tensors in this course?

- Example: Consider a crystal with the following conductivity

$$\sigma = \sigma_0 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



and an electric field

$$\mathbf{E} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$$

The current is then

$$\mathbf{J} = \sigma_0 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \sigma_0 \begin{bmatrix} 2 & 2 & 1 \end{bmatrix}^T$$

i.e. NOT in the same direction as the E-field

Rough guide to this course

- Description of the dielectric response : $\mathbf{J} = \sigma \bullet \mathbf{E}$
- Wave equation for dispersive media, e.g. crystals and plasmas
- Dissipation from e.g. collisions or Landau damping (Plemej formula)
  - Study damping: Assume small damping and expand
    - 0^{th} order: undamped plane wave (no dissipation)
    - 1^{st} order: calculate damping rate of the wave
- Next: insert the response $\mathbf{J}^* \bullet \mathbf{E} = \mathbf{E}^* \bullet \sigma \bullet \mathbf{E}$ in the energy conservation equation to study
  - spatial and temporal damping
  - non-homogeneous media
  - emission processes

Exercise 1.1: Tensor representations

Examples of tensor expressions:

$$\mathbf{A} \bullet \mathbf{B} = A_i B_i$$

$$(\mathbf{A} \times \mathbf{B})_i = \varepsilon_{ijk} A_j B_k \quad \Longleftrightarrow \quad \mathbf{A} \times \mathbf{B} = \mathbf{e}_i \varepsilon_{ijk} A_j B_k$$

$$(\nabla)_i = \frac{\partial}{\partial x_i} \quad \Longleftrightarrow \quad \nabla = \mathbf{e}_i \frac{\partial}{\partial x_i}$$

- **Exercise**: Express in tensor notation:

$$\nabla \times \mathbf{E} = \ldots$$

$$\nabla \times (\nabla \times \mathbf{E}) = \ldots$$

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = \ldots$$

$$\nabla \times (\mathbf{x} \times \mathbf{E}) = \ldots$$

Exercise 1.2: Plane waves and eigenmodes

A plane wave can be written as

$$\mathbf{E} = \Re\left\{\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}\right\} = \Re\left\{\left|\mathbf{E}_0\right| e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t + i\phi}\right\} = \left|\mathbf{E}_0\right|\cos\left(\mathbf{k}\cdot\mathbf{x} - \omega t + \phi\right)$$

where $\mathbf{E}_0 = \left|\mathbf{E}_0\right| e^{i\phi}$ is a complex amplitude, and the magnitude of the wave vector is $|\mathbf{k}|=k$.

a) What is the phase velocity of this wave?

b) Express an eigenmode

$$E(x,t) = E_0 \cos(kx)\cos(\omega t)$$

in terms of a sum of plane waves. Here E_0, k and ω are real.

Note: $2\cos(x) = e^{ix} + e^{-ix}$

Exercise 1.3: Time-averaged work

Consider an electric field

$$\mathbf{E} = \mathbf{e}_x \Re\{E\} \quad , \quad E = E_0 \mathrm{e}^{\mathrm{i}\phi} \quad , \quad \phi = kx - \omega t$$

and a current

$$\mathbf{J} = \mathbf{e}_x \Re\{J\} \quad , \quad J = J_0 \mathrm{e}^{\mathrm{i}\phi + i\delta}$$

where E_0 and J_0 are real amplitudes and the phase difference δ is real.

a) What is the work performed by the electric field on the charged particles that carry the current?

b) What is the time average of this work?

c) What is the phase difference, δ, that minimizes and maximizes the time averaged work, and what δ gives zero time-averaged work?

d) Also at these minima and maxima, when is the energy transferred from the wave to the particles (absorption) and vice versa (emission)?

Exercise 1.4: Hermitian and anti-hermitian tensors

In 2 dimensions, what number of real parameters is needed to describe…

a) any hermitian 2-tensor?

b) any anti-hermitian 2-tensor?

Note: if T^H is the hermitian and T^A the antihermitian parts of T then:

$$T^H = \frac{1}{2}\left(T + T^\dagger\right)$$

$$T^A = \frac{1}{2}\left(T - T^\dagger\right)$$

$$T^\dagger = transpose\left(T^*\right)$$

Exercise 1.5: Time-averaging and Fourier representations

a) Express the time-averaged energy density of an electric field $E(t)$ using a Fourier representation. Here the time-averaging should be performed over all times t in $(-\infty,\infty)$.

b) Use this expression to evaluate the energy density of the field

$$E(t) = E_0 \exp\left[-i\omega_m t\right]$$

Hint: Use Parseval's theorem:

$$\int_{-\infty}^{\infty} \left| f(t)^2 \right| dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \hat{f}(\omega)^2 \right| d\omega$$

$$\hat{f}(\omega) = \mathbf{F}\left\{ f(t) \right\}$$

Exercise 1.6: Conductivity and work

Consider a current $\mathbf{J}(\mathbf{x},t)$ driven linearly by the electric field $\mathbf{E}(\mathbf{x},t)$ such

$$\mathbf{J}(\mathbf{k},\omega) = \sigma \bullet \mathbf{E}(\mathbf{k},\omega)$$

where σ is the conductivity tensor (2-tensor, i.e. has matrix representations with components σ_{ij}).

How does the work performed by electric field $\mathbf{E}$ on a current $\mathbf{J}$ depend on the hermitian and anti-hermitian parts of the conductivity tensor?

Hint: Use Plancherel's theorem

$$\int_{-\infty}^{\infty} f(t) g^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}^*(\omega) \hat{g}(\omega) d\omega$$

Here g^* is the complex conjugate of g.