AUTOMATIC CONTROL
KTH

Nonlinear Control, EL2620 / 2E1262

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1. (a) Equilibria $(0,0), (-1,-1), (1,1)$ with corresponding Jacobians $A = [-2\ 1; -1\ 1]$, $A = [0\ 1; -1\ 1]$ (same for latter two). Corresponding eigenvalues are $-0.5 \pm \sqrt{5}/2$ and $0.5 \pm i\sqrt{3}/2$. Thus, origin is an unstable saddle point and the two other are unstable focus points.

   (b) E.g., the control $u = -x_2$ will make the origin a stable node with eigenvalue $-1, -1$.

   (c) We try the Lyapunov candidate $V = 0.5(x_1^2 + x_2^2) \geq 0$ and get $\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = -2x_1^2 1 + x_1^2 + x_1x_2 - x_1x_2 = -\frac{2x_1^2}{1+x_1^2} \leq 0$ which is zero whenever $x_1 = 0$. Need LaSalle to show that the origin $x_1 = x_2 = 0$ is the only invariant solution with $\dot{V} = 0$. From the differential equation we get that $\dot{x}_1 = x_2$ when $x_1 = 0$ and hence $x_1 = 0$ is invariant only when also $x_2 = 0$. Since also V is radially unbounded we have proven global asymptotic stability of the origin.

2. (a) (i) At $x_1 = -2$ we have $\dot{x}_1 = x_2 - 2(-3 - x_2^2) > 0 \forall x_2$ and hence all trajectories point inwards from this line. Similar reasoning for $x_1 = 2, x_2 = -2, x_2 = 2$ gives that all trajectories point inwards and hence the region is invariant.

   (ii) An invariant region must contain a stable stationary solution. The only equilbrium within the region is $x_1 = x_2 = 0$ which is unstable with eigenvalue $1 \pm i$, i.e., unstable focus. Thus, there must be a stable limit cycle within the region (2nd order systems can only have steady-states or limit cycles as stationary solutions).

   (b) We employ the control Lyapunov function $V = 0.5(x_1^2 + x_2^2)$ which yields

$$\dot{V} = \frac{-x_1x_2}{1+x_2^2} + \frac{-x_1x_2}{1+x_1^2} + x_1u$$

   By choosing the control law

$$u = x_2\left(\frac{1}{1+x_1^2} + \frac{1}{1+x_2^2}\right) - x_1$$

   we get $\dot{V} = -x_1^2$. Thus $\dot{V} = 0$ when $x_1 = 0$, but since $\dot{x}_1 \neq 0$ if $x_1 = 0, x_2 \neq 0$ only the origin is invariant with $\dot{V} = 0$. LaSalle proves global asymptotic stability of the origin.

3. (a) (i) Choose $z_1 = x_2, z_2 = \dot{x}_2$ to avoid involving u in transformed variables. This yields $\dot{z}_1 = z_2$ and $\dot{z}_2 = -\dot{x}_1 - 2x_2\dot{x}_2 = -\sin(x_1) - x_2^2 + 2x_1x_2 + 2x_2^3 - u$ and choosing

$$u = -\sin(x_1) - x_2^2 + 2x_1x_2 + 2x_2^3 - v$$

   yields $\dot{z}_1 = z_2, \dot{z}_2 = v$ which is linear.

(ii) The system has relative degree one since $\dot{y} = sin(x_1) + x_2^2 + u$ and the controller $u = -\sin(x_1) - x_2^2 + v$ yields a linear input-output relationship $\dot{y} = v$. The zero dynamics are given by $\dot{x}_2 = -x_2^2$ which is unstable. Since the zero dynamics are unstable, input-output linearizing control should not be used for this system.

(b) The sliding set is $\sigma = x_1 + x_2 = 0$ and this can be shown to be globally attracting by employing $V = 0.5\sigma^2$ which yields $\dot{V} = -\sigma^2 - \sigma\texttt{sign}(\sigma) \leq 0$ and equal to zero only when $\sigma = 0$. On the sliding set $\dot{\sigma} = \dot{x}_1 + \dot{x}_2 = x_2 - x_1 - 2x_2 + u_{eq} = -x_1 - x_2 + u_{eq} = u_{eq}$ and hence the equivalent control $u = 0$.

4. (a) Without the saturation and deadband, the controller is given by $F(s) = K$. The transfer function $G_c(s)$ from r to y is given by

$$G_c(s) = \frac{KG(s)}{1+KG(s)} = \frac{\frac{K}{\left(s+\frac{1}{2}\right)^2}}{1+\frac{K}{\left(s+\frac{1}{2}\right)^2}} = \frac{K}{\left(s+\frac{1}{2}\right)^2 + K} = \frac{K}{s^2+s+\frac{1}{4}+K}. \quad (1)$$

By the Routh-Hurwitz criterion, the poles of $G_c(s)$ lie in the left half complex plane if and only if $K > -\frac{1}{4}$. Alternatively, one can solve (1) and obtain

$$s = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{4} - K} = -\frac{1}{2} \pm \sqrt{-K},$$

which satisfies $\Re(s) < 0$ if and only if $K > -\frac{1}{4}$. Hence the feedback loop is stable for any positive value of K.

(b) First we note that $G(s)$ has a double pole in $s = -1$, and hence $G(s)$ is stable. We now calculate the gain of F, and obtain

$$\gamma(F) = \sup_{e\in\mathcal{L}_2} \frac{\|Fe\|_2}{\|e\|_2} = \frac{H}{D+\frac{H}{K}} = \frac{KH}{KD+H} = \frac{K}{1+\frac{KD}{H}},$$

which is the highest amplification of F, obtained at the point for which the controller saturates.

The gain of $G(s)$ is given by

$$\gamma(G(s)) = \sup_{\omega\in[0,\infty)} |G(i\omega)| = \sup_{\omega\in[0,\infty)} \left|\frac{1}{\left(i\omega+\frac{1}{2}\right)^2}\right| = \sup_{\omega\in[0,\infty)} \frac{1}{\omega^2+\frac{1}{4}} = 4.$$

By the small gain theorem, the feedback loop is stable if $\gamma(F)\gamma(G(s)) < 1$, which gives

$$\frac{4K}{1+\frac{KD}{H}} < 1 \Leftrightarrow K < \frac{H}{4H-D}.$$

(c) We have

$$G(i\omega) = \frac{1}{\left(i\omega+\frac{1}{2}\right)^2} = \frac{1}{\left(\frac{1}{4}-\omega^2\right)+i\omega} = \frac{\left(\frac{1}{4}-\omega^2\right)-i\omega}{\left(\frac{1}{4}-\omega^2\right)^2+\omega^2},$$

from which we get

$$\Re(G(i\omega)) = \frac{\left(\frac{1}{4}-\omega^2\right)}{\left(\frac{1}{4}-\omega^2\right)^2+\omega^2} < 0 \quad \text{when} \quad \omega^2 > \frac{1}{4}.$$

Hence, $G(s)$ is not passive, and we cannot guarantee stability of the feedback loop by the passivity theorem.

(d) From (b) we already know that

$$\frac{F(e)}{e} < \frac{KH}{KD+H}.$$

Furthermore

$$\frac{F(e)}{e} > 0,$$

so $F(e)$ is bounded by the sector $[k_1, k_2] = [0, \frac{KH}{KD+H}]$. By the circle criterion, the feedback loop is stable if the Nyquist curve of $G(s)$ does not encircle or intersect the smallest circle containing the line segment $[-\frac{1}{k_1}, -\frac{1}{k_2}] = [-\infty, -\frac{KD+H}{KH}]$. This criterion corresponds to

$$\Re(G(i\omega)) > -\frac{KD+H}{KH} \; \forall \omega \geq 0. \tag{2}$$

Recall the expression for $\Re(G(i\omega))$ from (c):

$$\Re(G(i\omega)) = \frac{\left(\frac{1}{4}-\omega^2\right)}{\left(\frac{1}{4}-\omega^2\right)^2+\omega^2} = \frac{\left(\frac{1}{4}-\omega^2\right)}{\left(\frac{1}{4}+\omega^2\right)^2}. \tag{3}$$

We need to obtain a lower bound for $\Re(G(i\omega))$ to guarantee that (2) is satisfied. The extreme values of $\Re(G(i\omega))$ are given either by the critical points, or the end points of the interval $[0, \infty)$. Differentiating (3) and setting the derivative to zero yields

$$\frac{\partial}{\partial\omega}\Re(G(i\omega)) = \frac{-2\omega\left(\frac{1}{4}+\omega^2\right)^2 - 4\omega\left(\frac{1}{4}+\omega^2\right)\left(\frac{1}{4}-\omega^2\right)}{\left(\frac{1}{4}+\omega^2\right)^4} = 0,$$

which gives $\omega = 0$ or

$$\left(\frac{1}{4}+\omega^2\right)^2\left(2\left(\frac{1}{4}+\omega^2\right)^2 + 4\left(\frac{1}{4}-\omega^2\right)^2\right) = \left(\frac{1}{4}+\omega^2\right)^2\left(\frac{3}{2}-2\omega^2\right) = 0,$$

which gives either $\omega^2 = -\frac{1}{4}$ or $\omega^2 = \frac{3}{4}$. The first solution can be discarded since it yields imaginary solutions roots. The second solution yields the positive root $\omega = \frac{\sqrt{3}}{2}$. Inserting the two real and positive roots in (3) yields

$$\Re(G(0)) = \frac{\left(\frac{1}{4}\right)}{\left(\frac{1}{4}\right)^2} = 4$$

$$\Re\Big(G\Big(i\frac{\sqrt{3}}{4}\Big)\Big) = \frac{\left(\frac{1}{4}-(\frac{\sqrt{3}}{2})^2\right)}{\left(\frac{1}{4}+(\frac{\sqrt{3}}{2})^2\right)^2} = \frac{\frac{1}{4}-\frac{3}{4}}{\left(\frac{1}{4}+\frac{3}{4}\right)^2} = -\frac{1}{2}.$$

We also have to check the limit when $\omega \to +\infty$:

$$\lim_{\omega \to +\infty} \Re(G(i\omega)) = \lim_{\omega \to +\infty} \frac{\left(\frac{1}{4} - \omega^2\right)}{\left(\frac{1}{4} + \omega^2\right)^2} = 0.$$

We see that $\Re(G(i\omega)) > -\frac{1}{2}$. Thus, the circle criterion gives

$$-\frac{1}{2} > -\frac{KD+H}{KH} \Leftrightarrow \frac{KD+H}{KH} > \frac{1}{2} \Leftrightarrow \frac{KH}{KD+H} < 2 \Leftrightarrow K < \frac{2H}{H-2D}.$$

Alternative solution: We can also solve the equation $\Re(G(i\omega)) = c$, and obtain the minimal c for which there exists a real and positive solution ω. We get from (3) that

$$\Re(G(i\omega)) = \frac{\left(\frac{1}{4} - \omega^2\right)}{\left(\frac{1}{4} + \omega^2\right)^2} = c \Leftrightarrow \left(\frac{1}{4} - \omega^2\right) = c\left(\frac{1}{4} + \omega^2\right)^2 \Leftrightarrow c\omega^4 + \frac{c+2}{2}\omega^2 + \frac{c-4}{16} = 0$$

Solving for ω^2 yields

$$\omega^2 = -\frac{c+2}{4c} \pm \frac{\sqrt{(c+2)^2 + c(4-c)}}{4c} = -\frac{c+2}{4c} \pm \frac{\sqrt{8c+4}}{4c}. \tag{4}$$

In order for the solution ω^2 to be real, we must have $8c + 4 \geq 0$, which gives $c \geq -\frac{1}{2}$. Inserting $c = -\frac{1}{2}$ in (4) yields $\omega^2 = \frac{3}{4}$, with the real and positive solution $\omega = \frac{\sqrt{3}}{2}$. Hence $\Re(G(i\omega)) > -\frac{1}{2}$.

5. (a) The optimal control problem is

$$\min_u \int_0^{24} u^2 dt$$

subject to $\dot{x} = -\delta x(t) + u(t)$, $x(0) = 0$, $x(24) = 12$. Here we have introduced $x = T - 10$.

(b) Without any heat loss, the amount of energy needed to heat the lecture hall to $x(24) = 12$ is the same indepdent of the shape of $u(t)$. However, since we have a square objective function it is optimal to heat with a constant $u = 12/24$

(c) Form the Hamiltonian

$$H = n_0 u^2 + \lambda(-0.02x(t) + u(t))$$

and the optimal solution for $\partial H/\partial u = 2n_0 u^*(t) + \lambda(t) = 0 \quad \Rightarrow$

$$u = -\frac{\lambda(t)}{2n_0}$$

To ensure minimum we require $\partial^2 H/\partial u^2 = 2n_0 > 0$ or $n_0 > 0$. The Lagrange multiplier $\lambda(t)$ is determined from

$$\dot{\lambda} = 0.02\lambda(t), \ \lambda(t_f) = \mu$$

which yields

$$\lambda(t) = \lambda_0 e^{0.02t} \quad \Rightarrow \quad u(t) = -\frac{\lambda_0}{2n_0} e^{0.02t}$$

The choice $\lambda_0 = \mu/e^{0.48}$ ensures $\lambda(t_f) = \mu$, but since μ is a free parameter we might as well keep λ_0 as the free parameter. To determine $\lambda_0/2n_0$ in $u(t)$ we solve the differential equation $x(t) = -\frac{\lambda_0}{2n_0} \int_0^{24} e^{-0.02\tau} e^{0.02(t-\tau)} d\tau = -\frac{\lambda_0}{2n_0} e^{0.02t} 25(1 - e^{-0.96})$ and to obtain $x(24) = 12$ we get

$$u(t) = \frac{12}{25} \frac{1}{e^{0.48} - e^{-0.48}}$$

The optimal heating corresponds to an input that gives a linear increase in the room temperature over the whole period. Comparing e.g. to the policy with constant heating over the period the saving is about 2%, i.e., not impressive but still. Also note that with no constraints on u and the objective of minimizing the energy consumption as such (and not the square), the optimal input would be a Dirac pulse at $t = 24h$.

(d) From (b) we have that a constant heating of $u = 0.5$ is needed to achieve the required temperature without any heat loss, and hence we can not satisfy the boundary condition $x(t_f) = 12$ with the constraint $u < 0.1$. The problem would not be feasible.