

Why tensors in this course?

- A key concept in this course is “the response of a media to electromagnetic fields”
- Given an electric field \mathbf{E} this response can be described in term of the relation with the induced current \mathbf{J} .
- For linear media the Fourier transformed relation can often be described by a conductivity tensor σ :

$$\mathbf{J} = \sigma \cdot \mathbf{E} \quad \text{or} \quad J_i = \sigma_{ij} E_j$$

- Example: In isotropic media the tensor is diagonal (no tensor description needed)

$$\sigma_{ij} = \sigma_0 \delta_{ij} \quad \longrightarrow \quad \mathbf{J} = \sigma_0 \mathbf{E} \quad \text{or} \quad J_i = \sigma_0 E_i$$

- Example: Uniaxial crystal conducts differently along the perpendicular to the crystal-plane. If the normal to the crystal-plane is in the z-directions then

$$\sigma_{ij} = \begin{bmatrix} \sigma_{\parallel} & 0 & 0 \\ 0 & \sigma_{\perp} & 0 \\ 0 & 0 & \sigma_{\perp} \end{bmatrix} = \sigma_{\parallel} \delta_{ij} + (\sigma_{\perp} - \sigma_{\parallel}) \delta_{i3} \delta_{j3} \quad \longrightarrow \quad J_{1,2} = \sigma_{\parallel} E_{1,2} \quad \& \quad J_3 = \sigma_{\perp} E_3$$

- In magnetised media the response often has off-diagonal components

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

Why tensors in this course?

- Example: Consider a crystal with the following conductivity

$$\sigma = \sigma_0 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

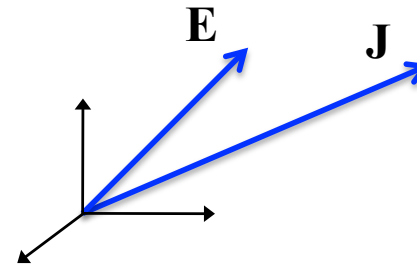
and an electric field

$$\mathbf{E} = [1 \quad 1 \quad 1]^T$$

The current is then

$$\mathbf{J} = \sigma_0 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \sigma_0 [2 \quad 2 \quad 1]^T$$

i.e. NOT in the same direction as the E-field



Rough guide to this course

- Description of the dielectric response : $\mathbf{J} = \sigma \cdot \mathbf{E}$
- Wave equation for dispersive media, e.g. crystals and plasmas
- Dissipation from e.g. collisions or Landau damping (Plemelj formula)
 - Study damping: Assume small damping and expand
 - 0th order: undamped plane wave (no dissipation)
 - 1st order: calculate damping rate of the wave
- Next: insert the response $\mathbf{J}^* \cdot \mathbf{E} = \mathbf{E}^* \cdot \sigma \cdot \mathbf{E}$ in the energy conservation equation to study
 - spatial and temporal damping
 - non-homogeneous media
 - emission processes

Exercise 1.1: Tensor representations

Examples of tensor expressions:

$$\mathbf{A} \cdot \mathbf{B} = A_i B_i$$

$$(\mathbf{A} \times \mathbf{B})_i = \varepsilon_{ijk} A_j B_k \quad \longleftrightarrow \quad \mathbf{A} \times \mathbf{B} = \mathbf{e}_i \varepsilon_{ijk} A_j B_k$$

$$(\nabla)_i = \frac{\partial}{\partial x_i} \quad \longleftrightarrow \quad \nabla = \mathbf{e}_i \frac{\partial}{\partial x_i}$$

- **Exercise:** Express in tensor notation:

$$\nabla \times \mathbf{E} = \dots$$

$$\nabla \times (\nabla \times \mathbf{E}) = \dots$$

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = \dots$$

$$\nabla \times (\mathbf{x} \times \mathbf{E}) = \dots$$

Exercise 1.2: Plane waves and eigenmodes

A plane wave can be written as

$$\mathbf{E} = \Re\left\{\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}\right\} = \Re\left\{|\mathbf{E}_0| e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t + i\phi}\right\} = |\mathbf{E}_0| \cos(\mathbf{k}\cdot\mathbf{x} - \omega t + \phi)$$

where $\mathbf{E}_0 = |\mathbf{E}_0| e^{i\phi}$ is a complex amplitude, and the magnitude of the wave vector is $|\mathbf{k}|=k$.

- a) What is the phase velocity of this wave?

- b) Express an eigenmode

$$E(x,t) = E_0 \cos(kx) \cos(\omega t)$$

in terms of a sum of plane waves. Here E_0 , k and ω are real.

Note: $2\cos(x) = e^{ix} + e^{-ix}$

Exercise 1.3: Time-averaged work

Consider an electric field

$$\mathbf{E} = \mathbf{e}_x \Re\{E\} \quad , \quad E = E_0 e^{i\phi} \quad , \quad \phi = kx - \omega t$$

and a current

$$\mathbf{J} = \mathbf{e}_x \Re\{J\} \quad , \quad J = J_0 e^{i\phi + i\delta}$$

where E_0 and J_0 are real amplitudes and the phase difference δ is real.

- a) What is the work performed by the electric field on the charged particles that carry the current?
- b) What is the time average of this work?
- c) What is the phase difference, δ , that minimizes and maximizes the time averaged work, and what δ gives zero time-averaged work?
- d) Also at these minima and maxima, when is the energy transferred from the wave to the particles (absorption) and vice versa (emission)?

Exercise 1.4: Hermitian and anti-hermitian tensors

In 2 dimensions, what number of real parameters is needed to describe...

- a) any hermitian 2-tensor?
- b) any anti-hermitian 2-tensor?

Exercise 1.5: Time-averaging and Fourier representations

a) Express the time-averaged energy density of an electric field $E(t)$ using a Fourier representation. Here the time-averaging should be performed over all times t in $(-\infty, \infty)$.

b) Use this expression to evaluate the energy density of the field

$$E(t) = E_0 e^{-i\omega t}$$

Hint: Use Parseval's theorem:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega$$

$$\hat{f}(\omega) = \mathbf{F}\{f(t)\}$$

Exercise 1.6: Conductivity and work

Consider a current $\mathbf{J}(\mathbf{x},t)$ driven linearly by the electric field $\mathbf{E}(\mathbf{x},t)$ such

$$\mathbf{J}(\mathbf{k},\omega) = \boldsymbol{\sigma} \cdot \mathbf{E}(\mathbf{k},\omega)$$

where $\boldsymbol{\sigma}$ is the conductivity tensor (2-tensor, i.e. has matrix representations with components σ_{ij}).

How does the work performed by electric field \mathbf{E} on a current \mathbf{J} depend on the hermitian and anti-hermitian parts of the conductivity tensor?

Hint: Use Plancherel's theorem

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}^*(\omega)\hat{g}(\omega)d\omega$$