KTH Electrical Engineering

EL2620 Nonlinear Control

Exercises and Homework

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Automatic Control
KTH, Stockholm, Sweden

Contents

Preface

Many people have contributed to this material in nonlinear control. Originally, it was developed by Bo Bernhardsson, Karl Henrik Johansson, and Mikael Johansson. Later contributions have been added by Henning Schmidt, Krister Jacobsson, Bo Wahlberg, Elling Jacobsen, Torbjörn Nordling, Per Hägg and Farhad Farokhi. Exercises have also shamelessly been borrowed (stolen) from other sources, mainly from Karl Johan Åström's compendium in Nonlinear Control and Khalil's book Nonlinear Systems.

Per Hägg and Elling W. Jacobsen, September 2012

Chapter 1

Homeworks

1.1 Report Requirements

Your homework reports should be handed-in as a single PDF file containing all the typed answers, figures, tables, and etc. You can prepare this document using both Microsoft Word® or LaTeX. However, there is a set of requirements that your homework should satisfy in order to get considered and subsequently, approved. In what follows, the homework requirements are specified.

In your report, you should first describe the problem in your own words and break it down into subproblems, unless the problem was already broken down into subproblems in the description of the homework. In each subproblem you should define a set of questions or tasks which gradually lead you to a solution of the overall problem. The problem should be described in detail, so that the reader can understand it without reading the description of the homework. Then, solve the subproblems in succession and give the solution to the overall problem at the end, followed by a sentence or two about what you learned from solving the problem. You should lead the reader through a detailed solution where you motivate every step, state every theorem, and verify that the assumptions are fulfilled. The presentation should be detailed yet clear and concise. You should provide a reference (including the page number) for every theorem and result that you are using [2]. Any engineering student with a bachelor's degree should be able to understand and reproduce the results based on your report alone. You may not copy-and-paste any text, theorem, or figure except the figures used in the description of the homework. Only include figures and tables if they fosters the understanding of the solution. Each figure and table should include a caption, and all the figures and tables should be associated with a reference in the text and placed at the end of your report. Each equation should only be written once and referred to by reference later when needed. Equations that you do not refer to later may be left unlabeled. You are not allowed to have any appendix nor attach any Matlab or other code.

You report (excluding the figures and tables) should **not** be shorter than two pages and longer than four pages. The reports should contain the names of the students (who have collaborated as a group to solve the problems and write the report) and their email addresses. The solutions may **not** be copy-pasted from other sources such as textbooks, internet webpages, or other groups report. You should properly cite all the theorems, texts, or results that you use when creating the report. Plagiarism (i.e., the act of copying someone else ideas, text, or results in any form while presenting it as your own work, particularly without permission[1]) will be reported to the KTH Disciplinary Board.

Finally, note that deadlines listed on the course page are hard. We will not accept any delays in handing in reports or reviews.

1.1.1 Writing your report using Microsoft Word®

You should write your report as a double column document. In Microsoft Word® 2010 release (which I would only refer during this description), you can change the number of columns under the page layout tab. You

[1] http://en.wiktionary.org/wiki/plagiarism

Homework 1 in EL2620 Nonlinear Control

First name1 Last name1
person number
email

First name2 Last name2
person number
email

September 22, 2009

Problem

Describe the problem in your own words and break it down into subproblems, unless the problem was already broken down into subproblems in the description of the homework. In each subproblem you should define a set of questions or tasks which gradually lead you to a solution of the overall problem. The problem should be described in detail, so that the reader can understand it without reading the description of the homework. For an example, see Problem 1.

Problem 1 – Behavior of a nonlinear system

Study the behavior of the system

$$\frac{dx}{dt} = y \tag{1a}$$

$$\frac{dy}{dt} = -2x - 2y - 4x^2 \tag{1b}$$

with regard to its initial value and assess if the system is globally asymptotically stable.

Figure 1.1: The homework report title page.

should use A4 paper size to write your report (which is the default choice). The top, right, left, and bottom margins should not be less than 2.5 cm (which is again the default choice). Similarly, you can change the margin specifications under the page layout tab. We strongly suggest "Times New Roman" as the document font. However, in case you are missing it in your font library, we suggest you use another standard font; e.g., Arial or Courier New. Following this recommendation, you guarantee that we can open your document without any problem and read the contents. The preferred font size for the text is 11 pt. However, you can certainly use larger (or smaller) font sizes for the headings (or footnotes). Finally, make sure that the first page of the report contains the names of the students (who have collaborated as a group to solve the problems and write the report) and their email addresses. We strongly suggest you follow the template in Figure 1.1 for creating the first page of your report. A template can also be found at the course homepage.

1.1.2 Writing your report using LaTeX

LaTeX is a document preparation system. Two major advantage of LaTeX over commonly used WYSIWYG (what you see is what you get) word processors like Microsoft Word, `http://office.microsoft.com/word/`, is that it allows you to enter mathematical equations very fast as code and that the layout is separated from the text. It is commonly used in Engineering disciplines, in particular for writing technical reports and scientific articles, so we hope that you will enjoy using LaTeX. You should use LaTeX to prepare your homework reports. Moreover, you should follow the report template, which can be downloaded from the course page (both as a PDF and as a ZIP file containing all source files for compiling the PDF).

We recommend those of you that are not familiar with LaTeX to see [1] for an introduction. Typically the fastest way to find help on LaTeX is to search on Google, for example `latex figure position`. Most TeX related software and material can be found in the TeXUsers Group, `http://www.tug.org/`. Many different TeX distributions exists, but we recommend Windows users to install MikTeX `http://miktex.org/`, Unix users to install TeX Live `http://www.tug.org/texlive/` and Mac users to install MacTeX `http://www.tug.org/mactex/`. When preparing a document using LaTeX then you write your text in a `tex` file, i.e. a normal text file, which you then compile using `pdflatex` or `latex`. The former will directly produce a PDF, while the later produces a DVI, which you then convert to PostScript using `dvips`, which then is converted to a PDF using `ps2pdf`. Most people prefer to use a text editor that supports highlighting of LaTeX commands and press a button compilation, like Emacs `http://www.gnu.org/software/emacs/` (Unix, Windows, Mac), WinEdt `http://www.winedt.com/` (Windows), TextPad `http://www.textpad.com/`

(Windows), TeXShop `http://www.uoregon.edu/~koch/texshop/` (Mac), but you may start by using built-in programs like Notepad (Windows), pico (Unix), TextEdit (Mac). If you have trouble, please read the help files and documents coming with the software, search on the Internet, ask your course mates and read LaTeX books.

A short getting you started list:

1. Install a TeX distribution.
2. Download and unzip the report template.
3. Open report_template.tex using a built-in text editor.
4. Produce a PDF by issuing the command `pdflatex report_template.tex` at the command prompt (Windows) or in the terminal (Unix, Mac) in the directory where you unzipped the report template. Do it three times in a row to get the references correct. This compiles the report template and creates the report_template.log, report_template.aux and report_template.pdf files in the same directory. Open the PDF and verify that it is identical to the one on the course page.
5. Start writing you report by changing in the report template.

Including vector graphics from Matlab

A bitmap image, e.g. JPEG, TIFF or PNG, has a finite resolution, since the image consists of a limited number of pixels. Vector graphics, e.g. EPS and PDF, on the contrary consists of objects placed relatively to each other, which in practice means infinite resolution. Vector graphics can therefore be scaled without any quality problems, while a bitmap typically becomes blurred. You should therefore always use vector graphics when possible. Note that vector graphics are containers of objects and it is therefore possible to place a bitmap image as one of the objects, but you should never do this, since it will lead to problems when scaling the figure.

The current `pdflatex` compiler can include figures in PDF (Portable Document Format), JPEG (Joint Photographic Experts Group), PNG (Portable Network Graphics) and MetaPost, while the `latex` compiler only can include EPS (Encapsulated PostScript). Matlab figures (`http://www.mathworks.com`) consists of vector graphics, unless you load a bitmap image, so we strongly recommend that you save them as EPS or PDF. You need to set the paper size equal to the figure size or you will typically end up with a small figure centered on a large paper. The paper size 15×10 centimeters is set by issuing the command `set(gcf,'PaperUnits','centimeters','PaperSize',[15 10],'PaperPosition',[0 0 15 10]);`, see Matlab_code_for_stepresponse_figure.m, which is included in the ZIP file containing the report template. The paper size, however, only matters for the PDF, since every EPS contains a BoundingBox that is used by LaTeX to define the size of the figure. A simple way to avoid setting the paper size is therefore to save the Matlab figure as an EPS and then convert it to a PDF using `epstopdf`. Matlab figures can either be saved by clicking on `File` and selecting `Save As...` or by issuing the command `print -dpdf -noui -painters FileName` to save as a PDF or `print -depsc2 -noui -painters -tiff FileName` to save as an EPS. Note the option `-noui` that removes any user interface control objects from the figure.

1.1.3 Work in groups of two

You should do the homeworks in pairs (more than two students per group will not be accepted). If more than one student has done homework 1 alone, then a partner will be designated to those who have done it on their own.

1.1.4 Grading

Every subproblem is graded as either 1 or 0 points. One point is given if at least 95% of the solution is correct. To reach the required 75% correct you thus need 4 out of 5 points per homework. After checking your

Cover and grading guide for homework ____ in EL2620 Nonlinear Control

Fill in both name, personal number and email.

Author 1: ----------

Author 2: ----------

Grading of report:	Teaching assistant fill this					
	Pass				Fail	
Length of report (excluding this cover, figures and tables)	$(2-4)$	☐			$(<2, >4)$	☐
Report is typed	Yes	☐			No	☐
Title and author names given	Yes	☐			No	☐
Problem divided into well specified subproblems	Yes	☐	Accept	☐	No	☐
Problem understandable without checking other sources	Yes	☐			No	☐
Solutions exist (list missing solutions in the review statement)	Yes	☐	>75%	☐	<75%	☐
Solutions are correct (clearly mark all errors using blue ink)	Yes	☐	>75%	☐	<75%	☐
Solutions are detailed yet clear and concise	Yes	☐	Accept	☐	No	☐
Solutions understandable without checking other sources	Yes	☐			No	☐
Solutions can be reproduced based on report	Yes	☐			No	☐
All theorems/assumptions stated and fulfilled/verified	Yes	☐	Accept	☐	No	☐
Structure/language is consistent and easy to follow	Yes	☐	Accept	☐	No	☐
Figures/tables are clear and easy to understand	Yes	☐	Accept	☐	No	☐
Every figure/table is referred to in the text and has a caption	Yes	☐			No	☐
Appendix, Matlab or other code included	No	☐			Yes	☐
Text, theorems or solutions copy-pasted or plagiarism	No	☐			Yes	☐

If any of the fail boxes is marked then you need to correct all errors and implement all improvements suggested by the reviewer or motivate why you have not implemented a suggestion. You should then mark all changes in a distinguishable color (e.g., red or green) and hand it in. Plagiarism will be reported to the KTH Disiplinary Board.

Final grade (teaching assistant fill this):				
Attempt 1	Pass	☐	Fail	☐
Attempt 2	Pass	☐	Fail	☐

Figure 1.2: Cover and grading guide for the homeworks.

homework, we will provide a cover (see Figure 1.2) with a set of formal requirements. You need to get a pass on each of the formal requirements listed on the cover and grading guide in order to pass the homework. If any of the fail boxes is marked then you need to correct all errors and implement all improvements suggested by the reviewers or motivate why you have not implemented a suggestion. You should mark all the changes in your PDF file (with a color that is easy to distinguish from the rest of the text; e.g., red, green, etc) and then, hand in

within two weeks from the day the graded homeworks were available.

1.1.5 References

[1] Tobias Oetiker, Hubert Partl, Irene Hyna, and Elisabeth Schlegl. *The Not So Short Introduction to LaTeX 2ε*. Oetiker, OETIKER+PARTNER AG, Aarweg 15, 4600 Olten, Switzerland, 2008.
`http://www.ctan.org/info/lshort/`.
[2] Hassan K Khalil. *Nonlinear systems.* Prentice Hall, Upper Saddle river, 3. edition, 2002. ISBN 0-13-067389-7.

1.2 Homework 1

Suppose that there is a grassy island supporting populations of two species x and y. If the populations are large then it is reasonable to let the normalized populations be continuous functions of time (so that $x = 1$ might represent a population of 100 000 species). We propose the following simple model of the change in population:

$$\begin{aligned} \dot{x}(t) &= x(t)(a + bx(t) + cy(t)) \\ \dot{y}(t) &= y(t)(d + ex(t) + fy(t)), \end{aligned} \tag{1.1}$$

where x and y are non-negative functions and $a, b, \ldots, f$ are constants. Let us start with discussing the model heuristically to gain some insight. Consider the first equation when $y = 0$: $\dot{x} = x(a + bx)$. If the species x preys on y then the coefficient a must be non-positive ($a \leq 0$), since if $y = 0$ there is no available food so the population should die of starvation. If x instead eats grass then $a > 0$, so the population grows if the initial population is small. The coefficient b enables a non-zero equilibrium for the population of species x in the absence of the second species. We have $b < 0$ since the island is finite so large populations suffer from overcrowding. The coefficient c describes the effect of y on x. If $c > 0$ then the population growth increases (for example, if x feeds upon y), while if $c < 0$ then the population growth decreases (for example, if x and y compete for the same resource). Similar interpretations hold for d, e, and f.

1. [1p] Depending on the sign of c and e there are four different population models. Continue the discussion above and label these four models as

- predator-prey (x predator, y prey),
- prey-predator (x prey, y predator),
- competitive (x and y inhibit each other),
- symbiotic (x and y benefit each other).

2. [1p] Consider the population model (1.1) with $a = 3$, $b = f = -1$, and $d = 2$. Draw the phase portrait[2] for the following four cases

- $(c, e) = (-2, -1)$,
- $(c, e) = (-2, 1)$,
- $(c, e) = (2, -1)$,
- $(c, e) = (2, 1)$.

Determine (analytically) the type of each equilibrium in each case. Interpret your results and comment on their implications when the model is applied in a real investigation of the dynamics between two species.

3. [1p] Repeat the previous exercise for $a = e = 1$, $b = f = 0$, and $c = d = -1$.

4. [1p] Show that the x- and y-axes are invariant for all values of the parameters in (1.1).[3] Why is this a necessary feature of a population model? Assume $a = e = 1$, $b = f = 0$, and $c = d = -1$ in (1.1) and show that a periodic orbit exists. (Hint: find a closed and bounded trajectory that is an invariant set and contains no equilibrium point.[4])

5. [1p] Generalize the population model (1.1) to $N > 2$ species.

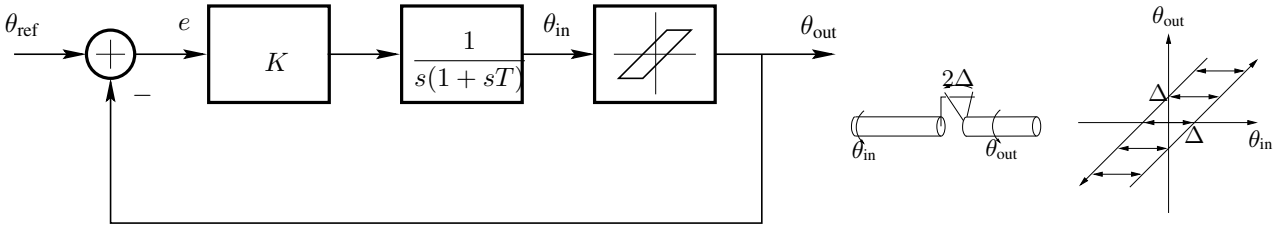
[2] It might be a good idea to first make a sketch by hand and then use pplane in Matlab.

[3] A set $W \subset \mathbb{R}^n$ is invariant if $z(0) \in W$ implies $z(t) \in W$ for all $t \geq 0$.

[4] The Poincaré-Bendixson criterion then ensures that the trajectory is a periodic orbit.

1.3 Homework 2

The influence of back-lash in a control system will be discussed in this homework. Consider the drive line in a crane, where torque is generated by an electric motor, transformed through a gearbox, and finally used to lift a load. A block diagram describing the control of the angular position of the lifting axis θ_{out} is shown below to the left. Here the first block represents the P-controller ($K > 0$), the second the dynamics in the motor ($T > 0$), and the third the back-lash between the gear teeth in the gearbox.

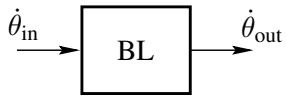


The back-lash is sketched in the middle above and its characteristic is shown to the right. This back-lash model gives a relation between the angles θ_{in} and θ_{out}. Another possibility is to model the back-lash as the relation between the corresponding angular velocities $\dot{\theta}_{\text{in}}$ and $\dot{\theta}_{\text{out}}$. This model, denoted BL, is given as

$$\dot{\theta}_{\text{out}} = \begin{cases} \dot{\theta}_{\text{in}}, & \text{in contact} \\ 0, & \text{otherwise,} \end{cases} \tag{1.2}$$

where "in contact" corresponds to that $|\theta_{\text{in}} - \theta_{\text{out}}| = \Delta$ and $\dot{\theta}_{\text{in}}\,(\theta_{\text{in}} - \theta_{\text{out}}) > 0$.

1. [1p] Consider the back-lash model (1.2) described as a block:



Assume

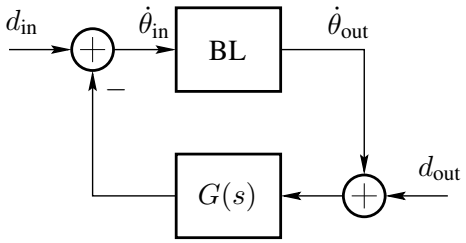
$$\dot{\theta}_{\text{in}}(t) = \begin{cases} 1, & t \in [0, 1] \\ -1, & t \in [1, 2] \\ 0, & \text{otherwise} \end{cases}$$

is the input to BL (in open-loop). Sketch θ_{in}, θ_{out}, $\dot{\theta}_{\text{in}}$, and $\dot{\theta}_{\text{out}}$ for $\theta_{\text{in}}(0) = 0$ and $\theta_{\text{out}}(0) = -\Delta$ in a diagram.

2. [1p] Show that the gain of BL is equal to $\gamma(\text{BL}) = 1$. Show that BL is passive. Motivate why BL can be bounded by a sector $[k_1, k_2] = [0, 1]$.[5]

[5] You may argue that BL as defined here is not a memoryless nonlinearity, which is required for the application of the Circle Criterion in the lecture notes. It is, however, possible to circumvent this problem. If you want to know how, check Theorem (126) on page 361 in Vidyasagar (1993).

3. [1p] Consider the back-lash model BL in a feedback loop:



Here d_{in} and d_{out} represent disturbances. Assume that $G(s)$ is an arbitrary transfer function (that is, not necessarily the one given by the crane control problem discussed previously).

a. Given $\gamma(\text{BL})$ derived in **2**, which constraints are imposed on $G(s)$ by the Small Gain Theorem in order to have a BIBO stable closed-loop system (from $(d_{\text{in}}, d_{\text{out}})$ to $(\dot{\theta}_{\text{in}}, \dot{\theta}_{\text{out}})$)?

b. Which constraints are imposed on $G(s)$ by the Passivity Theorem in order to have a BIBO stable closed-loop system?

c. Which constraints are imposed on $G(s)$ by the Circle Criterion in order to have a BIBO stable closed-loop system?

4. [1p] For the crane control system the transfer function in **3** is equal to

$$G(s) = \frac{K}{s(1+sT)}.$$

Motivate why BIBO stability cannot be concluded from the Small Gain Theorem or the Passivity Theorem. Let $T = 1$ and determine for which $K > 0$ the Circle Criterion leads to BIBO stability.

5. [1p] Simulate the crane control system in Simulink. Download the Simulink model `hw2.mdl` and the short macro `macro.m` from the course homepage to your current directory and start Matlab. Open the Simulink model by running

```
> hw2
```

and compare with the block diagrams in this homework. What disturbance is added in `hw2`? What Δ is chosen? Simulate the system with the controller $K = 0.25$ by running

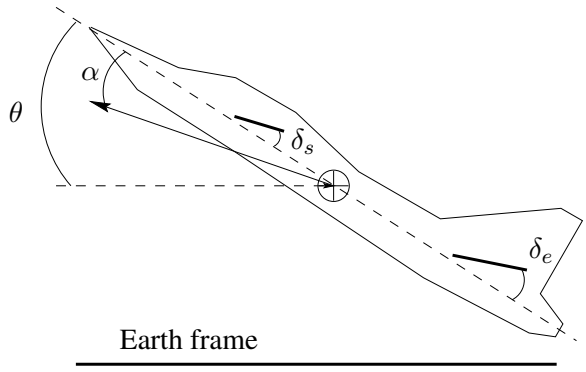
```
> K=0.25; macro
```

Does it seem like the closed-loop system is BIBO stable from d_{in} to $(\dot{\theta}_{\text{in}}, \dot{\theta}_{\text{out}})$? Why cannot the closed-loop system be BIBO stable from d_{in} to $(\theta_{\text{in}}, \theta_{\text{out}})$? Try other controller gains (for instance, $K = 0.5$ and 4). Compare with your conclusions in **4**. Compare your results to the case when the back-lash is neglected.

1.4 Homework 3

Simulation and control of the pitch angle dynamics of a JAS 39 Gripen is discussed in this homework, but first we give some background. The dynamics of an airplane is highly nonlinear. The control system is based on gain scheduling of speed and altitude to compensate for some of the nonlinearities. For the JAS 39 Gripen, linear models have been obtained for approximately 50 different operating conditions. The models are the result of extensive research including wind tunnel experiments. A linear controller is designed for each linear model, and a switching mechanism is used to switch between the different controllers. Many of the parameters in the models vary considerably within normal operation of the aircraft. Two extreme cases are "heavily loaded aircraft at low speed" and "unloaded aircraft at high speed". The case when the velocity of the aircraft is approximately equal to the speed of sound is also critical, because at higher velocities the aircraft is stable while for lower it is unstable. The velocity of an aircraft is specified by the Mach number $M = v/a$, which is the conventional velocity v normalized by the speed of sound a.

The following sketch of the JAS 39 Gripen illustrate the state variables of the system:



The direction in which the aircraft is pointing defines the pitch angle θ. The pitch angle does not necessarily equal the angle of the velocity direction of the aircraft (indicated by an arrow from the center of mass). The difference is the angle of attack α. JAS 39 Gripen has two surfaces for controlling the pitch angle: the spoiler and the elevator. Their corresponding angles are denoted δ_s and δ_e, respectively. The state-space model considered in the homework consists of the seven states

α	angle of attack
$q = \dot{\theta}$	pitch rate
θ	pitch angle
δ_e	elevator angle
δ_s	spoiler angle
x_e	internal elevator state
x_s	internal spoiler state

where the latter two states correspond to dynamics in the rudder servos. The control inputs are

u_e	elevator command
u_s	spoiler command

We consider a linear state-space model for normal load at $M = 0.6$ and altitude 1 km. This model has been released by SAAB Military Aircraft and discussed in a master thesis at Chalmers [1]. The model is given by

$$\dot{x} = Ax + Bu,$$

where

$$x = (\alpha, q, \theta, \delta_e, \delta_s, x_e, x_s)^T, \qquad u = (u_e, u_s)^T.$$

Download the Matlab files from the course homepage to your current directory and introduce the state-space model into the Matlab workspace by running

```
>> jasdata
```

1. [1p] Which are the dynamical modes of the JAS 39 Gripen state-space model, i.e., which are the eigenvalues of A? Is the model stable? Which eigenvalues correspond to the flight dynamics and which correspond to the rudder dynamics?

The aircraft is controlled by linear state feedback

$$u(t) = -Lx(t) + (K_f, K_f)^T u^f_{\text{pilot}}(t),$$

where the matrix L is derived using linear quadratic control theory and the scalar K_f is chosen such that the steady-state gain is correct. The internal elevator and spoiler states are not used in the controllers, so $L_{16} = L_{17} = L_{26} = L_{27} = 0$.[6] The feedback stabilizes the angle of attack α and the pitch rate q. The pitch angle θ is not stabilized (and not used in the controller since $L_{13} = L_{23} = 0$), but the control of this mode is left for the pilot. The signal u^f_{pilot} is the filtered pilot command signal:

$$u^f_{\text{pilot}} = \frac{1}{T_f s + 1} u_{\text{pilot}}.$$

Write

```
>> planemodel
```

to see a Simulink model of the aircraft. Match the blocks in the model with the equations above.

2. [1p] Choose a nominal design for the state feedback by typing

```
>> design1
```

Look at the L-matrix. Which states are used in the feedback loop? Which are the eigenvalues of $A-BL$? Why is there an eigenvalue close to the origin?

The pilot may be modeled as a PD-controller with a time delay of 0.3 s. Argue why this is a reasonable model. Let

$$u_{\text{pilot}} = K_p \frac{1 + T_d s}{1 + T_d/Ns} e^{-0.3s} (\theta_{\text{ref}} - \theta),$$

where θ_{ref} corresponds to the desired pitch angle. Set $K_p = 0.2$, $T_d = 0.5$, and $N = 10$. Simulate the closed-loop system including the pilot. Check the magnitudes of the rudder angles (`plot(t,x(:,[4 5]))`). Why is it important that the rudder angles are not too large?

The PD-controller pilot model can be seen as a rational pilot. In an emergency situation, however, the pilot may panic and try to compensate the error $\theta_{\text{ref}} - \theta$ with maximal command signals.[7] This behavior may induce oscillations in the system. They are called pilot induced oscillations (PIO) and got considerable attention after the crash in Stockholm in 1993. Next we do a simplified analysis illustrating what happened.

[6] In reality there are no measurements of the rudder angles δ_e and δ_s. They are estimated using a Kalman filter.

[7] Imagine yourself in a balancing act. When you are in control, you can keep the balance using very small motions, but as soon as you are a little out of control, your movements tends to be very large. This is typical for systems with slow unstable dynamics, which cannot react fast enough to the control commands signals.

3. [1p] In order to analyze the PIO mode, we will replace the PD-controller pilot by a relay model. The pilot then gives maximal joystick commands based on the sign of θ. Such a "relay pilot" can be found in the Simulink pilot model library; run

```
>> pilotlib
```

Plot the Nyquist curve of the linear system from u_{pilot} to θ. This can be done by deleting the feedback path from θ, connecting an input and an output at appropriate places (inputs and outputs are found in the `simulink` libraries), saving the system to a new file and using the `linmod` and `nyquist` commands.

Change pilot in the plane model by deleting the PD-controller pilot and inserting the "relay pilot". The describing function for a relay is

$$N(A) = \frac{4D}{\pi A}$$

What is D for the "relay pilot"? Use describing function analysis to predict amplitude and frequency of a possible limit cycle. Simulate the system. How good is the prediction?

As you saw, the amplitude of the PIO is moderate. This is because the flight condition is high speed and high altitude, and thus not extreme. Let us anyway discuss ways to reduce PIO.

4. [1p] Use `design2` to change L and K_f to a faster design. Is the PIO amplitude decreased? Make the pilot filter faster by reducing the filter time constant to $T_f = 0.03$ (`design3`). Is the PIO amplitude decreased? Discuss the results using the describing function method and thus plot the Nyquist curves from $u_{\texttt{pilot}}$ to θ. Are there any drawbacks with the design that gives smallest PIO amplitude?

5. [1p] Suggest a control strategy for reducing PIO in JAS 39 Gripen, with minimal influence on the pilots ability to manually control the plane. Analyze the performance of your strategy and compare it to the previous two designs. It should outperform the previous ones.

Extra [0p] There are no rate limitations on the rudders in the discussed aircraft model. Rate limitations were, however, part of the problems with the JAS 39 Gripen control system. Introduce rate limitations as in the article [2], and investigate what happens to the limit cycle. Try to understand the idea of the (patented) nonlinear filter.

References

[1] Axelsson, L., *Reglerstudier av back-up regulator för JAS 39 Gripen*, MSc Thesis, CTH/SAAB Flygdivision, 1992.

[2] Rundquist, L., K. Ståhl-Gunnarsson, and J. Enhagen, "Rate Limiters with Phase Compensation in JAS 39 Gripen," European Control Conference, 1997.

Chapter 2

Exercises

2.1 Nonlinear Models and Simulation

EXERCISE 1.1

The nonlinear dynamic equation for a pendulum is given by

$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta},$$

where l is the length of the pendulum, m is the mass of the bob, and θ is the angle subtended by the rod and the vertical axis through the pivot point, see Figure 1.1.

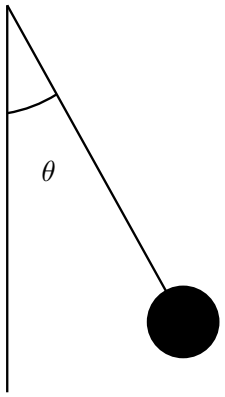


Figure 2.1: The pendulum in Exercise 1.1

(a) Choose appropriate state variables and write down the state equations.

(b) Find all equilibria of the system.

(c) Linearize the system around the equilibrium points, and determine whether the system equilibria are stable or not.

□

EXERCISE 1.2

We consider a bar rotating, with friction due to viscosity, around a point where a torque is applied. The non linear dynamic equation of the system is given by:

$$ml^2\ddot{\theta} = -mgl\sin(\theta) - kl^2\dot{\theta} + C$$

where θ is the angle between the bar and the vertical axis,
C is the torque ($C > 0$),
k is a viscous friction constant,
l is the length of the bar
and m is the mass of the bar.

(a) Choose appropriate state variables and write down the state equations.

(b) Find all equilibria of the system, assuming that $\frac{C}{mlg} < 1$.

(c) Linearize the system around the equilibrium points and determine the eigenvalues for each equilibrium.

□

EXERCISE 1.3

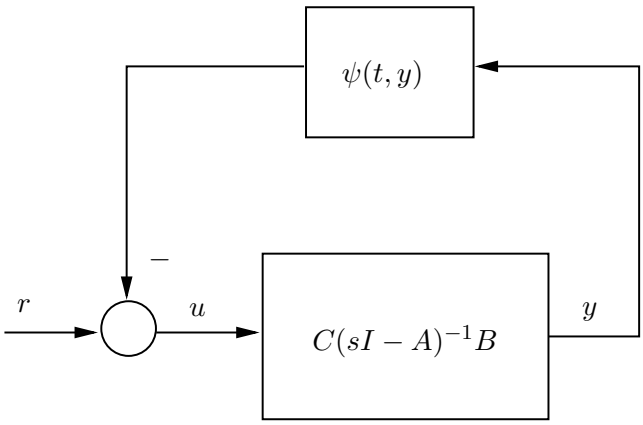


Figure 2.2: The feedback system in Exercise 1.3

Figure 2.2 shows a feedback connection of a linear time-invariant system and a nonlinear time-varying element. The variables r, u and y are vectors of the same dimension, and $\psi(t, y)$ is a vector-valued function.

(a) Find a state-space model with r as input and y as output.

(b) Rewrite the pendulum model from Exercise 1.1 into the feedback connection form described above.

□

EXERCISE 1.4

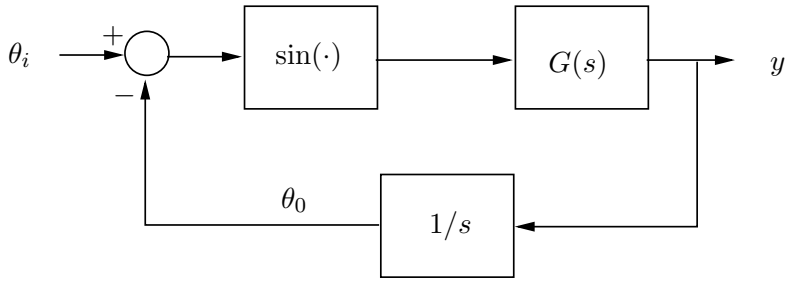


Figure 2.3: The phase-locked loop in Exercise 1.4

A phase-locked loop can be represented by the block diagram of Figure 2.3. Let $\{A, B, C\}$ be a state-space representation of the transfer function $G(s)$. Assume that all eigenvalues of A have negative real parts, $G(0) \neq 0$ and that θ_i is constant. Let z be the state of the realization $\{A, B, C\}$.

(a) Show that

$$\dot{z} = Az + B \sin e$$
$$\dot{e} = -Cz$$

is a state equation for the closed-loop system.

(b) Find all equilibrium points of the system.

(c) Show that if $G(s) = 1/(\tau s+1)$, the closed-loop model coincides with the model of a pendulum equation.

□

EXERCISE 1.5

The nonlinear dynamic equations for a single-link manipulator, see Figure 1.5, with flexible joints, damping ignored, is given by

$$I\ddot{q}_1 + MgL \sin q_1 + k(q_1 - q_2) = 0$$
$$J\ddot{q}_2 - k(q_1 - q_2) = u,$$

where q_1 and q_2 are angular positions, I and J are moments of inertia, k is a spring constant, M is the total mass, L is a distance, and u is a torque input. Choose state variables for this system and write down the state

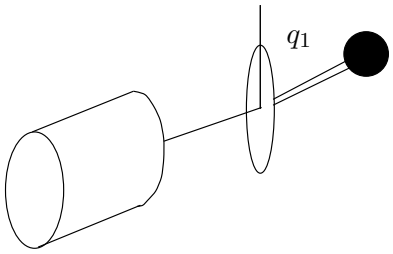


Figure 2.4: The flexible manipulator in Exercise 1.5

equations.

□

EXERCISE 1.6

A synchronous generator connected to an infinite bus can be modeled by

$$M\ddot{\delta} = P - D\dot{\delta} - \eta_1 E_q \sin\delta$$
$$\tau\dot{E}_q = -\eta_2 E_q + \eta_3 \cos\delta + E_{FD},$$

where δ is the angle in radians, E_q is voltage, P is mechanical input power, E_{FD} is field voltage (input), D is damping coefficient, M is inertial coefficient, τ is a time constant, and η_1, η_2, and η_3 are constant parameters.

(a) Using δ, $\dot{\delta}$, and E_q as state variables, find the state equation.

(b) Suppose that τ is relatively large so that $\dot{E}_q \approx 0$. Show that assuming E_q to be constant reduces the model to a pendulum equation.

(c) For the simplified model, derived in (b), find all equilibrium points.

□

EXERCISE 1.7

A Mass-Spring system is shown in Figure 2.5. The displacement, y, from a reference point is given by

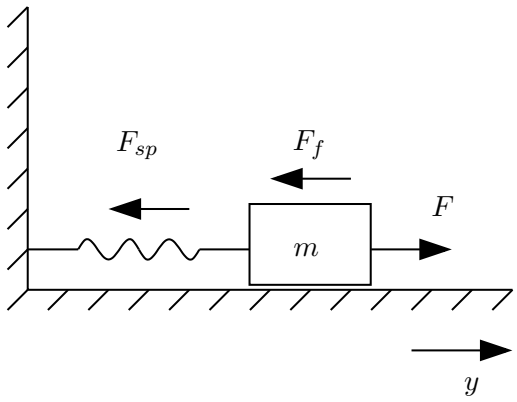


Figure 2.5: The mass and spring system in Exercise 1.7

$$m\ddot{y} + F_f + F_{sp} = F \tag{2.1}$$

where F_f is the friction force, F_{sp} is the spring force and F is the applied force. In reality both F_f and F_{sp} are nonlinear functions of y, however, we will use somewhat idealized relations here. First we assume a linear spring such that $F_{sp} = ky$. Then we assume that the friction force can be divided into two parts, friction due to viscosity F_{fv} (the mass moves in air) and dry friction against the surface, F_{fd}. We further assume that $F_{fv} = c\dot{y}$ and

$$F_{fd} = \begin{cases} -\mu_k mg & \dot{y} < 0 \quad \text{slipping friction} \\ F_s & \dot{y} = 0 \quad \text{rest friction} \\ \mu_k mg & \dot{y} > 0 \quad \text{slipping friction} \end{cases}$$

a) Assume $F = 0$, write the resulting state-space equation when

$$F_{sp} = ky$$

and

$$F_f = F_{fv} + F_{fd} = c\dot{y} + \eta(y, \dot{y})$$

where

$$\eta(y, \dot{y}) = \begin{cases} \mu_k mg\text{sign}(\dot{y}) & |\dot{y}| > 0 \quad \text{slipping friction} \\ -ky & \dot{y} = 0 \quad \text{rest friction} \end{cases}$$

b) Characterize the equilibria.

c) What can we benefit from the idealized computations above?

□

EXERCISE 1.8

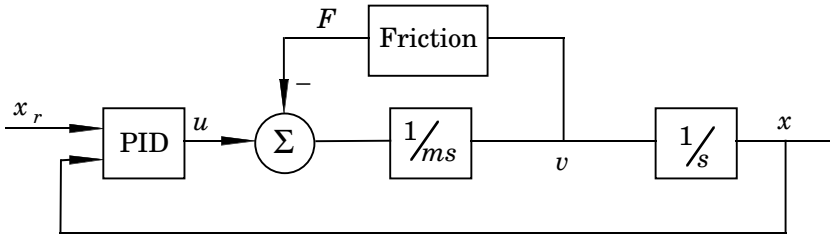


Figure 2.6: Control system with friction in Exercise 1.8.

Figure 2.6 shows a block diagram of a mechanical system with friction under PID control. The friction block is given by

$$F(v) = F_0\text{sign}(v)$$

Let $x_r = 0$ and rewrite the system equations into feedback connection form.

□

EXERCISE 1.9

Which of the following models of physical systems are nonlinear and why?

1. A motor with a valve. The valve has the characteristic $f(u) = u^2$.
2. An unknown system. The only thing you know is that when you send in a sine wave with a quite high amplitude the output signal is a sine-like wave with the tops "cut-off".
3. A falling apple. There is no air resistance.
4. A system with the following characteristics:
   in: x —> out: y
   in: a*x —> out: a*y
5. The distribution function you get after tossing a coin 25 times.
6. The control system for the aircraft JAS 39 Gripen.
7. A modern light-switch.

□

EXERCISE 1.10

(a) Consider the scalar nonlinear control system

$$\dot{x} = xu \tag{2.2}$$

where the $u = u(x)$ is a state-feedback control. Determine a condition that ensures that the origin is an asymptotically stable equilibrium.

(b) Does there exist a linear control law $u(x) = kx$ that ensures that (2.2) is asymptotically stable?

(c) Does there exist an affine control law $u(x) = k_1 x + k_2$ that ensures that (2.2) is asymptotically stable?

(d) Show that $u(x) = -x^2$ gives that (2.2) is globally asymptotically stable.

(e) We want to analyze an n-dimensional dynamical system of the form

$$\dot{x}(t) = (A + B\Delta C)x(t) \tag{2.3}$$

where Δ is a scalar but unknown constant. The matrices A, B, C are constant and of appropriate dimension. Show that the differential equation (2.3) can be analyzed as a linear system with a constant gain Δ in the feedback loop. Draw an illustrative block diagram.

□

2.2 Computer exercise: Simulation

EXERCISE 2.1

Consider the pendulum model given in Exercise 1.1.

(a) Make a simulation model of the system in Simulink. Simulate the system from various initial states. Is the system stable? Is the equilibrium point unique? Explain the physical intuition behind your findings.

(b) Use the function `linmod` in Matlab to find the linearized models for the equilibrium points. Compare with the linearizations that you derived in Exercise 1.1.

(c) Use a phase plane tool (such as `pptool` in ICTools or `pplane`) to construct the phase plane of the system. Compare with the results from (a).

□

EXERCISE 2.2

Consider the model of a motor with a nonlinear valve in Figure 2.7. The valve characteristic is given by $f(x) = x^2$.

(a) Simulate the system in Figure 2.7 using Simulink. Try different input signals r (both amplitude and shape).

(b) How is stability related to the input signals?

(c) Switch the nonlinearity to something else (e.g., a saturation). Repeat a and b.

□

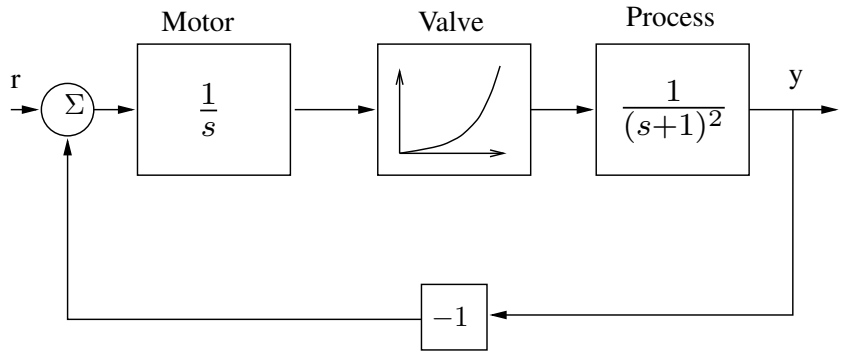


Figure 2.7: Block diagram of system in Exercise 2.2.

EXERCISE 2.3

Simulate the example from the lecture with two tanks, using the models

$$\dot{h} = (u - q)/A$$
$$q = a\sqrt{2g}\sqrt{h},$$

where h is the liquid level, u is the inflow to the tank, q the outflow, A the cross section area of the tank, a the area of the outflow and g the acceleration due to gravity, see Figure 2.8. Use a step input flow. Make step

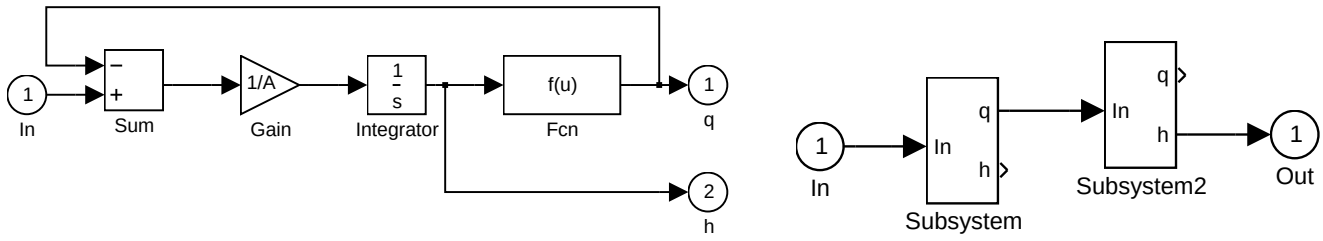


Figure 2.8: The flow system in Exercise 2.3

change in u from $u = 0$ to $u = 1$. Make step change from $u = 1$ to $u = 0$. Is the process linear? Linearize the system around $h_1 = h_2 = 0.1$. Use $A_1 = A_2 = 3 \times 10^{-3}$, $a_1 = a_2 = 7 \times 10^{-6}$. What are the time constants of the linearized system?

□

EXERCISE 2.4

Construct a rate limiter (i.e., a system that limit the rate of change of a signal) by adding one of the nonlinear models in Simulink to the block in Figure 2.9. Simulate and verify your results.

□

EXERCISE 2.5

Simulate the Lorentz system using the Differential Equation Editor `dee` in Matlab where it is provided as a demo. Change the initial condition of one state by 1%. How does that effect the simulation?

□

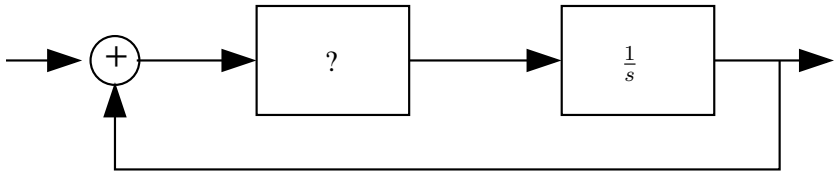


Figure 2.9: Block diagram of system in Exercise 2.4.

2.3 Linearization and Phase-Plane Analysis

EXERCISE 3.1

For each of the following systems, find and classify all equilibrium points.

(a)
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + x_1^3/6 - x_2$$

(b)
$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = 0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3$$

(c)
$$\dot{x}_1 = (1 - x_1)x_1 - 2x_1x_2/(1 + x_1)$$
$$\dot{x}_2 = (1 - x_2/(1 + x_1))x_2$$

(d)
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + x_2(1 - 3x_1^2 - 2x_2^2)$$

(e)
$$\dot{x}_1 = -x_1 + x_2(1 + x_1)$$
$$\dot{x}_2 = -x_1(1 + x_1)$$

(f)
$$\dot{x}_1 = (x_1 - x_2)(x_1^2 + x_2^2 - 1)$$
$$\dot{x}_2 = (x_1 + x_2)(x_1^2 + x_2^2 - 1)$$

□

EXERCISE 3.2

For all positive values of a, b and c, determine the equilibrium points of the system

$$\dot{x}_1 = ax_1 - x_1x_2$$
$$\dot{x}_2 = bx_1^2 - cx_2$$

and determine the type of equilibrium.

□

EXERCISE 3.3

For each of the following systems, construct the phase portrait, preferably using a computer program, and discuss the qualitative behaviour of the system.

(a) $$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - 2\tan^{-1}(x_1 + x_2) \end{aligned}$$

(b) $$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_2(1 - 3x_1^2 - 2x_2^2) \end{aligned}$$

(c) $$\begin{aligned} \dot{x}_1 &= 2x_1 - x_1x_2 \\ \dot{x}_2 &= 2x_1^2 - x_2 \end{aligned}$$

□

EXERCISE 3.4

Saturations constitute a severe restriction for stabilization of system. Figure 2.10 shows three phase portraits, each corresponding to one of the following linear systems under saturated feedback control.

(a) $$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 + x_2 - \text{sat}(2x_1 + 2x_2) \end{aligned}$$

(b) $$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + 2x_2 - \text{sat}(3x_2) \end{aligned}$$

(c) $$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 - 2x_2 - \text{sat}(-x_1 - x_2) \end{aligned}$$

Which phase portrait belongs to what system?

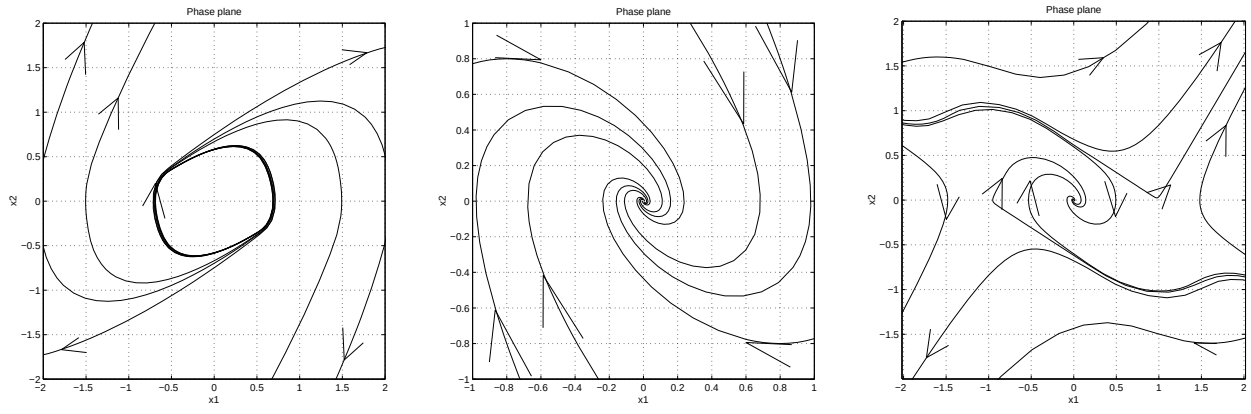


Figure 2.10: Phase portraits for saturated linear systems in Exercise 3.4

□

EXERCISE 3.5

The following system

$$\begin{aligned}\dot{x}_1 &= (u - x_1)(1 + x_2^2)\\ \dot{x}_2 &= (x_1 - 2x_2)(1 + x_1^2)\\ y &= x_2\end{aligned}$$

is controlled by the output feedback

$$u = -Ky$$

(a) For all values of the gain K, determine the equilibrium points of the closed loop system.

(b) Determine the equilibrium character of the origin for all values of the parameter K. Determine in particular for what values the equilibrium of the closed loop system is (locally) asymptotically stable.

□

EXERCISE 3.6

As an application of phase plane analysis, consider the model of a synchronous generator derived in Exercise 1.5c

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \frac{P}{M} - \frac{D}{M}x_2 - \frac{\eta_1}{M}E_q \sin x_1.\end{aligned}$$

Determine all equilibrium points of the system, and determine the type of equilibrium.

□

EXERCISE 3.7

Linearize

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1(1 - x_1^2 - x_2^2)\\ \dot{x}_2 &= -x_1 + x_2(1 - x_1^2 - x_2^2)\end{aligned}$$

around the trajectory $(x_1, x_2) = (\sin t, \cos t)$.

□

EXERCISE 3.8

Linearize the ball-on-beam equation

$$\frac{7}{5}\ddot{x} - x\dot{\phi}^2 = g \sin\phi + \frac{2r}{5}\ddot{\phi},$$

around the trajectory

$$\big(\phi(t), x(t)\big) = \left(\phi_0, \frac{5g}{7}\sin(\phi_0) \cdot \frac{t^2}{2}\right)$$

□

EXERCISE 3.9

Use a simple trigonometry identity to help find a nominal solution corresponding to $u(t) = \sin(3t), y(0) = 0, \dot{y}(0) = 1$ for the eqution

$$\ddot{y} + \frac{4}{3}y^3(t) = -\frac{1}{3}u(t).$$

Linearize the equation around this nominal solution.

□

EXERCISE 3.10

A regular bicycle is mainly controlled by turning the handle bars.[1] Let the tilt sideways of the bicycle be θ radians and the turning angle of the front wheel be β radians. The tilt of the bike obeys the following nonlinear differential equation:

$$\ddot{\theta} = \frac{mg\ell}{J}\sin\theta + \frac{m\ell V_0^2}{bJ}\cos\theta\tan\beta + \frac{am\ell V_0}{bJ}\cdot\frac{\cos\theta}{\cos^2\beta}u$$
$$\dot{\beta} = u,$$

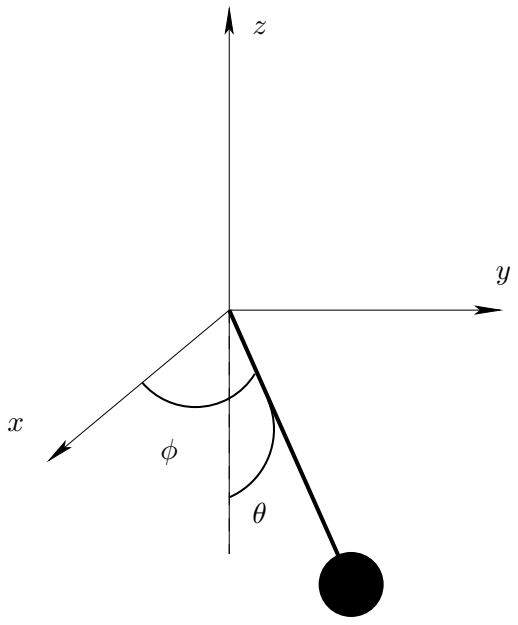
where $V_0 > 0$ is the (constant) velocity of the bicycle, and m, g, ℓ, J, a, and b are other positive constants. The control u is the angular velocity applied at the handle bars. To gain some understanding of the principal behavior of the bicycle, we study its linearization. Linearize the tilt equation around the equilibrium point $(\theta, \beta, u) = (0, 0, 0)$.

Derive the transfer function $G(s)$ from u to θ. Determine the poles and zeros of $G(s)$. Is the bike locally stable?

□

EXERCISE 3.11

Consider the spherical pendulum depicted below (similar to the one Professor Calculus uses in Tintin):



[1] The rider can also change the center of mass, but we will ignore that control actuation here.

Here θ denotes the angular position of the pendulum with respect to the z-axis and ϕ denotes the angular position in the x-y plane. A (normalized) model of the spherical pendulum is given by

$$\begin{aligned}\ddot{\theta} - \dot{\phi}^2 \sin\theta\cos\theta + \sin\theta &= 0\\ \ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta &= 0\end{aligned}$$

(a) Specify the pendulum dynamics on first-order form

$$\dot{x} = f(x)$$

and give condition on when the first-order form is equivalent to the original equations.

(b) Determine all equilibria for the pendulum system. (You don't have to determine stability properties.) *[Hint: Consider the original equations.]*

(c) Show that $(\theta(t), \phi(t)) = (\pi/3, t\sqrt{2})$ is a trajectory of the pendulum system.

(d) Linearize the pendulum system about the trajectory in (c).

□

2.4 Lyapunov Stability

EXERCISE 4.1

Consider the scalar system

$$\dot{x} = ax^3$$

(a) Show that Lyapunov's linearization method fails to determine stability of the origin.

(b) Use the Lyapunov function

$$V(x) = x^4$$

to show that the system is stable for $a < 0$ and unstable for $a > 0$.

(c) What can you say about the system for $a = 0$?

□

EXERCISE 4.2

Consider the pendulum equation

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{l}x_2.\end{aligned}$$

(a) Assume zero friction, *i.e.* let $k = 0$, and show that the origin is stable. (*Hint*. Show that the energy of the pendulum is constant along all system trajectories.)

(b) Can the pendulum energy be used to show asymptotic stability for the pendulum with non-zero friction, $k \neq 0$? If not, modify the Lyapunov function to show asymptotic stability of the origin.

□

EXERCISE 4.3

Consider the system

$$\ddot{x} + d\dot{x}^3 + kx = 0.$$

Show that

$$V(x) = \frac{1}{2}\left(kx^2 + \dot{x}^2\right)$$

is a Lyapunov function. Is the system locally stable, locally asymptotically stable, and globally asymptotically stable?

□

EXERCISE 4.4

Consider the linear system

$$\dot{x} = Ax = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} x$$

(a) Compute the eigenvalues of A and verify that the system is asymptotically stable

(b) From the lectures, we know that an equivalent characterization of stability can be obtained by considering the Lyapunov equation

$$A^T P + PA = -Q$$

where $Q = Q^T$ is any positive definite matrix. The system is asymptotically stable if and only if the solution P to the Lyapunov equation is positive definite.

(i) Let

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

Verify by completing squares that $V(x) = x^T P x$ is a positive definite function if and only if

$$p_{11} > 0, \qquad p_{11}p_{22} - p_{12}^2 > 0$$

(ii) Solve the Lyapunov function with Q as the identity matrix. Is the solution P a positive definite matrix?

(c) Solve the Lyapunov equation in Matlab.

□

EXERCISE 4.5

As you know, the system

$$\dot{x}(t) = Ax(t), \qquad t \geq 0,$$

is asymptotically stable if all eigenvalues of A have negative real parts. It might be tempted to conjecture that the time-varying system

$$\dot{x}(t) = A(t)x(t), \qquad t \geq 0, \tag{2.4}$$

is asymptotically stable if the eigenvalues of $A(t)$ have negative real parts for all $t \geq 0$. This is *not* true.

(a) Show this by explicitly deriving the solution of

$$\dot{x} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} x, \qquad t \geq 0.$$

(b) The system (2.4) is however stable if the eigenvalues of $A(t) + A^T(t)$ have negative real parts for all $t \geq 0$. Prove this by showing that $V = x^T x$ is a Lyapunov function.

□

EXERCISE 4.6

A student is confronted with the nonlinear differential equation

$$\ddot{x} + \frac{2x}{(1+x^2)^2} = 0$$

and is asked to determine whether or not the equation is stable. The students think "this is an undamped mass-spring system – the spring is nonlinear with a spring constant of $2/(1+x^2)^2$". The student re-writes the system as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{-2x_1}{(1+x_1^2)^2} \end{aligned}$$

and constructs the obvious Lyapunov function

$$V(x) = \int_0^{x_1} \frac{2\zeta}{(1+\zeta^2)^2} d\zeta + \frac{1}{2}x_2^2.$$

The student declares, "V is positive definite, because everywhere in $\mathbb{R}^2$, $V(x) \geq 0$, and $V(x) = 0$ only if $x = 0$." The student ascertains that $\dot{V} \leq 0$ everywhere in $\mathbb{R}^2$ and concludes, "the conditions for Lyapunov's theorem are satisfied, so the system is globally stable about $x = 0$".

(a) Sadly, there is a mistake in the student's reasoning. What is the mistake?

(b) Perhaps the student has merely made a poor choice of Lyapunov function, and the system really is globally asymptotically stable. Is there some other Lyapunov function that can be used to show global stability? Find such a function, or show that no such function exists.

□

EXERCISE 4.7

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1^7 \left(x_1^4 + 2x_2^2 - 10\right) \\ \dot{x}_2 &= -x_1^3 - 3x_2^5 \left(x_1^4 + 2x_2^2 - 10\right) \end{aligned} \tag{2.5}$$

(a) For which values of C is the set $\left\{x \in \mathbb{R}^2 | x_1^4 + 2x_2^2 \leq C\right\}$ invariant with respect to (2.5)?

(b) Use LaSalle's invariant set theorem to analyze the behavior of the system. (Hint: Use $V(x) = (x_1^4 + 2x_2^2 - 10)^2$).

(c) Repeat (a) and (b), but now with the set $\{x \in \mathbb{R}^2 | \epsilon \leq x_1^4 + 2x_2^2 \leq C\}$. What can be said about the stability of the origin?

□

EXERCISE 4.8

Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2x_1 - 2x_2 - 4x_1^3$$

Use the function

$$V(x) = 4x_1^2 + 2x_2^2 + 4x_1^4$$

to show that

(a) the system is globally stable around the origin.

(b) the origin is globally asymptotically stable.

□

EXERCISE 4.9

Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - \text{sat}(2x_1 + x_2).$$

(a) Show that the origin is asymptotically stable.

(b) Show that all trajectories starting in the first quadrant to the right of the curve

$$x_1 x_2 = c$$

for sufficiently large c, cannot reach the origin. (*Hint:* Consider $V(x) = x_1 x_2$; calculate $\dot{V}(x)$ and show that on the curve $V(x) = c$, the derivative $\dot{V}(x) > 0$ when c is sufficiently large.)

(c) Show that the origin is not globally asymptotically stable.

□

EXERCISE 4.10

In general, it is non-trivial to find a Lyapunov function for a given nonlinear system. Several different methods have been derived for specific classes of systems. In this exercise, we will investigate the following method, known as Krasovskii's method.

Consider systems on the form

$$\dot{x} = f(x)$$

with $f(0) = 0$. Assume that $f(x)$ is continuously differentiable and that its Jacobian, $\partial f / \partial x$, satisfies

$$P\frac{\partial f}{\partial x}(x) + \left(\frac{\partial f}{\partial x}(x)\right)^T P \leq -I$$

for all $x \in \mathbb{R}^n$, and some matrix $P = P^T > 0$. Then, the origin is globally asymptotically stable with $V(x) = f^T(x) P f(x)$ as Lyapunov function.

Prove the validity of the method in the following steps.

(a) Verify that $f(x)$ can be written as

$$f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x) \cdot x \, d\sigma.$$

and use this representation to show that the assumptions imply

$$x^T P f(x) + f^T(x) P x \leq -x^T x, \qquad \forall x \in \mathbb{R}^n$$

(b) Show that $V(x) = f^T(x) P f(x)$ is positive definite for all $x \in \mathbb{R}^n$.

(c) Show that $V(x)$ is radially unbounded.

(d) Using $V(x)$ as a Lyapunov function candidate, show that the origin is globally asymptotically stable.

□

2.5 Input-Output Stability

EXERCISE 5.1

The norms used in the definitions of stability need not be the usual Euclidian norm. If the state-space is of finite dimension n (*i.e.*, the state vector has n components), stability and its type are independent of the choice of norm (all norms are "equivalent"), although a particular choice of norm may make analysis easier. For $n = 2$, draw the unit balls corresponding to the following norms.

(a) $||x||^2 = x_1^2 + x_2^2$ (Euclidian norm)

(b) $||x||^2 = x_1^2 + 5x_2^2$

(c) $||x|| = |x_1| + |x_2|$

(d) $||x|| = \sup(|x_1|, |x_2|)$

Recall that a "ball" $B(x_0, R)$, of center x_0 and radius R, is the set of x such that $||x - x_0|| \leq R$, and that the unit ball is $B(0, 1)$.

□

EXERCISE 5.2

Compute the norms $\|\cdot\|_\infty$ and $\|\cdot\|_2$ for the signals

(a)

$$y(t) = \begin{cases} a \sin(t) & t > 0 \\ 0 & t \leq 0 \end{cases}$$

(b)

$$y(t) = \begin{cases} \frac{1}{t} & t > 1 \\ 0 & t \leq 1 \end{cases}$$

(c)

$$y(t) = \begin{cases} e^{-t}(1 - e^{-t}) & t > 0 \\ 0 & t \leq 0 \end{cases}$$

□

EXERCISE 5.3

Consider the linear system

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}.$$

Compute the gain of $\|G\|$ for all $\omega_0 > 0$ and $\zeta > 0$.

□

EXERCISE 5.4

Consider a linear time invariant system $G(s)$ interconnected with a static nonlinearity $\psi(y)$ in the standard form. Compare the Nyquist, Small Gain, Circle Criterion and Passivity theorem with respect to the following issues.

(a) What are the restrictions that must be imposed on $\psi(y)$ in order to apply the different stability criteria?

(b) What restrictions must be imposed on the Nyquist curve of the linear system in order to apply the stability criteria above?

□

EXERCISE 5.5

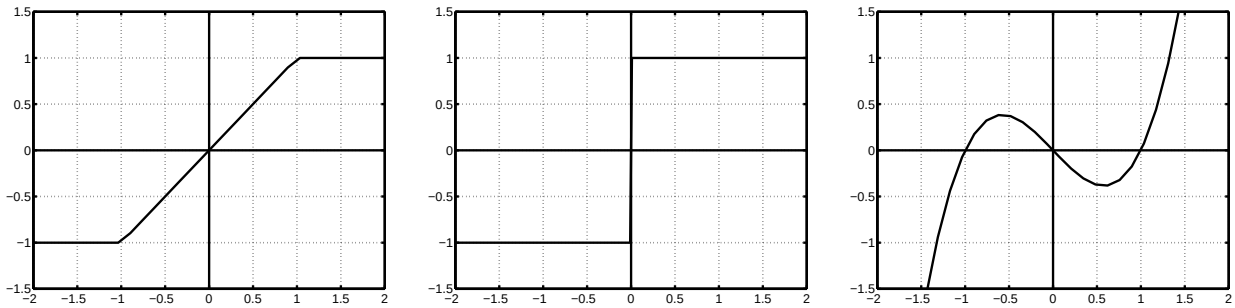


Figure 2.11: Static nonlinearities in Exercise 5.5.

Consider the static nonlinearities shown in Figure 2.11. For each nonlinearity,

(a) determine the minimal sector $[\alpha, \beta]$,

(b) determine the gain of the nonlinearity,

(c) determine if the nonlinearity is passive.

□

EXERCISE 5.6

The Nyquist curve of

$$G(s) = \frac{4}{(s-1)(s/3+1)(s/5+1)}$$

is shown in Figure 2.12. Determine a possible stability sector (α, β).

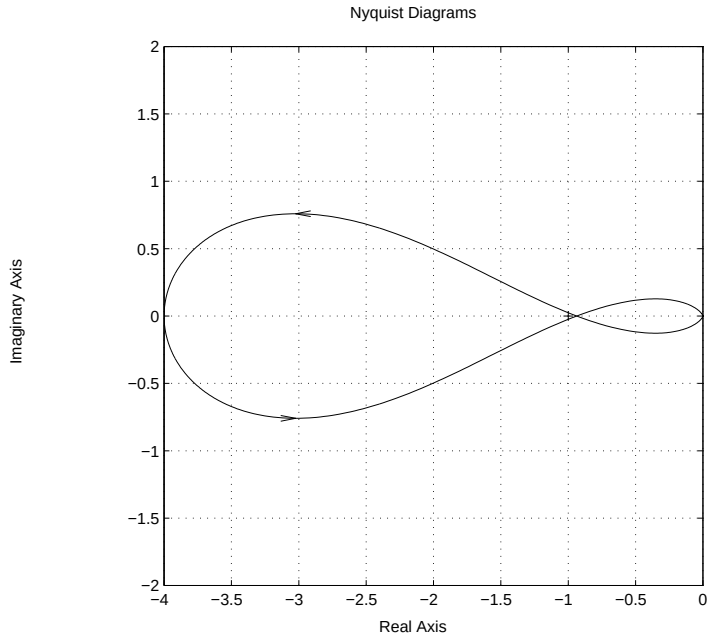


Figure 2.12: The Nyquist-curve in Exercise 5.6

□

EXERCISE 5.7

The singular values of a matrix A are denoted $\sigma_i(A)$.

(a) Use matlab to compute $\sigma(A)$ for

$$A = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}.$$

(b) The maximal singular value is defined by

$$\sigma_1(A) = \sup_x \frac{|Ax|}{|x|}.$$

Show that $\sigma_1(AB) \leq \sigma_1(A)\sigma_1(B)$.

□

EXERCISE 5.8

In the previous chapter, we have seen how we can use Lyapunov functions to prove stability of systems. In this exercise, we shall see how another type of auxillary functions, called *storage functions*, can be used to assess passivity of a system.

Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{2.6}$$

with zero initial conditions, $x(0) = 0$.

(a) Show that if we can find a *storage function* $V(x, u)$ with the following properties

- $V(x, u)$ is continuously differentiable.
- $V(0) = 0$ and $V(x, u) \geq 0$ for $x \neq 0$.
- $u^T y \geq \dot{V}(x, u)$.

then, the system (2.6) is passive.

(b) Show that

$$V(x) = \frac{1}{2} x^T P x$$

is OK as a storage function where

$$P : \begin{cases} A^T P + PA = -Q \\ B^T P = C \end{cases}$$

and P and Q are symmetric positive definite matrices.

□

EXERCISE 5.9

Let P be the solution to

$$A^T P + PA = -I,$$

where A is a asymptotically stable matrix. Show that $G(s) = B^T P(sI - A)^{-1} B$ is passive. (*Hint.* Use the function $V(x) = x^T P x$.)

□

EXERCISE 5.10

A DC motor is characterized by

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -\omega + \eta, \end{aligned}$$

where θ is the shaft angle and η is the input voltage. The dynamic controller

$$\begin{aligned} \dot{z} &= 2(\theta - z) - \text{sat}(\theta - z) \\ \eta &= z - 2\theta \end{aligned}$$

is used to control the shaft position. Use any method you like to prove that $\theta(t)$ and $\omega(t)$ converge to zero as $t \to \infty$.

□

EXERCISE 5.11

(a) Let $u_c(t)$ be an arbitrary function of time and let $H(\cdot)$ be a passive system. Show that

$$y(t) = u_c(t) \cdot H(u_c(t)u(t))$$

is passive from u to y.

(b) Show that the following adaptive system is stable

$$e(t) = G(s)\left(\theta(t) - \theta^0\right) u_c(t)$$
$$\dot{\theta}(t) = -\gamma u_c(t) e(t),$$

if $\gamma > 0$ and $G(s)$ is strictly passive.

□

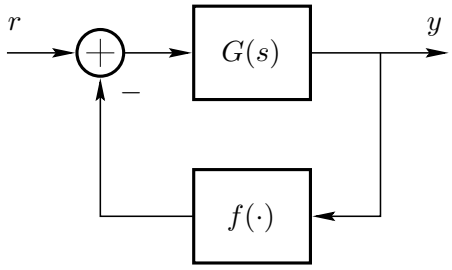
EXERCISE 5.12

(a) Consider the feedback system below with

$$G(s) = \frac{\Delta}{s(s+1)}$$

and

$$f(y) = K \arctan(y)$$



For what values of the uncertain (but constant) parameters $\Delta > 0$ and $K > 0$ does BIBO stability for the feedback system follow from the Circle Criterion?

(b) For which values of $\Delta > 0$ and $K > 0$ does the direct application of the Small Gain Theorem prove BIBO stability for the feedback system in (b)? (*Hint:* Is the Small Gain Theorem applicable?)

□

EXERCISE 5.13

Consider a second-order differential equation

$$\dot{x} = f(x),$$

where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a $\mathbf{C}^1$ (continuously differentiable) function such that $f(0) = 0$.

Determine if the following statements are true or false. You have to motivate your answers to get credits. The motivation can for example be a short proof, a counter example (Swedish: *motexempel*), or a reference to a result in the lecture notes.

(a) Suppose the differential equation (5.13) has more than one equilibria, then none of them can be globally asymptotically stable.

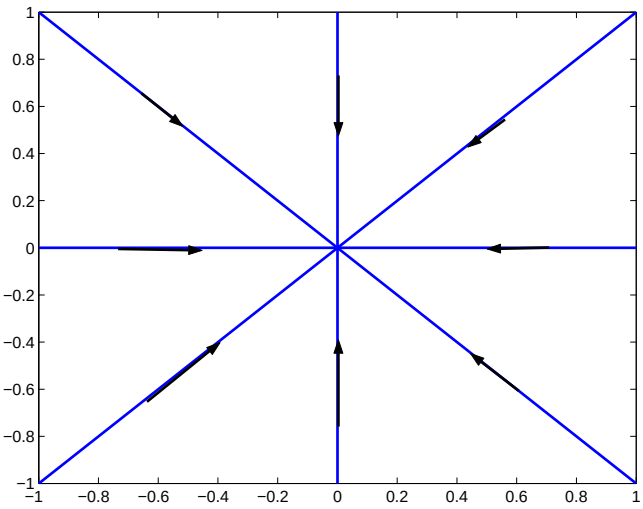
(b) The differential equation (5.13) cannot have a periodic solution.

(c) If the eigenvalues of

$$\frac{\partial f}{\partial x}(0)$$

are strictly in the left half-plane, then the nonlinear system (5.13) is asymptotically stable.

(d) There exists f such that the differential equation (5.13) have a phase portrait that looks like this:



(e) For any initial condition $x(0) = x_0 \neq 0$, the solution $x(t)$ of (5.13) cannot reach $x = 0$ in finite time, that is, there does not exist $0 < T < \infty$ such that $x(T) = 0$. *[**Hint:** Since f is $\mathbf{C}^1$, both the equations $\dot{x} = f(x)$ and $\dot{x} = -f(x)$ have unique solutions. What about a solution ending (for $\dot{x} = f(x)$) and starting (for $\dot{x} = -f(x)$) in $x_0 = 0$?]*

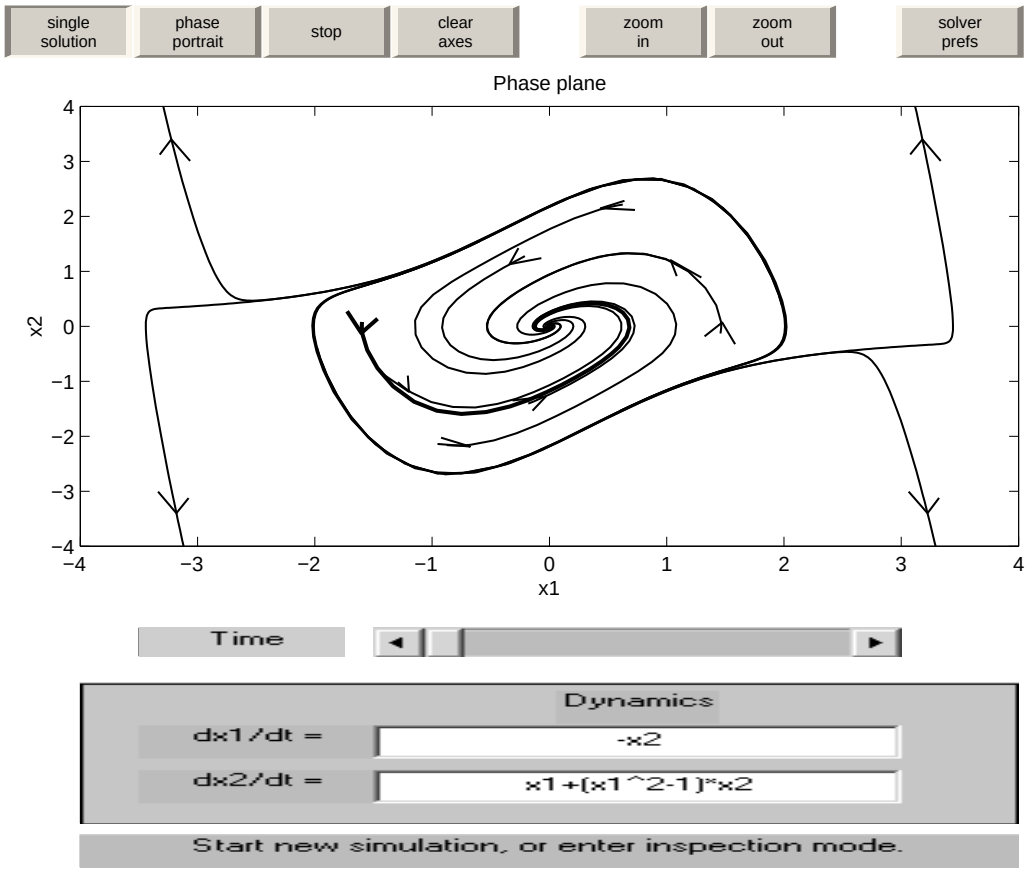
□

EXERCISE 5.14

We wish to estimate the domain of attraction D of the origin for the system

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = x_1 + (x_1^2 - 1)x_2,$$

see phase portrait below:



(a) Show that the origin is a locally asymptotically stable equilibrium. Linearize the system at the origin and determine the system matrix A.

(b) Find a Lyapunov function of the form $V(x) = x^T P x$ by solving the Lyapunov equation

$$PA + A^T P = -2I,$$

for an unknown positive definite symmetric matrix P, where A is the matrix in (a).

(c) We want to find an as large region $\Pi \subset \mathbb{R}^2$ as possible such that for all $x \in \Pi$,

$$\dot{V} = \frac{dV}{dx} \cdot \frac{dx}{dt} < 0.$$

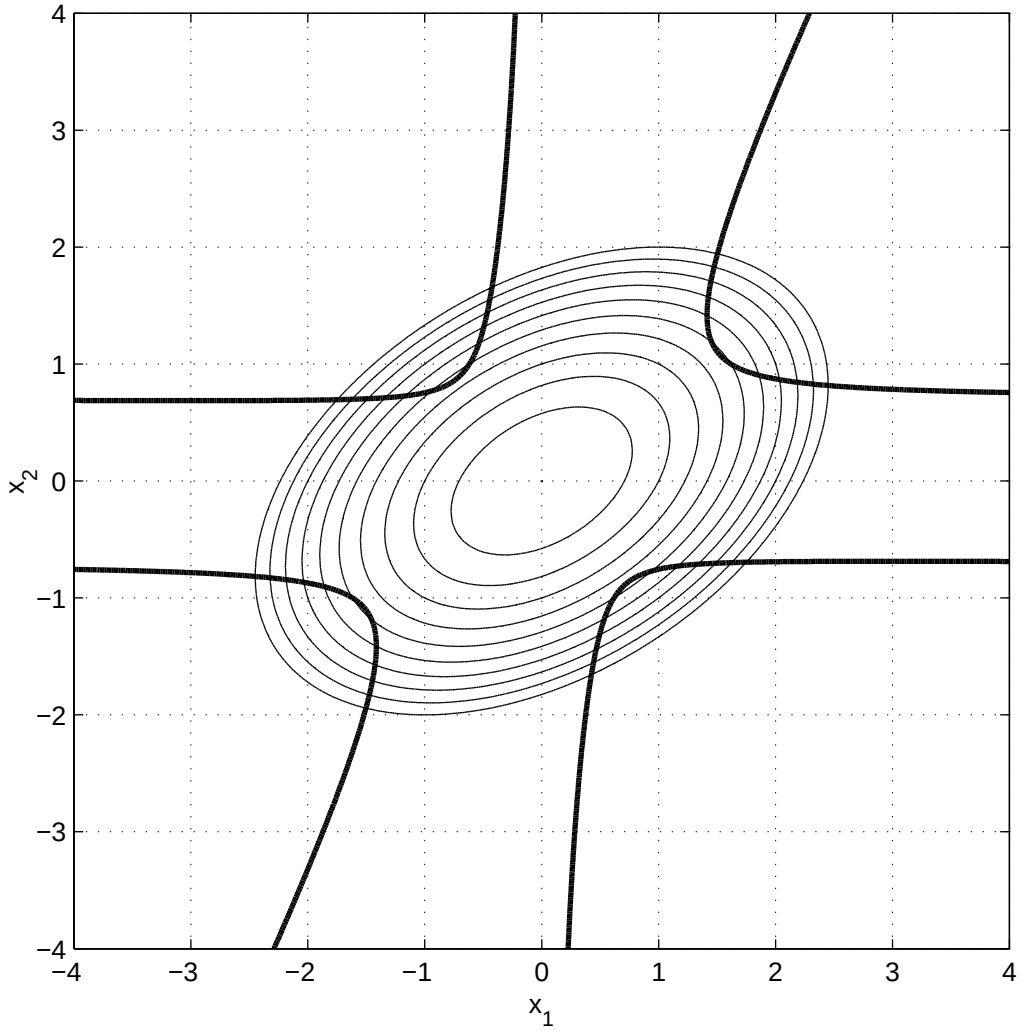
Show that

$$\dot{V} = -2x_1^2 - 2x_2^2 - 2x_1^3 x_2 + 4x_1^2 x_2^2.$$

(d) Let

$$\Omega_c = \{z \in \mathbb{R}^2 : \ V(z) \leq c\}.$$

We want to find $c > 0$, as large as possible, such that $\Omega_c \subset \Pi$, where Π is given in (c). Conclude that Ω_c is an estimate of the domain of attraction D. You may base your argument on the plot below. The ellipsoids are the boundaries of Ω_c for $c = 1, 2, \ldots, 10$. The other curves show the level curve for $\dot{V} = 0$.



□

EXERCISE 5.15

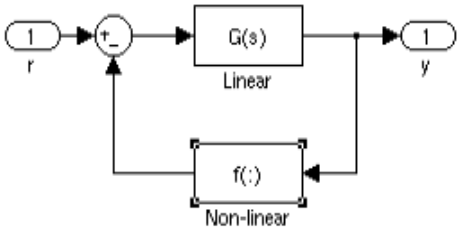


Figure 2.13: System model used in exercise

The system considered in the following tasks, contains of a linear part G(s) connected with a statical feedback-loop f(:) as seen in Figure 2.13.

Which statements are true, and which are false ?
If false, then justify why it is so ?

1. The system is BIBO-stable if the small gain theorem shows stability for the system.
2. The system is unstable if Small Gain theorem, Circle Criteria and Passivity theorem are proved not stable.
3. The system is BIBO-stable if the Circle criteria proves BIBO-stability, even if the Small gain theorem, and Passivity theorem don't fulfill it's requrement for BIBO-stability.
4. If G(s) is passive and f(:) is passive, the system is BIBO-stable.
5. For the system to be *strictly* passive, the nyquist curve for $G(i\omega)$ has to lie in the R.H.P and not intersect the $\Im$-axis, for all ω.
6. Using the Circle criteria: If the nyquist curve of $G(i\omega)$ stays inside the circle defined by the points $-1/k_1$ and $-1/k_2$, the system is stable.
   $k_1 \leq \frac{f(y)}{y} \leq k_2$

□

2.6 Describing Function Analysis

EXERCISE 6.1

Show that the describing function for a relay with hysteresis in Figure 2.14 satisfies

$$-\frac{1}{Y_N(C)} = -\frac{\pi C}{4H}\left(\left(1-\left(\frac{D}{C}\right)^2\right)^{1/2} + i\frac{D}{C}\right).$$

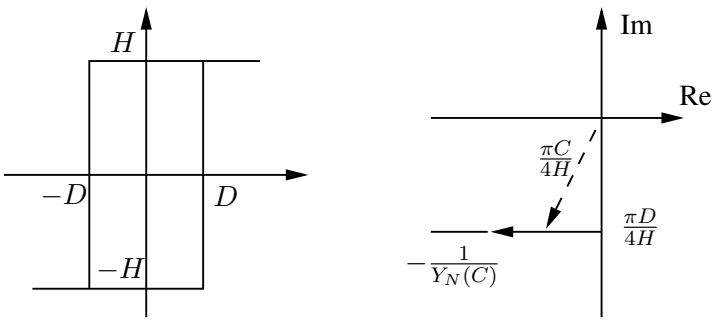


Figure 2.14: The relay with hysteresis in Exercise 6.1.

□

EXERCISE 6.2

Match each of the odd, static nonlinearities in Figure 2.15 with one of the describing functions in Figure 2.16. (*Hint.* Use the interpretation of the describing function $N(A)$ as "equivalent gain" for sinusoidal inputs with amplitude A.)

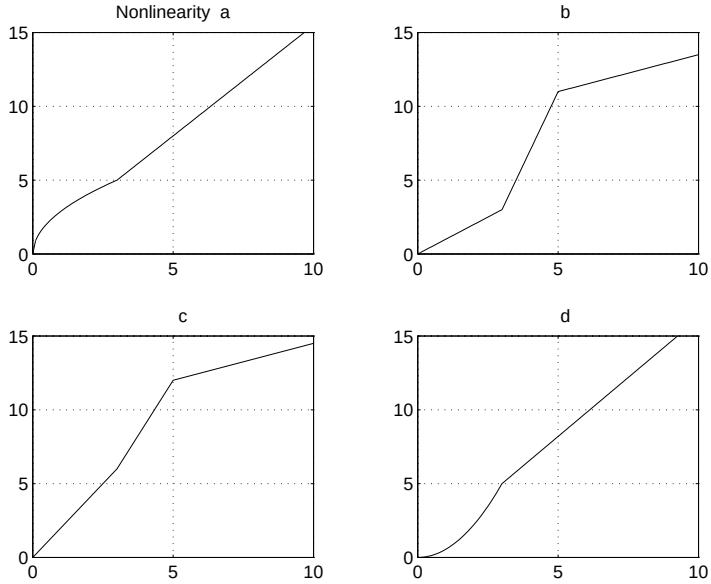


Figure 2.15: Nonlinearities in Exercise 6.2.

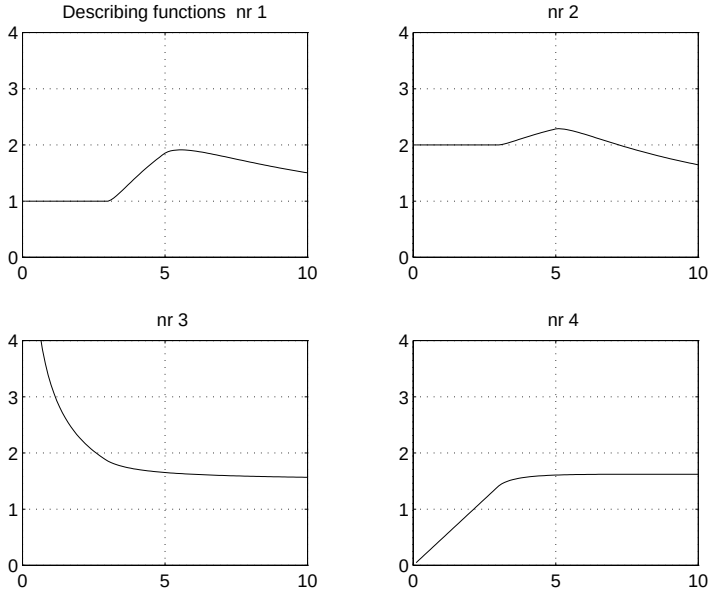


Figure 2.16: Describing functions in Exercise 6.2.

□

EXERCISE 6.3

Compute the describing functions for

(a) the saturation,

(b) the deadzone, and

(c) the piece-wise linear function

in Figure 2.17. (*Hint:* Use (a) in (b) and (c).)

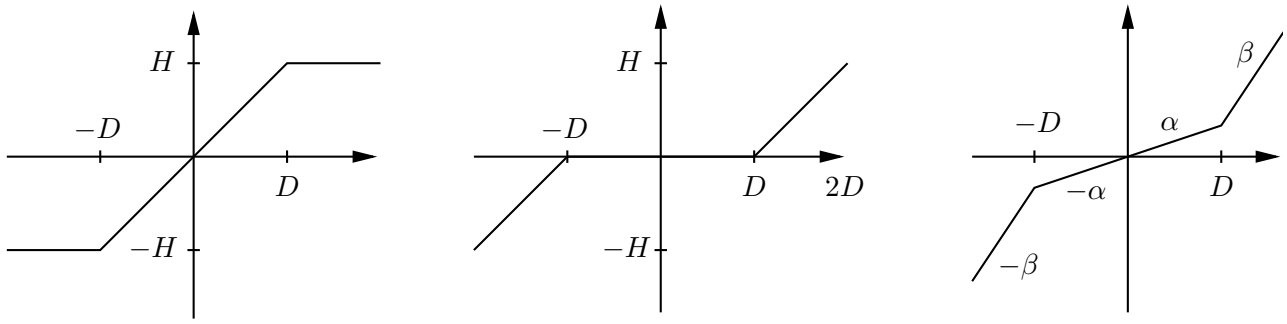


Figure 2.17: The static nonlinearities in Exercise 6.3

□

EXERCISE 6.4

If the describing function for the static nonlinearity $f(x)$ is $Y_N(C)$, then show that the describing function for $Df(x/D)$ equals $Y_N(C/D)$, where D is a constant.

□

EXERCISE 6.5

Compute the describing function for a static nonlinearity of the form

$$f(x) = k_1 x + k_2 x^2 + k_3 x^3.$$

How does the term $k_2 x^2$ influence the analysis?

□

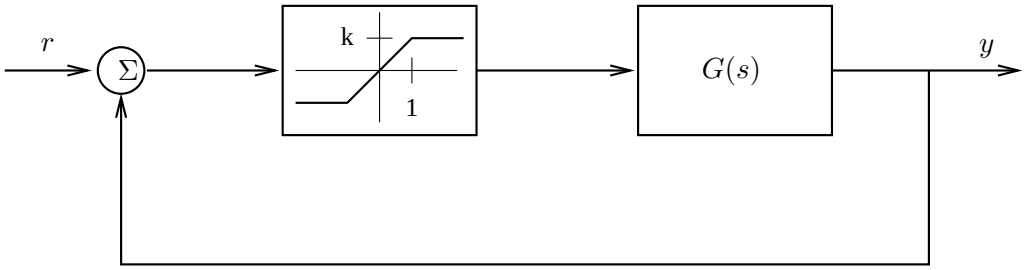


Figure 2.18: Electronic oscillator.

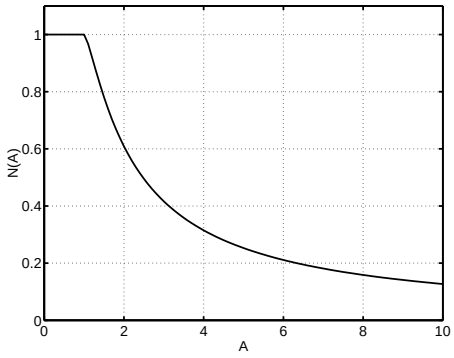


Figure 2.19: Normalized describing function.

EXERCISE 6.6

Consider the system in Figure 2.18, which is typical of the dynamics of electronic oscillators used in laboratories. Let

$$G(s) = \frac{-5s}{s^2 + s + 25}$$

(a) Assess intuitively the possibility of a limit cycle, by assuming that the system is started at some small initial state, and notice that the system can neither stay small (because of instability) nor at saturation values (by applying the final value theorem of linear control).

(b) Use describing function to predict whether the system exhibits a limit cycle, depending on the saturation level H. In such cases, determine the frequency and amplitude of the limit cycle. The describing function of a saturation is plotted in Figure 2.19.

(c) Use the extended Nyquist criterion to assess whether the limit cycle is stable or unstable.

□

EXERCISE 6.7

Consider a servo motor with transfer function

$$G(s) = \frac{4}{s(s+1)(s+2)}$$

controlled by a relay with a dead-zone a as shown in Figure 2.20.

(a) Show that the describing function for the relay with dead-zone a is given by

$$N(A) = \begin{cases} 0 & A < a \\ \frac{4}{\pi A}\sqrt{1 - \frac{a^2}{A^2}} & A \geq a \end{cases}$$

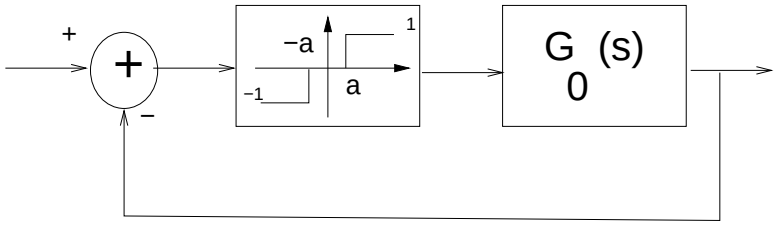


Figure 2.20: Position servo in Exercise 6.7.

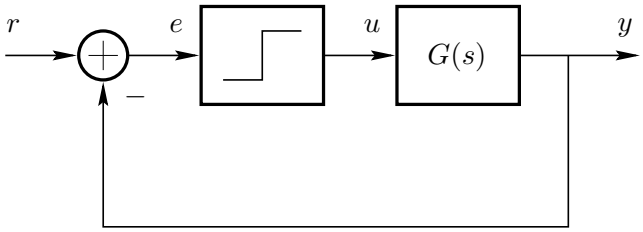


Figure 2.21: An auto-tuning experiment: linear system under relay feedback.

(b) How should the parameter a be choosen so that the describing function method predicts that sustained oscillations are avoided in the closed loop system?

□

EXERCISE 6.8

The Ziegler-Nichols frequency response method suggest PID parameters based on a system's ultimate gain K_u and ultimate period T_u according to the following table. The method provides a convenient method for tuning

Parameter	Value
K	$0.6K_u$
T_i	$0.5T_u$
T_d	$0.125T_u$

Table 2.1: Tuning rules for Ziegler-Nichol's method.

PID controllers, since K_u and T_u can be estimated through simple experiments. Once K_u and T_u have been determined, the controller parameters are directly given by the formulas above.

(a) Show that the parameters K_u and T_u can be determined from the sustained oscillations that may occur in the process under relay feedback. Use the describing function method to give a formula for computing K_u and T_u based on oscillation data. (amplitude A and angular frequency ω of the oscillation). Let the relay amplitude be D.

Recall that the ultimate gain and ultimate period are defined in the following way. Let $G(s)$ be the systems transfer function, and ω_u be the frequency where the system transfer function has a phase lag of -180 degrees. Then we have

$$T_u = 2\pi/\omega_u$$
$$K_u = 1/|G(i\omega_u)|$$

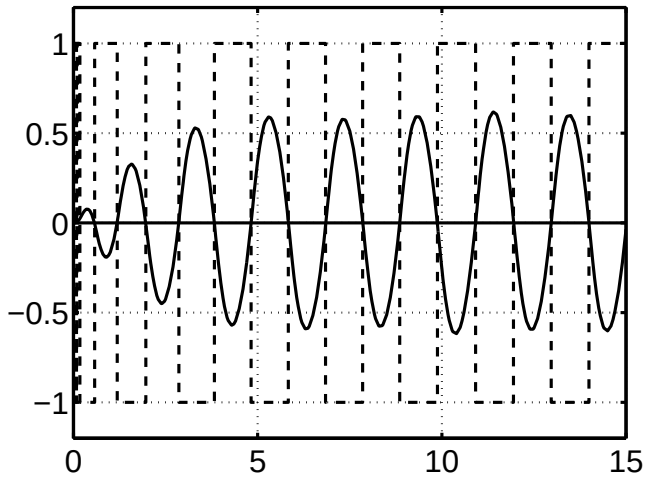


Figure 2.22: Input and output of system under relay feedback.

(b) What parameters would the relay method give for the process

$$G(s) = \frac{50}{s(s+1)(s+10)}$$

which is simulated in Figure 2.22 with $d = 1$? Compare what you obtain from analytical computations ($K_u = 0.46, T_u = 1.99$)

□

EXERCISE 6.9

In many cases, it is desirable to limit the high frequency content in a signal. Usually, such filtering is performed using a linear low-pass filter. Unfortunately, this type of filtering introduces phase lags. If the limiter is positioned within a feedback loop, this phase lag can be harmful for system stability.

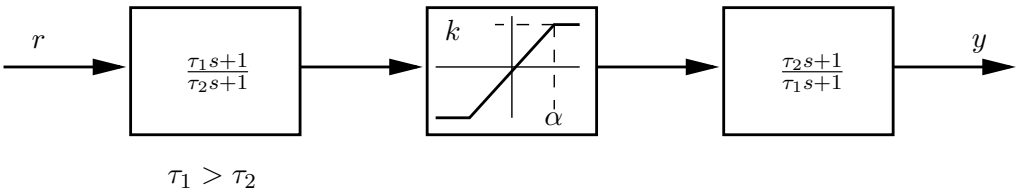


Figure 2.23: The nonlinear lowpass filter suggested in Exercise 6.9.

Figure 2.23 shows an alternative way of limiting the high frequency content of a signal. The system is composed of a high pass filter, a saturation, and a lowpass filter. Show that the system can be viewed as a nonlinear lowpass filter that attenuates high-frequency inputs without introducing a phase lag.

□

EXERCISE 6.10

(a) The power output from a cellular phone is an important system variable, since the power consumption should be kept as low as possible to make the battery last longer. Information from the base station about the received power is sent back to the transmitting phone and is used to control the power output. A simple model for such a power control system is as follows:

$$\begin{aligned} \dot{x}(t) &= au(t) \\ u(t) &= -\operatorname{sgn} y(t-L) \\ y(t) &= bx(t). \end{aligned}$$

Here x is the power output of the cellular phone (normalized around zero) and u is the control signal, which either increase or decrease the power at a fixed rate $a > 0$. The measured power y at the base station is proportional to x with proportional constant $b > 0$. The measured power is being transmitted back to the cellular phone after a time delay $L > 0$.

Draw a diagram illustrating the system. Use describing function analysis to predict amplitude, frequency, and stability of possible power oscillations.

□

EXERCISE 6.11

The Clegg integrator was invented by J. C. Clegg in 1958. It is simply an integrator with a state that is set to zero whenever the input crosses zero. Let e be the input to the Clegg integrator and x the integrator state. Then, the Clegg integrator can be described by the following two equations:

$$\begin{aligned} \dot{x}(t) &= e(t) \\ x(t+) &= 0, \qquad\qquad \text{if } e(t) = 0, \end{aligned}$$

where the plus sign in $x(t+)$ indicates that x is set to zero directly after e becomes zero.

(a) Sketch the output of the Clegg integrator for a sinusoidal input $e(t) = A\sin\omega t$. Assume that $x(0) = 0$.

(b) Show that the describing function for the Clegg integrator is

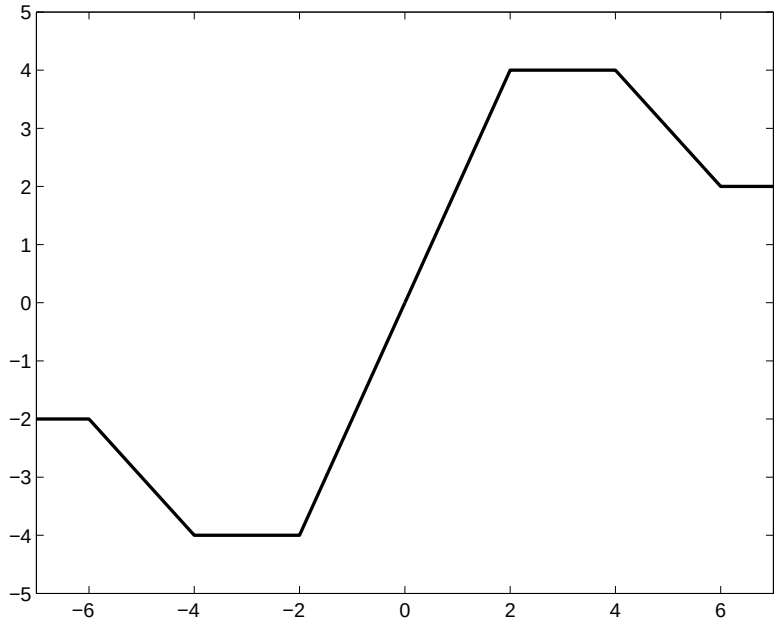
$$N(A,\omega) = \frac{4}{\pi\omega} - \frac{i}{\omega}$$

(c) The describing function gives (as you know) the amplification and phase shift of a sinusoidal input $e(t) = A\sin\omega t$. Draw the Nyquist diagram for the ordinary integrator ($G(s) = 1/s$) together with the describing function for the Clegg integrator. Comment on similarities and differences in their gain and phase characteristics. What is the main advantage of using the Clegg integrator instead of an ordinary integrator (for example, in a PID controller) and vice versa?

□

EXERCISE 6.12

Consider the odd static nonlinearity f below.



(a) Sketch the describing function for f. You only need to draw an approximate sketch.

(b) Consider a feedback system that consists of a linear system $G(s)$ in negative feedback connection with f, that is,

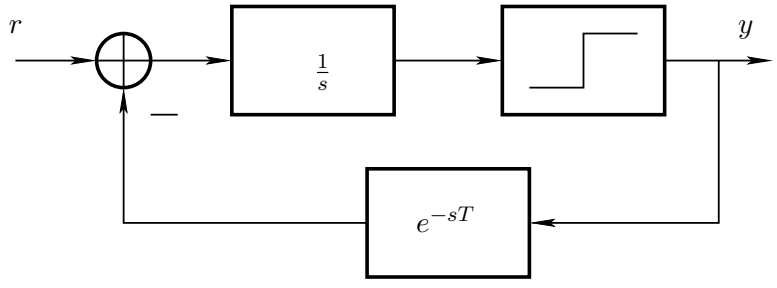
$$y = Gu = -Gf(y).$$

Based on the describing function method, specify a transfer function $G(s)$ such that the closed-loop system is likely to have an oscillation.

□

EXERCISE 6.13

A simple model of a $\Sigma\Delta$-modulator is shown below.



It works as an AD-converter and converts the analog signal $r : [0, \infty) \mapsto \mathbb{R}$ to the digital signal $y : [0, \infty) \mapsto \{-1, 1\}$. The relay block (one-bit quantizer) represents the sign function $\mathrm{sgn}(\cdot)$ and e^{-sT} represents a time delay of T units.

(a) Let $r(t) = 0.5$ and $T = 1$. Then the average of y will converge to 0.5. Illustrate this by plotting r and y as functions of time in steady state.

(b) Let $r(t) = 0$ and $T = 1$. Use describing function analysis to show that the system is likely to have a (symmetric) periodic solution. Determine an estimate of the period time of the oscillation.

(c) Let $r(t) = 0$ and $T > 0$. Show that (for any non-zero initial condition of the integrator) the period time of the oscillation will be equal to $4T$.

(d) Let $r(t) = 0$ and $T = 1$ (as in (b)). Suppose that in an implementation the relay is replaced by a saturation with slope $k > 0$, that is,

$$\text{sat}(kx) = \begin{cases} 1, & kx > 1 \\ kx, & k|x| \leq 1 \\ -1, & kx < -1 \end{cases}$$

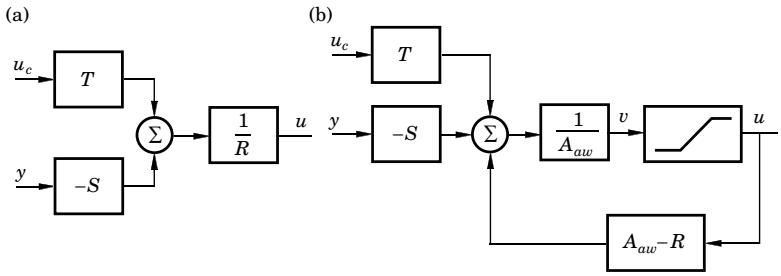
where x denotes the output of the integrator. For which values of $k > 0$ does describing function analysis suggest that there will be an oscillation?

□

2.7 Anti-windup

EXERCISE 7.1

Consider the antiwindup scheme in polynomial form described in the figure together with a process $A(s)y = B(s)u$. Put $u_c = 0$. Make a block transformation to write the system in standard feedback form with lower



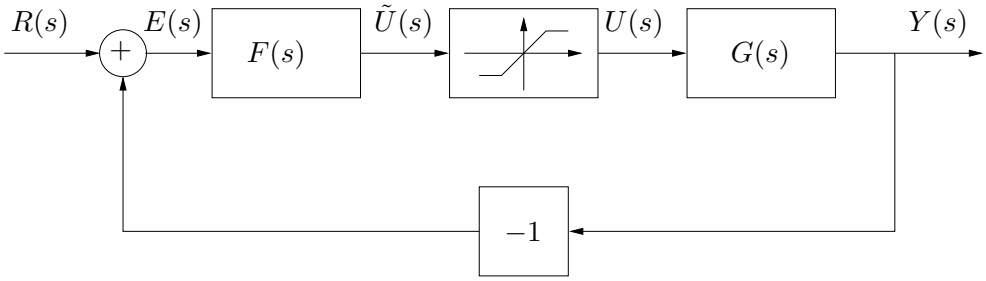
block $P = \frac{AR+BS}{AA_{aw}} - 1$. Use the circle criterion (or passivity) to conclude that the system is globally asymptotically stable if A is stable and the following condition holds:

$$Re\left(\frac{AR + BS}{AA_{aw}}(i\omega)\right) \geq \epsilon > 0, \quad \forall\omega.$$

□

EXERCISE 7.2

Consider the servo system below



The transfer function of the DC motor is given by

$$G(s) = \frac{1}{s(s+0.1)},$$

and the output signal of the amplifier is limited. The system is controlled by a PID regulator, given by

$$F(s) = K_c(1 + \frac{1}{T_i s} + T_d s)$$

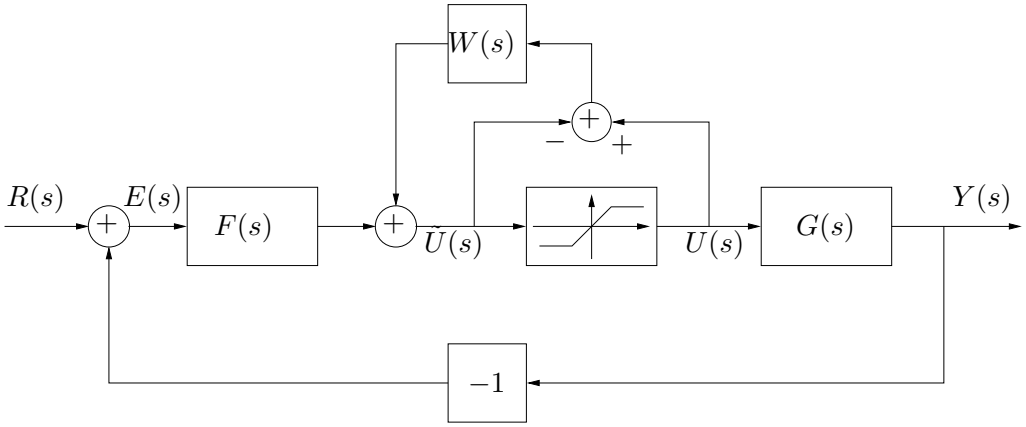
where $K_c = 1$, $T_d = 2$ and $T_i \in [1, 15]$. Using the describing function method one can predict autooscillations if the parameter of the integral part is set to $T_i < 10$. In this exercise we will study two different modifications of the PID regulator in order two avoid the autooscillations. These modifications ar such that integrator constants less than 10 can be used.

(a) PID regulator in series with phase advancing term, i.e.

$$F(s) = K_c \left(1 + \frac{1}{T_i s} + T_d s\right) N \frac{s+b}{s+bN}.$$

Assume that K_c, T_i, and T_d are chosen positive and such that the closed loop system is stable when the saturation is not present. Determine the values for b and N, such that the describing function method does not predict autooscillations. (Hint: $b = 0.1$ is a possible choice.)

(b) A different alternative to avoid autooscillations is to feedback the difference between the calculated control signal $\tilde{u}(t)$ and the saturated control signal $u(t)$ as in the figure below.



Assume that $W(s) = 1/(sT_t)$ and that

$$F(s) = K_c \left(1 + \frac{1}{T_i s} + T_d s\right)$$

where $K_c = 0.95$, $T_i = 3.8$, and $T_d = 1.68$. If the saturation is neglected, the closed loop systems poles are given by the zeros of the characteristic polynomial

$$(s + \alpha\omega_0)(s^2 + 2\zeta\omega_0 s + \omega_0^2)$$

where $\omega_0 = 0.5$, $\zeta = 0.7$, and $\alpha = 2$. Determine the value of T_t for which the describing function method does not predict autooscillations.

(c) What are advantages and drawbacks of the methods in (a) and (b).

□

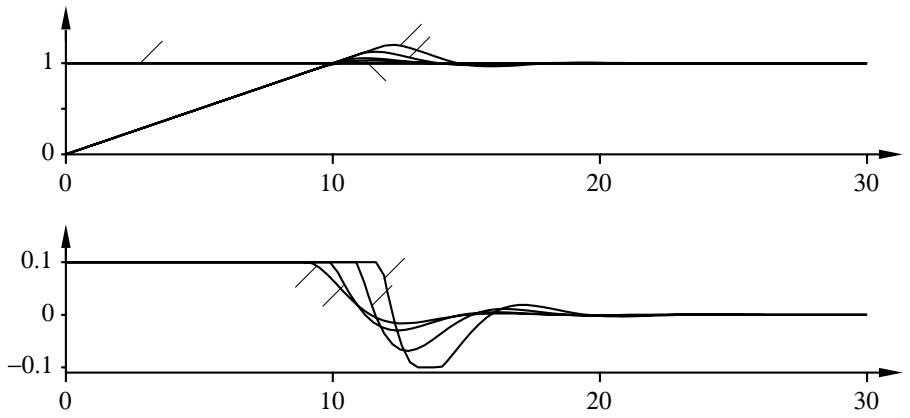
EXERCISE 7.3

Rewrite the blockdiagram of a PI controller with antiwindup (Lecture 7 slide 5) on state-space form as in slide 10.

□

EXERCISE 7.4

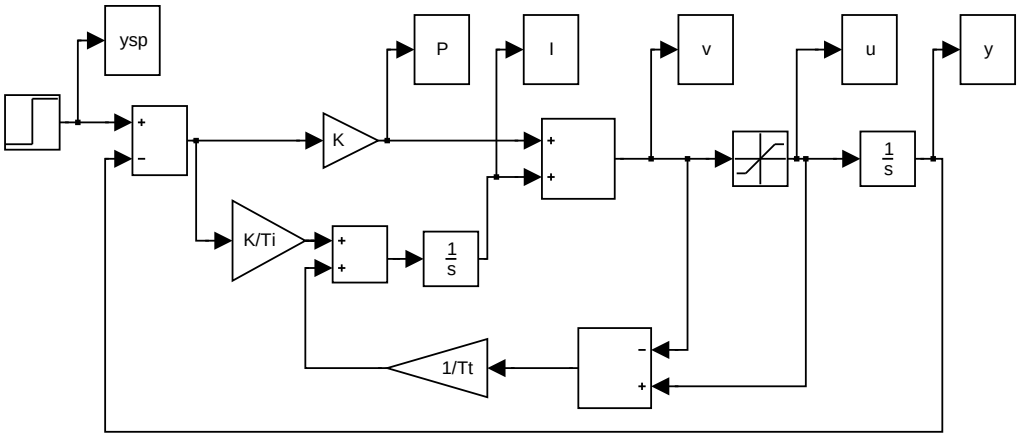
Consider a PID controller with anti-windup (see Lecture 7). The following plots illustrate control of an integrator process with four different choices of the tracking time constant T_t. The upper plot shows the desired set-point (the straight line) together with four curves for the output y (curves with more or less overshoot). The lower plot shows four curves for the control signal u. Redraw the curves in your solutions and combine $T_t = 0.1$, 1, 2, and 3 with the corresponding curves. (Note that y for two of the choices are almost identical.) Motivate.



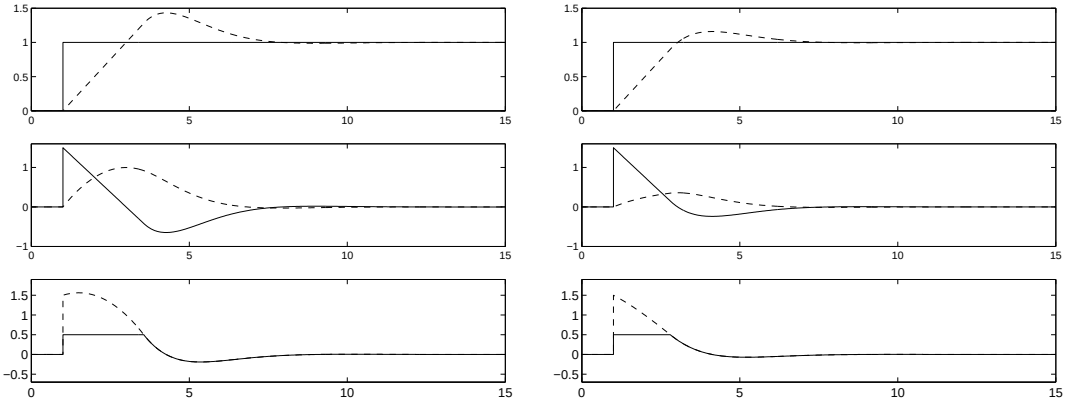
□

EXERCISE 7.5

The block diagram shows a control system with anti-windup. The process (as you see) is given by an integrator.



(a) The three left plots below show set-point experiments for the system without anti-windup ($T_t = 1000$). The three right plots show set-point experiments with anti-windup ($T_t = T_i$). Match the plots with the six signals indicated in the block-diagram. Also, specify the values of K, T_i, and the saturation level. (These parameters are the same in both simulations.) Motivate your answers.



(b) Let $y_{\text{sp}} = 0$. The block diagram can then be described as a linear part $v = G(s)u$ in feedback connection with a static nonlinear part $u = \text{sat}_\alpha(v)$, where $\alpha > 0$ is the saturation level from (a). Specify $G(s)$.

(c) Let again $y_{\text{sp}} = 0$. For certain values of T_i and T_t, the linear system in (b) is given by

$$G(s) = -\frac{(K-1)s + K}{s(s+1)}.$$

Use the Circle Criterion to determine for which $K > 1$ the closed-loop system is BIBO stable (from a perturbation in u to v).

□

2.8 Friction, Backlash and Quantization

EXERCISE 8.1

The following model for friction is described in a recent PhD thesis

$$\begin{aligned} \frac{dz}{dt} &= v - \frac{|v|}{g(v)} z \\ F &= \sigma_0 z + \sigma_1(v)\frac{dz}{dt} + F_v v, \end{aligned}$$

where σ_0, F_v are positive constants and $g(v)$ and $\sigma_1(v)$ are positive functions of velocity.

(a) What friction force does the model give for constant velocity?

(b) Prove that the map from v to z is passive if $z(0) = 0$.

(c) Prove that the map from v to F is passive if $z(0) = 0$ if $0 \leq \sigma_1(v) \leq 4\sigma_0 g(v)/|v|$.

□

EXERCISE 8.2

Derive the describing function (v input, F output) for

(a) Coulomb friction, $F = F_0 \text{sign}\,(v)$

(b) Coulomb + linear viscous friction $F = F_0 \text{sign}\,(v) + F_v v$

(c) as in b) but with stiction for $v = 0$.

□

EXERCISE 8.3

If v is not directly measurable the adaptive friction compensation scheme in the lectures must be changed. Consider the following double observer scheme:

$$\begin{aligned} \widehat{F} &= (z_F + K_F|\widehat{v}|)\text{sign}(\widehat{v}) \\ \dot{z}_F &= -K_F(u - \widehat{F})\text{sign}(\widehat{v}) \\ \widehat{v} &= z_v + K_v x \\ \dot{z}_v &= -\widehat{F} + u - K_v \widehat{v}. \end{aligned}$$

Show that linearisation of the error equations

$$\begin{aligned} \dot{\widehat{e}}_v &= \dot{v} - \dot{\widehat{v}} = g_v(e_v, e_F, v, F) \\ \dot{\widehat{e}}_F &= \dot{F} - \dot{\widehat{F}} = g_F(e_v, e_F, v, F) \end{aligned}$$

gives Jacobian

$$A = \begin{bmatrix} -K_v & -1 \\ -K_v K_F & 0 \end{bmatrix}.$$

Conclude that local error convergence is achieved if $K_v > 0$ and $K_F < 0$.

□

EXERCISE 8.4

Show that the describing function for quantization is given by

$$N(A) = \begin{cases} 0 & A < \frac{D}{2} \\ \frac{4D}{\pi A} \sum_{i=1}^{n} \sqrt{1 - \left(\frac{2i-1}{2A} D\right)^2} & \frac{2n-1}{2} D < A < \frac{2n+1}{2} D \end{cases}$$

(*Hint: Use one of the nonlinearities from lecture 8 and superposition.*)

□

EXERCISE 8.5

Show that a saturation is a passive element.

□

EXERCISE 8.6

Consider the mass-spring system with dry friction

$$\ddot{y} + c\dot{y} + ky + \eta(y, \dot{y}) = 0$$

where η is defined as

$$\eta(y, \dot{y}) = \begin{cases} \mu_k mg \text{sign}(\dot{y}) & \text{for } |\dot{y}| > 0 \\ -ky & \text{for } \dot{y} = 0 \text{ and } |y| \leq \mu_s mg/k \\ -\mu_s mg \text{sign}(y) & \text{for } \dot{y} = 0 \text{ and } |y| > \mu_s mg/k \end{cases}$$

Construct the phase potrait and discuss its qualitative behavior. (*Hint:* Use piecewise linear analysis.)

□

EXERCISE 8.7

The accuracy of a crude A/D converter can be improved by adding a high-frequency dither signal before quantization and lowpass filtering the discretized signal, see Figure 2.24 Compute the stationary value y_0 of the

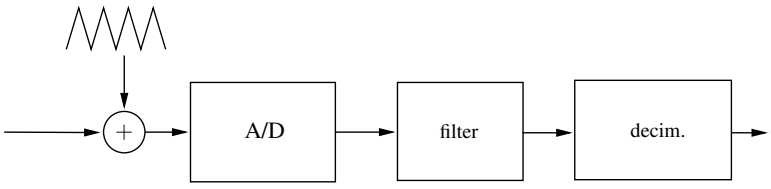


Figure 2.24: A/D converter in Exercise 8.7

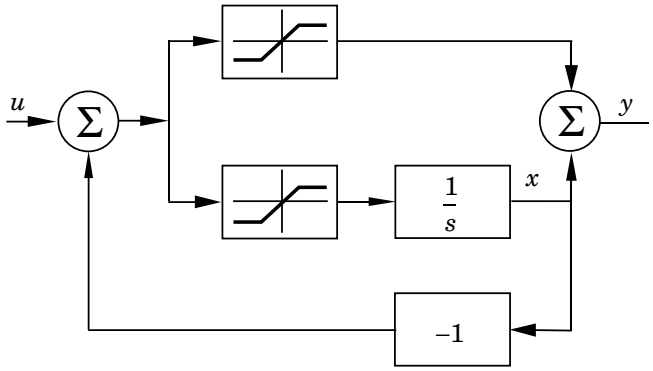
output if the input is a constant u_0. The dither signal is a triangle wave with zero mean and amplitude $D/2$ where D is the quantization level.

□

EXERCISE 8.8

(a) Write down the equations describing the following system, where the saturation blocks are defined as

$$\mathrm{sat}(u-x) = \begin{cases} -1, & u-x \le -1 \\ u-x, & |u-x| \le 1 \\ 1, & u-x \ge 1. \end{cases}$$



The system is a jump and rate limiter: y is equal to u if u changes slowly. If u makes an abrupt change, then y converges to u after a while. Conclude this by simply deriving the equations for the system for $|u-x| \le 1$ and for $|u-x| \ge 1$, when u makes a step change.

□

EXERCISE 8.9

The describing function for backlash is shown below:

$N(A) = \frac{b_1 + ia_1}{A}$

$a_1 = -\frac{4D}{\pi}\left(1 - \frac{D}{A}\right)$

$b_1 = \frac{A}{\pi}\left[\frac{\pi}{2} + \arcsin\left(1 - \frac{2D}{A}\right) + 2\left(1 - \frac{2D}{A}\right)\sqrt{\frac{D}{A}\left(1 - \frac{D}{A}\right)}\right]$

a) Calculate $\lim_{A\to\infty} -\frac{1}{N(A)}$ and explain what the result means.

b) Explain why N(A)=0 when A < D

□

2.9 High-gain and Sliding mode

EXERCISE 9.1

In some cases, the main nonlinearity of a system can be isolated to a static nonlinearity on the input. This is, for example, the case when a linear process is controlled using a actuator with a nonlinear characteristic. A simple design methodology is then to design a controller $C(s)$ for the linear process and cancel the effect of the actuator nonlinearity by feeding the computed control through the inverse of the actuator nonlinearity, see Figure 2.25. Compute the inverse of the following common actuator characteristics

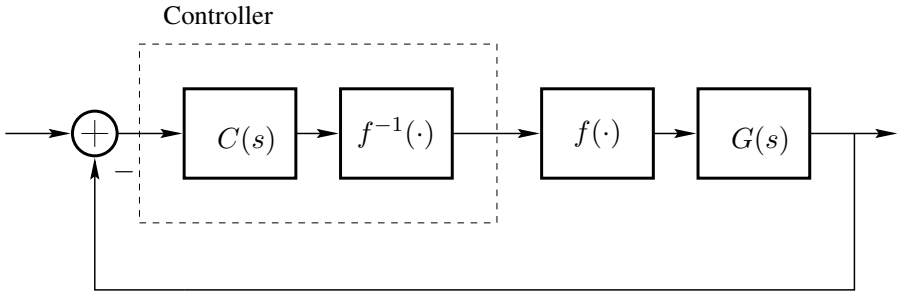


Figure 2.25: Compensation of input nonlinearity by inversion.

(a) The quadratic (common in valves)

$$f(v) = v^2, \qquad v \geq 0$$

(b) The piecewise linear characteristic

$$f(v) = \begin{cases} k_1 v & |v| \leq d \\ (k_1 - k_2)d + k_2 v & |v| > d \end{cases}$$

Use your result to derive the inverse of the important special case of a dead zone.

(c) A backlash nonlinearity.

□

EXERCISE 9.2

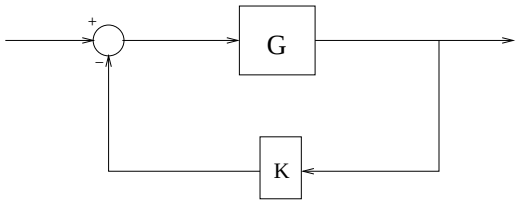


Figure 2.26: System in exercise 9.2

What should K be in order for the closed-loop system(see fig. 2.26) to have low sensitivity towards open-loop perturbations when $G = 500$ and the distorsion $\frac{dG}{G} = 0.15$?
Here, low sensitivity means: $\frac{dG_{cl}}{G_{cl}} \leq 0.1\%$

□

EXERCISE 9.3

Consider the linear system

$$\dot{x}(t) = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} x(t)$$

which is controlled by a relay.

(a) Sketch the phase-plane directions in the points.

$$(x_1, x_2) = (\pm 2, \pm\varepsilon), (\pm 0.5, \pm\varepsilon)$$

where $\varepsilon \to 0$.

(b) What is the equivalent control u_{eq} on the switching surface?

□

EXERCISE 9.4

Consider a linear system

$$\dot{x}_1 = ax_2 + bu$$
$$\dot{x}_2 = x_1$$

with nominal parameter values $a = 1$, $b = 1$. The system equations were obtained by linearization of a nonlinear system, which has the consequence that the parameters a and b vary with operating region.

(a) One of the design parameters in the design of a sliding mode controller is the choice of sliding surface. Which of the following sliding surfaces will result in a stable sliding mode for the above system?

(i) $\sigma(x) = x_1 - x_2$
(ii) $\sigma(x) = x_1 + 2x_2$
(iii) $\sigma(x) = x_1$

(b) Let the sliding mode be $\sigma(x) = x_1 + x_2$. Construct a sliding mode controller for the system.

(c) How large variations in the parameters a and b can the controller designed in (b) tolerate in order to still guarantee a stable closed loop system?

□

EXERCISE 9.5

We have seen how it in many cases can be of interest to control the system into a set, rather than to the origin. One example of this is sliding mode control, where the system state is forced into an invariant set, chosen in such a way that if the state is forced onto this set, the closed loop dynamics are exponentially stable. In this example, we will use similar ideas to design a controller that "swings" up an inverted pendulum from its stable equilibrium (hanging downwards) to its upright position.

Let the pendulum dynamics be given by

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{mgl}{J_p}\sin(x_1) - \frac{ml}{J_p}\cos(x_1)u\end{aligned}$$

(a) Denote the total energy of the pendulum by E and determine the value E_0 corresponding to the pendulum standing in the upright position.

(b) Investigate whether the control strategy

$$u = k(E - E_0)\text{sign}(x_2\cos(x_1))$$

forces the value of E towards E_0. (*Hint.* Use a Lyapunov function argument with $V(x) = (E(x) - E_0)^2$

(c) Draw a phase portrait of the system and discuss the qualitative behavior of the closed loop system. In particular, will the suggested control stabilize the unstable equilibrium point?

□

EXERCISE 9.6

An uncertain nonlinear system is given by

$$\dot{x} = -a\sin(x(t)) + u$$

where $a \in [1, 2]$ and $x(0) = 0$.

(a) Consider the open loop system. Sketch the step response for $a = 1$ and $a = 2$.

(b) Now apply a P controller with gain $k \gg a$. What is the static gain of the closed system.

(c) What are the benefits/problems with highgain control?

□

EXERCISE 9.7

A nonlinear process is denoted by P and a model of the process is denoted $\hat{P}$. Assume that the nonlinear model $\hat{P}(u(t))$ is described by a linear system in parallel with a nonlinear system, thus, the output is given by

$$y(t) = \hat{P}(u(t)) = L(p)u(t) + N(u(t)).$$

Here, $L(p)$ denotes a linear system and $N()$ denotes a nonlinear system.

(a) Show that the inverse of $\hat{P}$ can be represented by the block diagram in Figure 2.27.

(b) Draw a block diagram of an Internal Model Controller using the inverse of $\hat{P}$ as above such that the desired closed loop system is given by T.

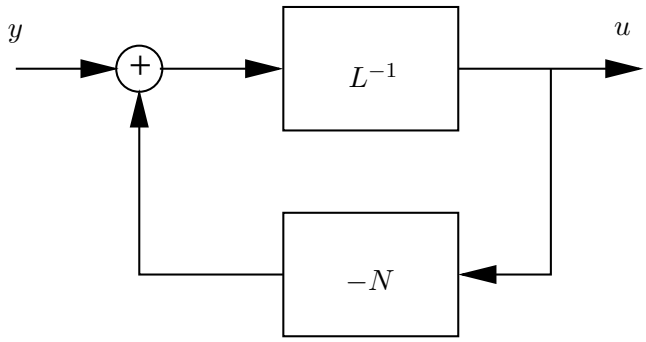


Figure 2.27: Inversion of parallel system

□

EXERCISE 9.8

Consider a prey (Swedish: *byte*) and a predator (*rovdjur*) population with densities x_1 and x_2, respectively. Assume that the predator population is harvested when they are sufficiently many, that is, when $x_2 > \alpha$ for some constant $\alpha > 0$. Think for instance on rabbits, foxes, and human hunters. The hunters only hunt if the density of foxes is sufficiently high. For a particular set of populations, if $x_2 < \alpha$ the dynamics is given by

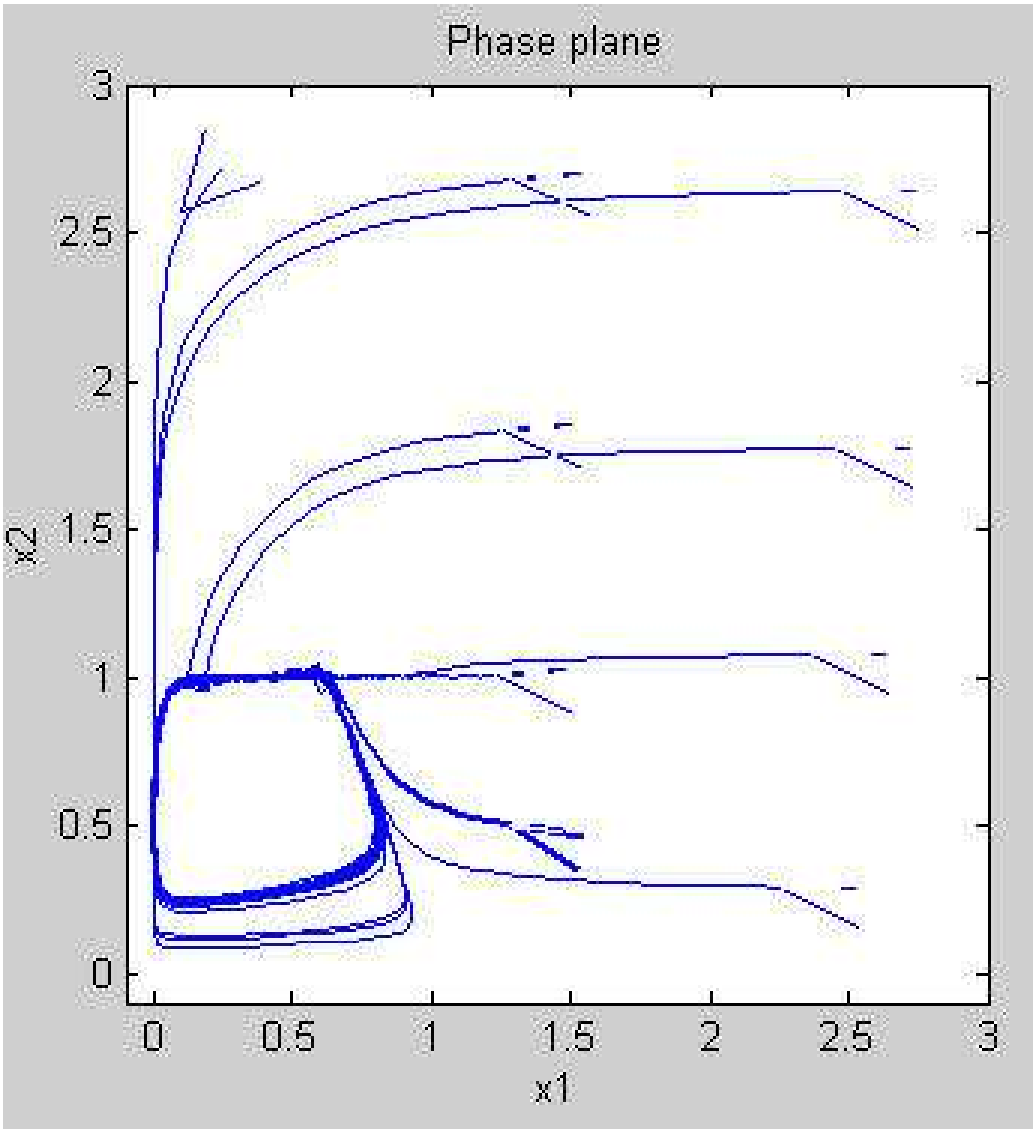
$$\begin{aligned}\dot{x}_1 &= 5x_1(1-x_1) - \frac{20x_1x_2}{2+10x_1}\\ \dot{x}_2 &= \frac{16x_1x_2}{2+10x_1} - \frac{6x_2}{10}\end{aligned}$$

and if $x_2 > \alpha$ the dynamics is given by

$$\begin{aligned}\dot{x}_1 &= 5x_1(1-x_1) - \frac{20x_1x_2}{2+10x_1}\\ \dot{x}_2 &= \frac{16x_1x_2}{2+10x_1} - \frac{6x_2}{10} - \beta x_2\end{aligned}$$

where $\beta > 0$ is a parameter reflecting harvesting effort. In the following, we assume that $\alpha = \beta = 1$.

A (low quality) phase plane plot in ICTools is shown below.



(a) Derive all equilibria in the region $\{x \in \mathbb{R}^2 : x_1 \geq 0, 0 \leq x_2 \leq 1\}$. Linearize the system about these equilibria.

(b) Derive all equilibria in the region $\{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 > 1\}$. Linearize the system about these equilibria.

(c) Show that the population dynamics model can be written as

$$\begin{aligned} \dot{x} &= f(x, u) \\ u &= -\operatorname{sgn} \sigma(x). \end{aligned}$$

Determine f and σ.

(d) Derive the sliding mode at $x_2 = 1$ (see the phase plane plot), for example, using the equivalent control method. Describe what the sliding mode correspond to in physical terms (that is, in behavior of prey, predator, etc.).

□

EXERCISE 9.9

Consider the system

$$\dot{x}_1 = 2x_1 - x_2 + u$$
$$\dot{x}_2 = x_1$$

Design a sliding mode controller such that the origin can be shown to be globally asymptotically stable.

□

EXERCISE 9.10

Consider the system

$$\dot{x}_1 = -2x_1 - \frac{x_2}{1+x_2^2} + u$$
$$\dot{x}_2 = \frac{x_1}{1+x_1^2}$$

(a) Consider the manifold

$$S = \{(x_1, x_2) | x_1 + ax_2 = 0\}$$

How will the choice of a affect the dynamics on the sliding manifold.

(b) For the sliding manifold in (a), determine a controller that makes the manifold globally asymptotically stable.

(c) Give the equivalent control on the sliding manifold.

□

2.10 Computer Exercises: Sliding Mode, IMC and Gain scheduling

EXERCISE 10.1

Consider concentration control for a fluid that flows through a pipe, with no mixing, and through a tank, with perfect mixing. A schematic diagram of the process is shown in Figure 2.28(left). The concentration at the inlet of the pipe is c_{in}. Let the pipe volume be V_d and let the tank volume be V_m. Furthermore, let the flow be q and let the concentration in the tank at the outlet be c. A mass balance gives

$$V_m \frac{dc(t)}{dt} = q(t)(c_{in}(t-L) - c(t))$$

where $L = V_d/q(t)$.

(a) Show that for fixed $q(t)$, the system from input c_{in} to output c can be represented by a linear transfer function

$$G(s) = \frac{K}{sT+1} e^{-sL}$$

where L and T depend on $q(t)$.

(b) Use Ziegler-Nichols time response method and your model knowledge from (a) to determine a gain scheduled PI-controller from the step response in Figure 2.29. For this particular system, $V_d = V_m = 1$. Recall that the Ziegler-Nichols step response method relies on the parameters L and $a = KL/T$ defined in Figure 2.28(right) . (The line is tangent to the point where the step response has maximum slope).

Given these process parameters, the method suggest PID controller gains according to Table 2.2.

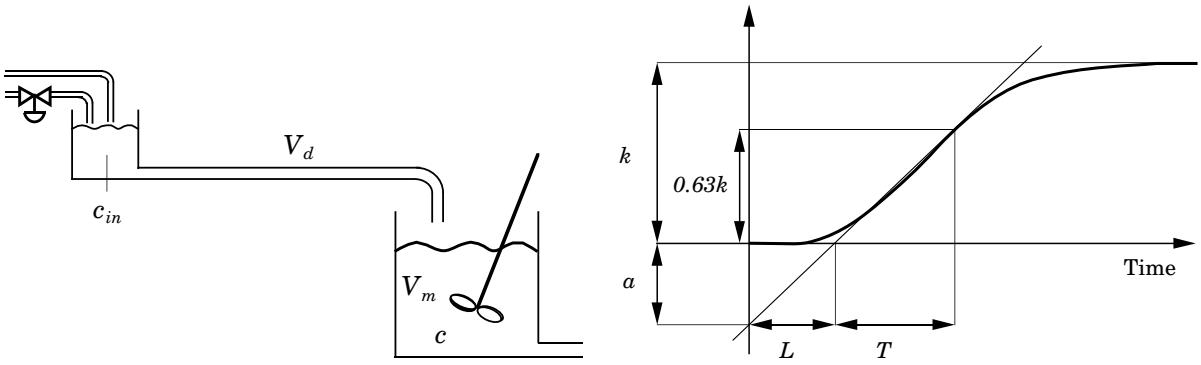


Figure 2.28: Schematic of the concentration control system (left). Parameters in Ziegler-Nichols step response method (right).

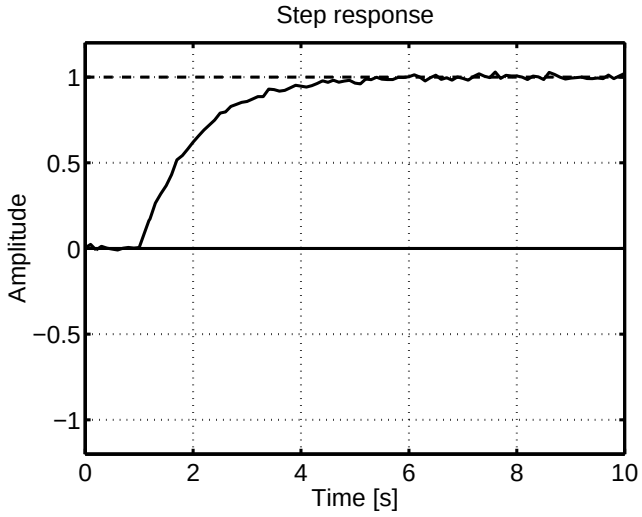


Figure 2.29: Experimental step response for $q(t) = 1$.

(c) Compare, using simulations, the performance of your gain scheduled controller with a controller designed for fixed $q = q_0 = 1$. Let q vary like sine-wave with amplitude 0.5, frequency 0.01 rad/s and a DC-offset of 1. The reference can be represented by a square-wave with an amplitude of 0.2 and a frequency of 0.01 Hz.

□

EXERCISE 10.2

Consider the linear system

$$G(s) = \frac{1}{s^2 + s}$$

Controller	K_p	T_i	T_d
P	$1/a$		
PI	$0.9/a$	$3L$	
PID	$1.2/a$	$2L$	$L/2$

Table 2.2: PID parameters suggested by Ziegler-Nichols step response method.

and simulate an on-off control. Hence, let the controller be relay with output ± 1. Plot the step response for a step starting at $t = 0$ (Simulation is troublesome if $r = 0$). What happens if there is a measurement disturbance (a step with amplitude 0.1) appears at t = 20? What about the control signal?

□

EXERCISE 10.3

Consider the linear system in Exercise 9.4 where the system is given by

$$\begin{aligned}\dot{x}_1 &= ax_2 + bu \\ \dot{x}_2 &= x_1\end{aligned}$$

with nominal parameter values $a = 1$, $b = 1$. Using the sliding surface $\sigma(x) = x_1 + x_2$ and $\hat{a} = \hat{b} = 1$ we get the sliding controller

$$u = -(x_1 + x_2) - \mu \text{sign}(x_1 + x_2).$$

Implement the controller and try different values of a, b and μ. Plot x_1 on one axis and x_2 on the other. Try different initial conditions, far away and close to the line $x_2 = -x_1$ which is the sliding surface.

Please note that when simulating systems controlled by a sliding mode controller Matlab/Simulink may have difficulties simulating the system when x_1 and x_2 approaches zero. In these cases, stop the simulation and plot what you get instead of waiting too long for the result.

□

EXERCISE 10.4

Consider the pendulum given by

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -a \sin(x_1(t) + \delta) - bx_2(t) + cu(t)\end{aligned}$$

where $x_1 = \theta - \delta$ and $x_2 = \dot{\theta}$. A sliding surface $\sigma(x) = x_1 + x_2$, gives the sliding mode controller

$$u = -\frac{1}{\hat{c}}(x_2 - \hat{a} \sin(x_1 + \delta) - \hat{b}x_2) - \frac{\mu}{\hat{c}} \text{sign}(x_1 + x_2).$$

Simulate the controller using $a = 1.5$, $b = 0.75$, $c = 0.75$ and $\delta = \frac{\pi}{4}$. For the controller use $\hat{a} = \hat{b} = \hat{c} = \mu = 1$. Try different values of μ.

□

2.11 Gain scheduling and Lyapunov based methods

EXERCISE 11.1

Controlling the air-fuel-ratio (A/F) in a car engine using a linear lambda sensor uses the linear model

$$y = \frac{k}{\tau s + 1} e^{-sT}$$

where k and τ are constants ($k = 1$ and $\tau = 70$ ms). The time-delay T on the other hand varies depending on the revolutions per minute (rpm) of the engine.

(a) Calculate an Internal Model Controller (IMC) for a fixed time-delay. The desired closed loop transfer function is given by

$$H(s) = \frac{1}{(\lambda s + 1)^2}.$$

Use a first order Padé approximation

$$e^{-sT} = \frac{1 - sT/2}{1 + sT/2},$$

to represent the time-delay. Hence, the time delay is represented by a right half plane zero and a left half plane pole.

(b) *Computer exercise:* Examples of time-delay values are shown in Table 2.3 (in reality we have many more points) Tune a controller for each time delay, such that there is no overshoot for a step in the

Table 2.3: *Time delay as a function of the rpm*

rpm	825	2000	6000
T (ms)	300	155	105

reference (try a reference step from 1 to 1.2). What is λ in each case? Check the sensitivity function, what is the max $|S(i\omega)|$? Make a linear interpolation of the λ parameter between rpm 825 and 2000. Try that controller the for 1200 rpm (the actual time delay is then 215 ms).

□

EXERCISE 11.2

Consider the system

$$\begin{aligned} \dot{x} &= -x + z_2 x^2 \\ \dot{z}_1 &= z_2 \\ \dot{z}_2 &= u \end{aligned}$$

with $x(0) = x_0$ and $z(0) = z_0$ where $z = (z_1 \ z_2)^T$.

(a) As a first idea we could let $z_2 \to 0$ as fast as possible using the controller $u = -az$. What is the problem with this approach? (*Hint: Check if the system may suffer from finite escape time*)

(b) Consider the more general form of the system

$$\begin{aligned}\dot{x} &= f_0(x) + y^T f(x,z)\\ \dot{z} &= Az + Bu\\ y &= Cz\end{aligned}$$

and use the controller

$$u = Kz - \frac{1}{2}\frac{\partial V}{\partial x} f(x,z) \tag{2.7}$$

where $V(x)$ is the Lyapunov function for the system $\dot{x} = f_0(x)$. The matrix K satisfies the positive real condition

$$\begin{aligned}(A+BK)^T P + P(A+BK) &= -Q\\ \text{and } PB &= C^T\end{aligned}$$

for some $P > 0$ and $Q \geq 0$.

(c) Study the global asymptotic stability using the Lyapunov function

$$W(x,z) = V(x) + z^T P z.$$

□

EXERCISE 11.3

Consider the uncertain system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_1^3 + x_2^3 + (1+a)u\end{aligned}$$

where a is an unknown constant which satisfy and $|a| \leq \frac{1}{2}$.

□

EXERCISE 11.4

Consider the uncertain system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= (1+a_1)(x_1^3 + x_2^3) + (1+a_2)u\end{aligned}$$

where a_1 and a_2 are unknown constants which satisfy $|a_1| \leq 1$ and $|a_2| \leq \frac{1}{2}$.

□

EXERCISE 11.5

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u + \delta(x)\end{aligned}$$

where δ is unknown but we know an estimate of ρ such that $|\delta(x)| \leq \rho\|x\|_2$. Let $u = \psi(x) = -x_1 - x_2$ be the nominal stabilizing controller and

$$v = \begin{cases} -\rho\|x\|_2 \frac{w}{\|w\|_2} & \text{if } \rho\|x\|_2\|w\|_2 \geq \epsilon \\ -\rho^2\|x\|_2^2 \frac{w}{\epsilon} & \text{if } \rho\|x\|_2\|w\|_2 < \epsilon \end{cases}$$

where $w^T = 2x^T PB$ and $V(x) = x^T Px$ is a Lyapunov function for the nominal closed loop system.

Show that when $\delta(x) = 2(x_1 + x_2)$ and $\rho = 2\sqrt{2}$ the origin is unstable.

□

EXERCISE 11.6

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2\\ \dot{x}_2 &= u.\end{aligned}$$

Compute a controller using backstepping to globally stabilize the origin.

□

EXERCISE 11.7

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= u.\end{aligned}$$

Compute a controller using backstepping to globally stabilize the origin.

□

EXERCISE 11.8

Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1^2 x_2\\ \dot{x}_2 &= u\end{aligned}$$

compute a controller to obtain global stabilization using the backstepping approach.

□

EXERCISE 11.9

Consider the system

$$\dot{x}_1 = x_1 x_2$$
$$\dot{x}_2 = x_1 + u$$

compute a controller using backstepping to globally stabilize the origin.

□

EXERCISE 11.10

Consider the system

$$\dot{x}_1 = x_2 + a + (x_1 - a^{1/3})^3$$
$$\dot{x}_2 = x_1 + u$$

where a is a constant. Compute a controller using backstepping to globally stabilize the origin.

□

EXERCISE 11.11

The equations of motion of an m-link robot is described by

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u$$

where q is an m-dimensional vector of generalized coordinates representing joint positions. The control torque input is represented by u and $M(q)$ is a symmetric inertia matrix which is positive definite $\forall q \in R^m$. The term $C(q, \dot{q})\dot{q}$ accounts for centrifugal and Coriolis forces. The matrix C has the property that $\dot{M} - sC$ is a skew-symmetric matrix $\forall q, \dot{q} \in R^m$, where $\dot{M}$ is the total derivative of M with respect to t. The term $g(q)$, which accounts for gravity forces, is given by $g(q) = \left[\frac{\partial P(q)}{\partial q}\right]^T$ where $P(q)$ is the total potential energy of the links due to gravity. We will assume that $P(q)$ is positive definite function of q and $g(q) = 0$ has an isolated root at $q = 0$.

(a) With $u = 0$ use the total energy

$$V(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$$

as a Lyapunov function candidate to show that the origin $(q = 0, \dot{q} = 0)$ is stable.

(b) With $u = -K_d\dot{q}$ where K_d is a positive diagonal matrix, show that the origin is asymptotically stable.

(c) With $u = g(q) - K_p(q - q^*) - K_d\dot{q}$ where K_p and K_d are positive diagonal matrices. and q^* is a desired robot position in R^m, show that the point $(q = q^*, \dot{q} = 0)$ is an asymptotically stable equilibrium point.

□

EXERCISE 11.12

(a) Consider the system

$$\dot{x}_1 = x_2 + f(x_1)$$
$$\dot{x}_2 = u,$$

where $f(x_1)$ is a $\mathbf{C}^1$ function with $f(0) = 0$. Show that the coordinate transformation

$$z_1 = x_1$$
$$z_2 = x_2 + f(x_1),$$

together with the control law

$$u = -z_1 - 2z_2 - z_2 f'(z_1)$$

gives an asymptotically stable linear system $\dot{z} = Az$.

(b) Find a state feedback controller $k : \mathbb{R}^2 \to \mathbb{R}$ for (11.12) such that the origin is asymptotically stable for the closed-loop system

$$\dot{x}_1 = x_2 + f(x_1)$$
$$\dot{x}_2 = k(x).$$

(You may use your result in (a).) Find a (Lyapunov) function $V : \mathbb{R}^2 \to \mathbb{R}$ and use it to prove that $x = 0$ is a globally asymptotically stable equilibrium for (11.12).

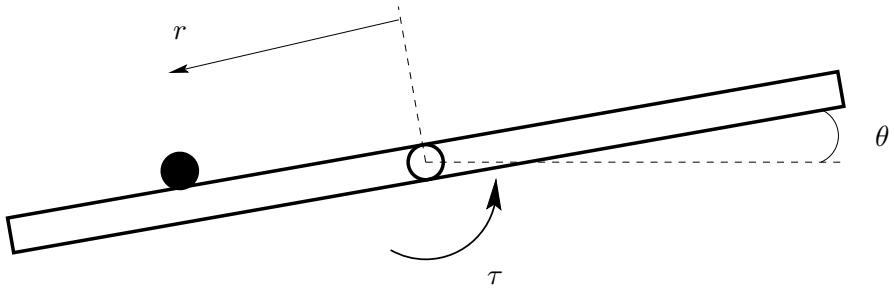
□

EXERCISE 11.13

The ball-and-beam system is given by the equations

$$0 = \ddot{r} + g \sin\theta + \beta\dot{r} - r\dot{\theta}^2$$
$$\tau = (r^2 + 1)\ddot{\theta} + 2r\dot{r}\dot{\theta} + gr\cos\theta,$$

where r is the position of the ball, θ the angle of the beam, τ the torque applied to the beam, $g > 0$ the gravity constant, and $\beta > 0$ the viscous friction constant.



(a) Transform the system into first-order state-space form $\dot{x} = f(x, u)$, where $x = (r, \dot{r}, \theta, \dot{\theta})^T$ and $u = \tau$.

(b) Determine all equilibria of (11.13).

(c) Some of the equilibria in (b) correspond to that the beam is upside-down. Disregard these and linearize the system about one of the appropriate equilibria.

(d) Discuss how one can obtain a state feedback control law ($r, \dot{r}, \theta, \dot{\theta}$ are all measurable) that gives a locally asymptotically stable ball-and-beam system. You don't have to come up with an explicit solution.

(e) Consider only the first equation of (11.13) and assume that $\theta = \dot{\theta} = 0$. Show that $(r, \dot{r}) = (0, 0)$ is globally stable. What about asymptotic stability? (*Hint:* $V = (\beta r + \dot{r})^2/2 + \dot{r}^2/2$.)

□

EXERCISE 11.14

Consider the control system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + u \\ \dot{x}_2 &= x_1 \end{aligned}$$

(a) Sketch the phase portrait for $u = 0$.

(b) Based on the Lyapunov function candidate $V(x) = x_1^2 + x_2^2$, suggest a control law $u = g(x_1, x_2)$, such that the origin is globally asymptotically stable. You need to clearly motivate why the control law of your choice makes the system globally asymptotically stable.

(c) Derive a globally stabilizing control law $u = h(x_1, x_2)$ based on exact feedback linearization.

□

2.12 Feedback linearization

EXERCISE 12.1

Consider a mass with damping and spring described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\phi(x_1) - \psi(x_2) + u \end{aligned}$$

where x_1 denotes the position and x_2 denotes the velocity. The force imposed by the spring is given by the nonlinear function $\phi(x_1)$ and the damping is described by the nonlinear function $\psi(x_2)$. Compute the feedback such that the overall system is linear with two arbitrary poles.

□

EXERCISE 12.2

Consider the system

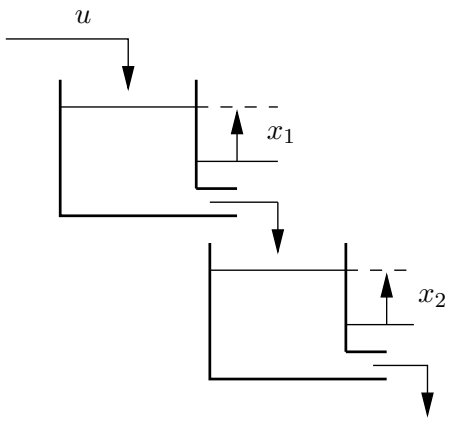
$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_2 \\ \dot{x}_2 &= u \\ y &= x_1 \end{aligned}$$

Determine a feedback control law, exactly linearizing the system.

□

EXERCISE 12.3

Consider the system with two water tanks



The dynamics are given by

$$\begin{aligned}\dot{x}_1 &= 1 + u - \sqrt{1 + x_1} \\ \dot{x}_2 &= \sqrt{1 + x_1} - \sqrt{1 + x_2} \\ y &= x_2\end{aligned}$$

(a) Show that the system has a strong relative degree of 2 ($r = 2$).

(b) Determine the coordinate transformation such that the system can be written as

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \alpha(z) + \beta(z)u \\ y &= z_1\end{aligned}$$

(c) Determine a feedback control law, exactly linearizing the system.

□

EXERCISE 12.4

Find a feedback law

$$u = h(x, v)$$

that renders the closed loop system from the new input, v to the output, y, linear. Is the zero dynamic stable or unstable?

(a) Consider

$$\begin{aligned}\dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= u \\ y &= x_1 + x_2\end{aligned}$$

(b) Consider

$$\begin{aligned}\dot{x}_1 &= -x_2^2 + u \\ \dot{x}_2 &= u \\ y &= x_1\end{aligned}$$

□

EXERCISE 12.5

Determine a feedback control law

$$\dot{x}_1 = -x_1 + 7x_2$$

$$\dot{x}_2 = -x_2 + \cos x_1 + u$$

exactly linearizing the system.

□

EXERCISE 12.6

Determine a feedback control law

$$\begin{aligned}\dot{x}_1 &= x_3 + 8x_2 \\ \dot{x}_2 &= -x_2 + x_3 \\ \dot{x}_3 &= -x_3 + x_1^4 - x_1^2 + u \\ y &= x_1\end{aligned}$$

exactly linearizing the system.

□

EXERCISE 12.7

An important class of nonlinear systems can be written on the form

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= f(x) + g(x)u\end{aligned}$$

Assume that the full state x is available for measurement.

(a) Find a feedback

$$u = h(x, v)$$

that renders the closed loop system from the new input v to the state linear. What conditions do you have to impose on $f(x)$ and $g(x)$ in order to make the procedure well posed?

(b) Apply this procedure to design a feedback for the inverted pendulum

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= a\sin(x_1) + b\cos(x_2)u\end{aligned}$$

that makes the closed loop system behave as a linear system with a double pole in $s = -1$. Is the control well defined for all x? Can you explain this intuitively?

(c) One drawback with the above procedure is that it is very sensitive to modelling errors. Show that this is the case by designing a linearizing feedback for the system

$$\dot{x} = x^2 + u$$

that makes the closed loop system linear with a pole in -1. Apply the suggested control to the system

$$\dot{x} = (1+\epsilon)x^2 + u$$

and show that some solutions are unbounded irrespectively of $\epsilon \neq 0$.

□

2.13 Optimal Control

EXERCISE 13.1

For a fixed $t_f > 0$, find an optimal control law u^* that solves

$$\begin{aligned}\min_u &\int_0^{t_f} (x(t) + u2(t))\,dt\\ \text{s.t.}&\dot{x}(t) = x(t) + u(t) + 1,\\ &x(0) = 0.\end{aligned}$$

□

EXERCISE 13.2

Find the optimal control function of the control problem

$$\begin{aligned}\min_u &\int_0^{t_f} dt\\ \text{s.t.}&\dot{x}_1(t) = x_2(t),\\ &\dot{x}_2(t) = u(t),\\ &x_1(0) = x_0, \quad x_1(t_f) = 0,\\ &x_2(0) = v_0, \quad x_2(t_f) = 0,\\ &\|u(t)\| \leq 1 \text{ for all } t.\end{aligned}$$

□

EXERCISE 13.3

Neglecting air resistance and the curvature of the earth, the launching of a satellite can be described by the following equations

$$\begin{aligned}\dot{x}_1 &= x_3\\ \dot{x}_2 &= x_4\\ \dot{x}_3 &= \frac{F}{m}\cos u\\ \dot{x}_4 &= \frac{F}{m}\sin u - g\end{aligned}$$

where (x_1, x_2) is the coordinate and (x_3, x_4) the corresponding velocity. The signal u is the controlled angle. Suppose that $\phi(x(t_f))$ is the criterion to be minimized and that $\psi(x(t_f)) = 0$ is the end constraint. Show that the optimal control signal has the form

$$\tan u^*(t) = \frac{At + B}{Ct + D}$$

where A, B, C, D are constants (which you do not have to determine).

EXERCISE 13.4 □

Suppose more realistically that m and F vary. Let u_1 be the control signal in previous exercise and $u_2(t) = F(t)$. Suppose u_2 has limitations

$$0 \le u_2(t) \le u_{\max}$$

Let the mass $m = x_3(t)$ vary as

$$\dot{x}_3 = -\gamma u_2$$

Show that

$$\tan u_1 = \frac{\lambda_1}{\lambda_4} \quad \text{and} \quad u_2 = \begin{cases} u_{\max} & \sigma < 0\\ 0 & \sigma > 0\\ \star & \sigma = 0, \end{cases}$$

where $\star$ means that the solution is unknown. Determine equations for λ and σ. (You do not have to solve the equations).

EXERCISE 13.5 □

Consider the system

$$\begin{cases} \dot{x}_1 = x_2\\ \dot{x}_2 = -x_1 - x_2^3 + (1 + x_1)u \end{cases}$$

with initial conditions $x_1(0) = 1, x_2(0) = 1$ and let the criterion be

$$\min_{u:[0,1]\to\mathbf{R}} \int_0^1 (e^{x_1^2} + x_2^2 + u^2)\, dt.$$

Show that the extremals satisfy

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, \lambda_1, \lambda_2)\\ \dot{x}_2 &= f_2(x_1, x_2, \lambda_1, \lambda_2)\\ \dot{\lambda}_1 &= f_3(x_1, x_2, \lambda_1, \lambda_2)\\ \dot{\lambda}_2 &= f_4(x_1, x_2, \lambda_1, \lambda_2)\end{aligned}$$

Determine f_1, f_2, f_3, f_4. What conditions must λ_1, λ_2 satisfy at the end point?

□

EXERCISE 13.6

Consider the double integrator

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u, \qquad |u| \leq 1 \end{aligned}$$

with initial value $x(0) = x_0$. We are interested in finding the control that brings the system to rest ($x(t_f) = 0$) in minimum time. (You may think of this as a way of designing a controller that reacts quickly to set-point changes.)

(a) Show that the optimal control is of "bang-bang type" with at most one switch. (In other words, show that the optimal control is first maximal in one direction for some time, and then possibly changes sign to become maximal in the other direction for the remaining time)

(b) Show that the control can be expressed as a feedback law

$$u = \begin{cases} u_{\max} & \sigma(x) > 0 \\ -u_{\max} & \sigma(x) < 0 \end{cases}$$

Derive an explicit formula for the switch function $\sigma(x)$.

(c) Using a phase plane tool, construct the phase portrait of the closed-loop system under the optimal control.

□

EXERCISE 13.7

Consider the problem of controlling the double integrator

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u, \ \ |u| \leq 1 \end{cases}$$

from an arbitrary initial condition $x(0)$ to the origin so that the criterion

$$\int_0^{t_f} (1 + |u|)\, dt$$

is minimized (t_f is the first time so that $x(t) = 0$). Show that all extremals are of the form

$$u(t) = \begin{cases} -1 & 0 \leq t \leq t_1 \\ 0 & t_1 \leq t \leq t_2 \\ 1 & t_2 \leq t \leq t_f \end{cases}$$

or

$$u(t) = \begin{cases} 1 & 0 \leq t \leq t_1 \\ 0 & t_1 \leq t \leq t_2 \\ -1 & t_2 \leq t \leq t_f \end{cases}$$

for some t_1, t_2 with $0 \leq t_1 \leq t_2 \leq t_f$. (Some time interval can thus have length zero.) Assume that the problem is normal.

□

EXERCISE 13.8

Consider the problem of taking the state of the system

$$\dot{x} = \begin{pmatrix} -5 & 2 \\ -6 & 2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

from $x_0 = (0,0)^T$ to $x(t_f) = (1,1)^T$ in minimum time with $|u(t)| \leq 3$. Show that the optimal controller is either

$$u(t) = \begin{cases} -3 & 0 \leq t \leq t_1 \\ +3 & t_1 \leq t \leq t_f \end{cases}$$

or

$$u(t) = \begin{cases} +3 & 0 \leq t \leq t_1 \\ -3 & t_1 \leq t \leq t_f \end{cases}$$

for some t_1.

□

EXERCISE 13.9

Show that the minimum-time control of a linear system

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad |u| \leq 1, \quad x(t_f) = 0,$$

leads to extremals with

$$u(t) = \text{sign}\, \sigma(t)$$

Determine $\sigma(t)$.

□

EXERCISE 13.10

What is the conclusion from the maximum principle for the problem

$$\begin{aligned} &\min \int_0^1 u\, dt, \\ &\dot{x} = u \\ &x(0) = 0 \\ &x(1) = 1 \end{aligned}$$

□

EXERCISE 13.11

Consider an optimal heating problem where T is the temperature and P heat effect. Let

$$\begin{aligned} &\dot{T} = P - T \\ &0 \leq P \leq P_{\max} \\ &T(0) = 0, \quad T(1) = 1 \end{aligned}$$

and solve $\min \int_0^1 P(t)\, dt$.

□

EXERCISE 13.12

In this problem we study the linear quadratic optimal control problem

$$\min_{u:[0,t_f]\to\mathbb{R}^m} \frac{1}{2}\int_0^{t_f} \left[x(t)^T Q x(t) + u(t)^T R u(t)\right] dt$$

with

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0.$$

Suppose that $t_f > 0$ and x_0 are fixed and that the matrices $Q = Q^T$ and $R = R^T$ are positive definite. Then, the optimal control is given by

$$u^*(t) = -R^{-1}B^T S(t)x(t),$$

where the matrix $S(t) = S^T(t)$ satisfies the differential equation

$$-\dot{S}(t) = A^T S(t) + S(t)A - S(t)BR^{-1}B^T S(t) + Q,$$

where $S(t_f) = 0_{n\times n}$ is an $n \times n$ zero matrix.

(a) Determine the Hamiltonian function $H(x, u, \lambda)$.

(b) Derive the adjoint equation for the optimal control problem, that is, the differential equation for $\lambda(t)$ (including the final condition $\lambda(t_f)$).

(c) Show that the optimal control can be written as

$$u^*(t) = -R^{-1}B^T\lambda(t),$$

where λ is the solution to the adjoint equation in (b). (*Hint:* Derive the solution to $\partial H/\partial u = 0$.)

(d) Show that (13.12) is the optimal control with $S(t)$ given by (13.12). Do this by setting $\lambda(t) = S(t)x^*(t)$. Then derive

$$\dot{\lambda}(t) = \dot{S}(t)x^*(t) + S(t)\dot{x}^*(t) = \dot{S}(t)x(t) + S(t)[Ax^*(t) + Bu^*(t)]$$

and use (b) together with (c). You will end up with an equation

$$-\dot{S}(t)x^*(t) = [A^T S(t) + S(t)A - S(t)BR^{-1}B^T S(t) + Q]x^*(t)$$

from which you can conclude (13.12).

(e) What is the solution for the case $t_f = \infty$?

(f) Show that $H(x, u, \lambda)$ is constant along every optimal trajectory $(x^*(t), u^*(t))$. (*Hint:* Derive $\dot{H}(x^*(t), u^*(t), \lambda(t))$ and use that $\frac{\partial H}{\partial u}(x^*(t), u^*(t), \lambda(t)) = 0$ since u^* minimizes H.)

□

EXERCISE 13.13

The double integrator

$$\ddot{y} = u$$

should be controlled from $(y, \dot{y}) = (10, 0)$ to $(y, \dot{y}) = (0, 0)$ as fast as possible given the constraint $|u| \leq C$, $C > 0$. Show that the optimal control law is given by

$$u(t) = C \operatorname{sgn} p(t),$$

where p is a polynomial. Detemine the degree of p. [You do not have to derive p explicitly.]

□

EXERCISE 13.14

Determine the control law that minimizes

$$J(u) = \int_0^\infty \frac{5}{2} y^2 + 2u^2 dt$$

The system is given by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \end{aligned}$$

□

2.14 Fuzzy control

EXERCISE 14.1

Consider the inverted pendulum model that we have been using in previous exercises

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin(x_1) + \cos(x_1)u \end{aligned}$$

Here, the first state is the pendulum angle and the second state describes the angular velocity. Assume that we have defined fuzzy sets that describe out notions of the angle and angular velocity being "negative large", "negative small", "zero", "positive small" and "positive large". The fuzzy sets for the control signal (the acceleration of the pendulum pivot). The fuzzy rule base can be represented by the table below

	NL	NS	ZE	PS	PL
NL					
NS					
ZE			ZE		
PS					
PL					

Here the first row corresponds to the proposition "x_1 is negative large", the first row corresponds to "x_2 is negative large", etc. The already filled-in entry of the table specifies the rule

IF x_1 IS ZERO AND x_2 IS ZERO THEN u IS ZERO

(a) Fill in the remaining rules based on your intuition for controlling an inverted pendulum

(b) Implement the controller in Matlab/Simulink and see if your fuzzy controller performs better than the linear control law

$$u = -2x_1 - 2x_2$$

□

Chapter 3

Solutions

3.1 Solutions to Nonlinear Models and Simulation

SOLUTION 1.1

(a) Choose the angular position and velocity as state variables, *i.e.*, let

$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$

We obtain

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$$

(b) By setting the state derivatives to zero, we obtain

$$0 = x_1$$
$$0 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$$

We find the equilibrium points $(x_1, x_2) = (n\pi, 0)$ with $n = 0, \pm 1, \pm 2, \ldots$. The equilibrium points correspond to the pendulum hanging down (n even), or the pendulum balancing in the upright position (n odd). Linearization gives

$$\frac{d}{dt}\Delta x = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}(-1)^n & -\frac{k}{m} \end{pmatrix} \Delta x \tag{3.1}$$

The linearized system is stable for even n, and unstable for odd n. We can use Lyapunov's linearization method to conclude that the pendulum is LAS around the lower equilibrium point, and unstable around the upper equilibrium point.

□

SOLUTION 1.2

1. We can rewrite the equation:

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) - \frac{k}{m}\dot{\theta} + \frac{C}{ml^2}$$

Then we choose the angular position and velocity as state variables:

$$\begin{aligned} x_1 &= \theta \\ x_2 &= \dot{\theta} \end{aligned}$$

We obtain:

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 + \frac{C}{ml^2} \end{aligned}$$

2. By setting the state derivatives to zero, we obtain:

$$\begin{aligned} 0 &= x_2 \\ 0 &= -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 + \frac{C}{ml^2} \end{aligned}$$

So, at the equilibrium points, x_1 satisfies:

$$\sin(x_1) = \frac{C}{mlg}$$

So, all the equilibrium points are:

$$\begin{aligned} x_1 &= \arcsin(\frac{C}{mlg}) + 2n\pi \\ or \quad & -\arcsin(\frac{C}{mlg}) + (2n+1)\pi \\ x_2 &= 0 \end{aligned}$$

3. The linearization gives the A matrix:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos(x_1) & -\frac{k}{m} \end{pmatrix}$$

if $x_1 = \arcsin(\frac{C}{mlg}) + 2n\pi$ then

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\sqrt{1-(\frac{C}{mlg})^2} & -\frac{k}{m} \end{pmatrix}$$

and its eigenvalues are:

$$\lambda = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{(\frac{k}{m})^2 - 4\frac{g}{l}\sqrt{1-(\frac{C}{mlg})^2}}$$

if $x_1 = -\arcsin(\frac{C}{mlg}) + (2n+1)\pi$ then

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ +\frac{g}{l}\sqrt{1-(\frac{C}{mlg})^2} & -\frac{k}{m} \end{pmatrix}$$

and its eigenvalues are:

$$\lambda = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{(\frac{k}{m})^2 + 4\frac{g}{l}\sqrt{1-(\frac{C}{mlg})^2}}$$

□

SOLUTION 1.3

(a) Let

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

be a state-space realisation of the linear system. We have

$$u = r - \psi(t, y) = r - \psi(t, Cx)$$

and hence

$$\dot{x} = Ax - B\psi(t, Cx) + Br, \qquad y = Cx$$

(b) To separate the linear dynamics from the nonlinearities, write the pendulum state equations as

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{k}{m}x_2 - \frac{g}{l}\sin(x_1)\end{aligned}$$

and view $\sin(x_1)$ as an input, generated by a nonlinear feedback from $y = x_1$ (Compare with Figure 2.2). Introduce the state vector $x = (x_1, x_2)^T$, and re-write the equations as

$$\dot{x} = \begin{bmatrix}0 & 1\\ 0 & -k/m\end{bmatrix} x + \begin{bmatrix}0\\ g/l\end{bmatrix} u \tag{3.2}$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix} x \tag{3.3}$$

$$u = -\sin(y), \tag{3.4}$$

which is on the requested form.

□

SOLUTION 1.4

(a) Hint: $\dot{e} = -y = -Cx$.

(b) The equilibrium points are given by

$$0 = G(0)\sin e$$

from which we obtain

$$e = \pm n\pi, \quad n = 0, 1, 2, \ldots$$

(c) For $G(s) = 1/(\tau s + 1)$, we take $A = -1/\tau$, $B = 1/\tau$ and $C = 1$. Then

$$\begin{aligned}\dot{z} &= -\frac{1}{\tau}z + \frac{1}{\tau}\sin e\\ \dot{e} &= -z\end{aligned}$$

Now, let $x_1 = e$ and $x_2 = -z$, so that

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{1}{\tau}x_2 - \frac{1}{\tau}\sin x_1,\end{aligned}$$

which is the pendulum model.

□

SOLUTION 1.5

We choose angular positions and velocities as state variables. Letting $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$, $x_4 = \dot{q}_2$, we obtain

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{k}{J}(x_1 - x_3) + \frac{1}{J}u\end{aligned}$$

□

SOLUTION 1.6

(a) Let $x_1 = \delta$, $x_2 = \dot{\delta}$, $x_3 = E_q$ and $u = E_{FD}$. We obtain

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{P}{M} - \frac{D}{M}x_2 - \frac{\eta_1}{M}x_3 \sin x_1 \\ \dot{x}_3 &= -\frac{\eta_2}{\tau}x_3 + \frac{\eta_3}{\tau}\cos x_1 + \frac{1}{\tau}u\end{aligned}$$

(b) With E_q being constant, the model reduces to

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{P}{M} - \frac{D}{M}x_2 - \frac{\eta_1}{M}E_q \sin x_1\end{aligned}$$

which is the pendulum equation with input torque P/M.

(c) The equilibrium points of the simplified equation are given by $\sin x_1 = \frac{P}{\eta_1 E_q}$, $x_2 = 0$.

□

SOLUTION 1.9

1. Problems involving valves are typically nonlinear.
2. This is a typical nonlinear system. There is a saturation somewhere in the system.
3. A linear system since the acceleration (due to gravity) is constant.
4. There is nothing to indicate that this system is nonlinear, it has the characteristics of a LTI-system so far.
5. This function is stochastic and cannot be linear. But just because it is not linear can it then be classified as nonlinear?

6. The control system for JAS 39 Gripen consists of 50 different linear controllers between which the system switches around depending on the operating conditions. The airplane itself is highly nonlinear.
7. This is a nonlinear system. It can only take on two different values, on and off.

□

SOLUTION 1.10

(a) The system $\dot{x} = f(x) = xu(x)$ is locally asymptotically stable at $x = 0$ if

$$\left.\frac{df}{dx}\right|_{x=0} = u(0) < 0$$

(b) $u = kx$ gives $\dot{x} = kx^2$. If $k > 0$ then $x \to \infty$ for $x(0) > 0$ and if $k < 0$ then $x \to -\infty$ for $x(0) < 0$; hence the system is unstable. For $k = 0$, we have the equation $\dot{x} = 0$ which is not asymptotically stable.

(c) $k_1 = 0$ and $k_2 < 0$ (e.g., $k_2 = -1$) gives $\dot{x} = k_2 x$, which is (globally) asymptotically stable.

(d) The Lyapunov function $V(x) = x^2/2$ can be used to prove that $\dot{x} = -x^3$ is globally asymptotically stable.

(e) Follows from the feedback system

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\\ u &= \Delta y\end{aligned}$$

□

3.2 Solutions to Computer exercise: Simulation

SOLUTION 2.1

See Lecture 2 for some hints.

□

SOLUTION 2.2

Blank

□

SOLUTION 2.2

Blank

□

SOLUTION 2.3

Blank

□

SOLUTION 2.4

Blank

□

SOLUTION 2.5

Plot only one variable, e.g., x_1.

□

3.3 Solutions to Linearization and Phase-Plane Analysis

SOLUTION 3.1

(a) The equilibrium points are

$$(x_1, x_2) = (0, 0), (\sqrt{6}, 0), (-\sqrt{6}, 0),$$

which are stable focus, saddle, and saddle, respectively.

(b) The equilibrium points are

$$(x_1, x_2) = (0, 0), (-2.5505, -2.5505), (-7.4495, -7.4495),$$

which are stable node, saddle point, and stable focus, respectively.

(c) The equilibrium points are

$$(x_1, x_2) = (0, 0), (1, 0), (0, 1),$$

which are unstable node, saddle, and stable node, respectively.

(d) The equilibrium is

$$(x_1, x_2) = (0, 0),$$

which is an unstable focus.

(e) The equilibrium point is

$$(x_1, x_2) = (0, 0),$$

which is a stable focus.

(f) The system has an equilibrium set

$$x_1^2 + x_2^2 = 1$$

and an equilibrium point

$$(x_1, x_2) = (0, 0),$$

which is a stable focus.

□

SOLUTION 3.2

The three equilibrium points are

$$(x_1, x_2) = (0, 0), (\sqrt{(ac/b)}, a), (-\sqrt{(ac/b)}).$$

The first equilibrium point is a saddle. The other equilibria are stable nodes if $8a < c$ and stable focuses if $8a > c$.

□

SOLUTION 3.3

(a) The system has three equilibrium points

$$(x_1, x_2) = (0, 0), (a, 0), (-a, 0)$$

where a is the first positive root of

$$a - \tan(\frac{a}{2}) = 0$$

given, approximately, by $a = 2.33$. The origin is a stable node, while the other two equilibria are saddles.

(b) The system has the origin as a unique equilibrium point, being a saddle.

(c) The system has the equilibrium points

$$(x_1, x_2) = (0, 0), (1, 2), (-1, 2),$$

which are saddle, stable focus, and stable focus, respectively.

□

SOLUTION 3.4

Close to the origin, the saturation element opens in the linear region, and all system are assigned the same closed loop dynamics. Far away from the origin, the influence of the saturated control can be neglected, and the open loop dynamics governs the behaviour.

(a) System (a) has one stable and one unstable eigenvalue. For initial values close to the stable eigenvector, the state will move towards the origin. For initial values close to the unstable eigenvector, the system diverges towards infinity. This corresponds to the rightmost phase portrait.

(b) All eigenvalues of system (b) are unstable. Thus, for initial values sufficiently far from the origin, the system state will diverge. This corresponds to the leftmost phase portrait. Note how the region of attraction (the set of initial states, for which the state converges to the origin) is severly limited.

(c) System (c) is stable also in open loop. This corresponds to the phase portrait in the middle.

□

SOLUTION 3.5

(a) The equilibrium points are obtained by setting $\dot{x} = 0$. For $K \neq -2$, the origin is the unique equilibrium point. When $K = -2$, the line $x_1 = 2x_2$ is an equilibrium set.

(b) The Jacobian and its eigenvalues are given by

$$\frac{\partial f}{\partial x}(0) = \begin{bmatrix} -1 & -K \\ 1 & -2 \end{bmatrix}, \quad \lambda = -\frac{3}{2} \pm \sqrt{\frac{1}{4} - K}.$$

Thus, the closed loop system is asymptotically stable about the origin for $K > -2$. Depending on the value of K, we can origin has the following character

$$\begin{aligned} \frac{1}{4} < K & \qquad \text{stable focus} \\ -2 < K < \frac{1}{4} & \qquad \text{stable node} \\ K < -2 & \qquad \text{saddle.} \end{aligned}$$

□

SOLUTION 3.6

The equilibria are given by $\sin x_1^0 = \frac{P}{\eta E_q}$, $x_2^0 = 0$. The characteristic equation for the lineraisation becomes

$$\lambda^2 + \alpha\lambda + \beta = 0,$$

where $\alpha = \frac{D}{M} > 0$ and $\beta = \frac{\eta E_q}{M}\cos x_1^0$. Depending on α, β the equilibria are stable focus, stable nodes or saddle points.

□

SOLUTION 3.7

$\dot{\tilde{x}} = A(t)\tilde{x}$, where

$$A(t) = \begin{pmatrix} -2\sin^2(t) & 1-\sin(2t) \\ -1-\sin(2t) & -2\cos^2(t) \end{pmatrix}.$$

□

SOLUTION 3.8

$$\frac{7}{5}\ddot{\tilde{x}} = g\cos(\phi_0)\tilde{\phi} + \frac{2r}{5}\ddot{\tilde{\phi}}$$

□

SOLUTION 3.9

Using the identitiy

$$(\sin t)^3 = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t$$

we see that $u_0(t) = \sin(3t), y_0(t) = \sin t$ is a nominal solution. The linearisation is given by

$$\ddot{\tilde{y}} + 4\sin^2 t \cdot \tilde{y} = -\frac{1}{3}\tilde{u}.$$

□

SOLUTION 3.10

Introduce f through the equation $\ddot{\theta} = f(\theta, \dot{\theta}, \beta, \dot{\beta})$ and let c_1, c_2, c_3 denote the (positive) constants in f. Linearizing f around $\theta = \beta = \dot{\beta} = 0$ gives

$$\begin{aligned} f(\theta, \dot{\theta}, \beta, \dot{\beta}) &\approx \frac{\partial f}{\partial \theta}\theta + \frac{\partial f}{\partial \dot{\theta}}\dot{\theta} + \frac{\partial f}{\partial \beta}\beta + \frac{\partial f}{\partial \dot{\beta}}\dot{\beta} \\ &= c_1\theta + c_2\beta + c_3\dot{\beta} \end{aligned}$$

where $\dot{\beta} = u$. Hence, the linearized system is given by

$$\ddot{\theta} = c_1\theta + c_2\beta + c_3 u.$$

It follows that in the Laplace domain

$$s^2\theta = c_1\theta + c_2\beta + c_3 u = c_1\theta + (c_2/s + c_3)u$$

and thus

$$G(s) = \frac{c_3 s + c_2}{s(s^2 - c_1)}.$$

The poles are located in $0, \pm\sqrt{c_1}$ and the zero in $-c_2/c_3$. The bicycle is, as expected, unstable.

□

SOLUTION 3.11

(a) With $x = (\theta, \dot{\theta}, \phi, \dot{\phi})^T$, we have if $\sin\theta \neq 0$,

$$\dot{x} = \begin{pmatrix} x_2 \\ x_4^2 \sin x_1 \cos x_1 - \sin x_1 \\ x_4 \\ -2x_2 x_4 \cos x_1 / \sin x_1 \end{pmatrix}$$

(b) Setting all derivatives in the original equation equal to zero, yields $\sin\theta = 0$. Hence, the equilibria are determined by $\theta_k = k\pi$ with ϕ taking any value. (Note that the equilibria cannot be obtained directly from the first-order form in (a).)

(c) The solution $(\theta(t), \phi(t)) = (\pi/3, t\sqrt{2})$ fulfills the first pendulum equation, since

$$-2\sin\pi/3\cos\pi/3 + \sin\pi/3 = 0$$

The second equation is also satisfied.

(d) Denote the first pendulum equation by $f_1(z) = 0$ and the second by $f_2(z) = 0$ where $z = (\theta, \dot{\theta}, \ddot{\theta}, \phi, \dot{\phi}, \ddot{\phi})^T$. To linearize these equations, we write

$$\begin{aligned} 0 = f_1(z(t)) = f_1(z^0(t) + \delta z(t)) &\approx f_1(z^0(t)) + \left.\frac{df_1}{dz}\right|_{z=z^0(t)} \delta z(t) \\ &= \left.\frac{\partial f_1}{\partial z_1}\right|_{z=z^0(t)} \delta z_1(t) + \cdots + \left.\frac{\partial f_1}{\partial z_6}\right|_{z=z^0(t)} \delta z_6(t) \end{aligned}$$

and similar for f_2. Deriving the partial derivatives and using the definition of z ($\delta z_1 = \delta\theta$ etc.), we get

$$\begin{aligned} \frac{3}{2}\delta\theta + \delta\ddot{\theta} - \sqrt{\frac{3}{2}}\delta\dot{\phi} &= 0 \\ \sqrt{2}\delta\dot{\theta} + \frac{\sqrt{3}}{2}\delta\ddot{\phi} &= 0 \end{aligned}$$

□

3.4 Solutions to Lyapunov Stability

SOLUTION 4.1

(a) Linearization about the system around the origin yields

$$A = \frac{\partial f}{\partial x} = 3ax^2$$

Thus, at the origin we have $A = 0$. Since the linearization has one eigenvalue on the imaginary axis, linearization fails to determine stability of the origin.

(b) $V(0) = 0$, $V(x) \neq 0$ for $x \neq 0$, and $V(x) \to \infty$ as $x \to \infty$. Thus, $V(x)$ satisfies the conditions for being a Lyapunov function candidate. Its time derivative is

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = 4ax^6 \tag{3.5}$$

which is negative definite for $a < 0$, and positive definite for $a > 0$. The desired results now follow from Lyapunov's stability and instability theorems.

(c) For $a = 0$, the system is linear and given by

$$\dot{x} = 0$$

The system has solutions $x(t) = x_0$ for all t. Thus, the system is stable. A similar conclusion can be drawn from the Lyapunov function used in (b).

□

SOLUTION 4.2

(a) We use the pendulum's total energy

$$V(x) = gl(1 - \cos(x_1)) + \frac{l^2}{2}x_2^2$$

as Lyapunov function. We see that V is positive, and compute the time derivative

$$\frac{dV(x)}{dt} = \sum_i \frac{\partial V}{\partial x_i}\dot{x}_i = gl\sin(x_1)x_2 + x_2(-gl\sin(x_1)) = 0$$

From Lyapunov's theorem, we conclude that the origin is a stable equilibrium point. Since $\dot{V}(x) = 0$, we can also conclude that the origin is *not* asymptotically stable; trajectories starting at a level surface $V(x) = c$, remain on this surface for all future times.

(b) For $k \neq 0$, using $V(x)$ as above, similar calculations give

$$\frac{dV(x)}{dt} = -\frac{kl^2}{m}x_2^2$$

$\dot{V}(x)$ is negative semidefinite. It is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$. In other words $\dot{V}(x) = 0$ along the x_1 axis.

Now, consider a modified Lyapunov function

$$V(x) = \frac{g}{l}(1 - \cos(x_1)) + x^T P x$$

for some positive definite matrix P. In order for P to be positive definite, the elements of P must satisfy

$$p_{11} > 0; \quad p_{22} > 0; \quad p_{11}p_{22} - p_{12}^2 > 0$$

The derivative $\dot{V}(x)$ is given by

$$\begin{aligned}\dot{V}(x) = \frac{g}{l}(1 - p_{22})x_2 \sin x_1 - \frac{g}{l}p_{12}x_1 \sin x_1 + \\ + (p_{11} - p_{12}\frac{k}{m})x_1x_2 + (p_{12} - p_{22}\frac{k}{m})x_2^2\end{aligned}$$

We now want to choose p_{11}, p_{12} and p_{22} such that $\dot{V}(x)$ is negative definite. Since the cross terms $x_1 \sin x_1$ and x_1x_2 are sign indefinite, we cancel them out by letting $p_{22} = 1$, $p_{11} = p_{12}k/m$. With these choices, p_{12} must satsify $0 < p_{12} < k/m$. We let $p_{12} = 0.5k/m$, and obtain

$$\dot{V}(x) = -\frac{1}{2}\frac{g}{l}\frac{k}{m}x_1 \sin x_1 - \frac{1}{2}\frac{k}{m}x_2^2.$$

The term $x_1 \sin x_1 > 0$ for all $0 < |x_1| < \pi$. Within this strip, $V(x)$ is positive definite, and $\dot{V}(x)$ is negative definite. We conclude that the origin is asymptotically stable.

□

SOLUTION 4.3

With $V = kx^2/2 + \dot{x}^2/2$ we get $\dot{V} = -d\dot{x}^4 \leq 0$. Since $\dot{V} = 0$ only when $\dot{x} = 0$ and the system equation then gives $\ddot{x} = -kx \neq 0$ unless also $x = 0$, we conclude that $x = \dot{x} = 0$ is the only invariant set. The origin is globally asymptotically stable since the Lyapunov function is radially unbounded.

□

SOLUTION 4.4

(a) The eigenvalues of A are $\lambda = -1/2 \pm i\sqrt{3}/2$.

(b) (i) We have

$$\begin{aligned}V(x) = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2 = (\text{if } p_{11} \neq 0) \\ = p_{11}(x_1 + \frac{p_{12}}{p_{11}}x_2)^2 + (p_{22} - \frac{p_{12}^2}{p_{11}})x_2^2\end{aligned}$$

If $p_{11} > 0$ and $p_{11}p_{22} - p_{12}^2 > 0$, both terms are non-negative. Moreover, $V(x) \to \infty$ as $x \to \infty$, and $V(x) = 0 \Rightarrow x_1 = x_2 = 0$ (This proves the "if"-part). If the conditions on p_{ij} do not hold, it is easy to find x such that $V(x) < 0$ (proving the "only if"-part).

(ii) We want to solve

$$\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix} + \begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

Reading off the elements, we see that

$$\begin{cases}2p_{12} & = -1\\ p_{22} - p_{11} - p_{12} & = 0\\ -2p_{12} - 2p_{22} & = -1\end{cases}$$

which has the solution $p_{11} = 1.5$, $p_{12} = -0.5$ and $p_{22} = 1$. P is a positive definite matrix.

(c) Use the Matlab command `lyap(A',eye(2))`.

□

SOLUTION 4.7

□

SOLUTION 4.6

(a) The mistake is that V is not really positive definite. The student has forgotten to check that $\lim_{x\to\infty} V(x) = \infty$, which must hold for a positive definite function. In fact,

$$V(x_1, x_2) = \frac{x_1^2}{1+x_1^2} + \frac{1}{2}x_2^2$$

so that $\lim_{x\to\infty} = 1$. Consequently, V is not positive definite.

(b) No, the problem is not that the student was not clever enough to find a Lyapunov function. There is no Lyapunov function, since the system is not globally stable. Let's show this now. In part (a), you may have noticed that $\dot{V} = 0$ for all x. In other words, V is an "integral" of the motion; the trajectories lie on the curves of constant value of V, *i.e.*, we have

$$V(x) = \frac{1}{2}x_2^2 + \frac{x_1^2}{1+x_1^2} = V(x_0) = c$$

If $c > 1$ then $x(t)$ cannot change sign, since

$$x_2^2 = c - \frac{x_1^2}{1+x_1^2} \geq c - 1$$

In this case, we have $|x_2| \geq \sqrt{c-1}$. Since $\dot{x}_1 = x_2$, it follows that $|x_1| \to \infty$ as $t \to \infty$. Roughly speaking, if the system starts with more initial stored energy than can possibly be stored as potential energy in the spring, the trajectories will diverge.

□

SOLUTION 4.7

Left blanc

□

SOLUTION 4.8

Verify that $V(0) = 0$, $V(x) > 0$ for $x \neq 0$ and $V(x) \to \infty$ for $||x|| \to \infty$. Now,

(a) We have

$$\begin{aligned}\frac{d}{dt}V(x_1, x_2) &= 8x_1\dot{x}_1 + 4x_2\dot{x}_2 + 16x_1^3\dot{x}_1 = \\ &= 8x_1x_2 + 4x_2(-2x_1 - 2x_2 - 4x_1^3) + 16x_1^3x_2 = \\ &= -8x_2^2\end{aligned}$$

Since $\dot{V}(x) \leq 0$, we conclude global stability.

(b) The Lyapunov function has $\dot{V}(x) = 0$ for $x_2 = 0$. For $x_2 = 0$, we obtain

$$\dot{x}_2 = -x_1(2 + x_1^2).$$

which implies that if x_2 should remain zero, then x_1 has also to be zero. The invariance theorem from the lectures can now be used to conlclude global asymptotic stability of the origin.

□

SOLUTION 4.9

(a) For $|2x_1 + x_2| \leq 1$, we have

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x. \tag{3.6}$$

The system matrix is as. stable. Hence, the origin is locally asymptotically stable.

(b) We have $V(x) > 0$ in the first and third quadrant.

$$\dot{V}(x) = \dot{x}_1 x_2 + x_1 \dot{x}_2 = x_1^2 - x_1 \text{sat}(2x_1 + x_2) + x_2^2.$$

Now, let us evaluate $\dot{V}(x)$ on the strip $x_1 x_2 = c$ and suppose that $c > 0$ is chosen so large that the saturation is active, *i.e.*, $|2x_1 + x_2| > 1$. By inspection of the dynamics, we see that for sufficiently large c, no trajectories can escape from the first quadrant. We can therefore use the following Lyapunov argument. Consider the Lyapunov function derivative

$$\dot{V}(x) = x_1^2 - x_1 + \frac{c^2}{x_1^2}.$$

If $c \geq 1$, $\dot{V}(x)$ is positive for all $x_1 \geq 0$. Hence, all trajectories starting in the first quadrant to the right of the curve $x_1 x_2 = c$ cannot cross the curve. Consequently, they cannot reach the origin.

(c) It follows from (b) that the origin is not globally asymptotically stable.

□

SOLUTION 4.10

(a) Integration of the equality $\frac{d}{d\sigma} f(\sigma x) = \frac{\partial f}{\partial x}(\sigma x) \cdot x$ gives the equation

$$f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x) \cdot x \, d\sigma.$$

We get

$$\begin{aligned} x^T P f(x) + f^T(x) P x &= x^T P \int_0^1 \frac{\partial f}{\partial x}(\sigma x) x d\sigma + \int_0^1 x^T \left[\frac{\partial f}{\partial x}(\sigma x)\right]^T d\sigma P x \\ &= x^T \int_0^1 \left\{ P \frac{\partial f}{\partial x}(\sigma x) + \left[\frac{\partial f}{\partial x}(\sigma x)\right]^T P \right\} d\sigma x \leq -x^T x \end{aligned}$$

(b) Since P is positive definite, $V(x)$ is clearly positive semidefinite. To show that it is positive definite, we need to show that $f(x) = 0$ only when $x = 0$. But the inequality proved in (a) shows that if $f(p) = 0$ then

$$0 \leq -p^T p.$$

(c) Suppose that f is bounded, i.e. that $\|f(x)\| \leq c$ for all x. Then

$$\|x^T P f + f^T P x\| \leq 2c\|P\|\|x\|.$$

But this contradicts the inequality in (a) as $\|x\| \to \infty$.

(d) We have shown that V is positive definite and radially unbounded. Moreover

$$\dot{V} = \dot{x}^T \left[\frac{\partial f}{\partial x}\right]^T P f + f^T P \frac{\partial f}{\partial x}\dot{x} = f^T \left[P\frac{\partial f}{\partial x}(x) + \left(\frac{\partial f}{\partial x}(x)\right)^T P\right] f \leq -\|f(x)\|^2.$$

Hence $\dot{V}(x) < 0$ for all $x \neq 0$. Thus, the origin is globally asymptotically stable.

□

3.5 Solutions to Input-Output Stability

SOLUTION 5.1

See the Figure 3.1.

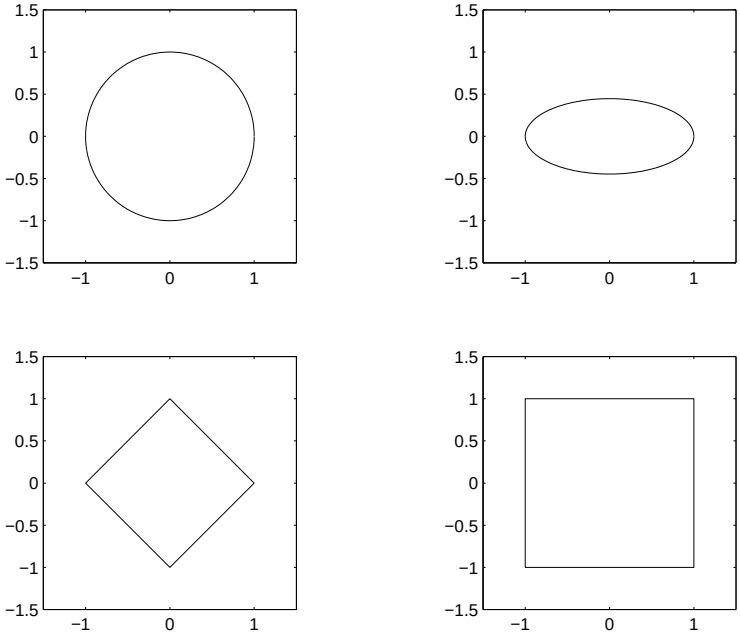


Figure 3.1: The balls $B(0, 1)$ in Exercise 4.1

□

SOLUTION 5.4

We use the definitions

$$\|y(t)\|_\infty = \sup_t |y(t)|$$

and

$$\|y(t)\|_2 = \sqrt{\int_{-\infty}^{\infty} |y(t)|^2 dt}$$

(a)

$$\|y(t)\|_2 = \int_0^\infty a^2 \sin^2(t) dt = \infty$$

, hence, the signal has infinite energy.

$$\|y(t)\|_\infty = \sup_t |asin(t)| = |a|$$

(b)

$$\|y(t)\|_2 = \int_1^\infty \frac{1}{t^2} dt = 1$$

and

$$\|y(t)\|_\infty = \sup_{t>1} |\frac{1}{t}| = 1$$

(c)

$$\|y(t)\|_2 = \int_0^\infty (e^{-t}(1-e^{-t}))^2 dt = \frac{\sqrt{3}}{6}$$

and

$$\|y(t)\|_\infty = \sup_{t>0} |e^{-t}(1-e^{-t})| = \sup_{0 \le x \le 1} |x(1-x)| = \frac{1}{4}$$

□

SOLUTION 5.3

$$\|G\| = 1, \qquad \zeta > \frac{1}{\sqrt{2}}$$

since no resonance peak, and

$$\|G\| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}, \qquad 0 < \zeta < \frac{1}{\sqrt{2}},$$

and in this case there is a resonance peak.

□

SOLUTION 5.4

Note that the requirement that the block $\psi(y)$ should be static is not a restriction for the Small Gain Theorem and the Passivity Theorem, since they can also cope with dynamic nonlinearities. For simplicity, we will only consider the case when the linear system is asymptotically stable.

(a) What are the restrictions that we must impose on the nonlinearities so that we can apply the various stability theorems?

The Nyquist Criterion $\psi(y)$ must be a linear function of y, *i.e.*, $\psi(y) = k_1 y$ for some constant k_1.

The Circle Criterion $\psi(y)$ must be contained in some sector $[k_1, k_2]$.

Small Gain Theorem $\psi(y)$ should be contained in a symmetric sector $[-k_2, k_2]$. The gain of the nonlinearity is then k_2.

The Passivity Theorem $\psi(y)$ is strictly passive for all nonlinearities in the sector $(\epsilon, 1/\epsilon)$ for some small ϵ.

These conditions are illustrated in Figure 3.2.

(b) If the above restrictions hold, we get the following conditions on the Nyquist curve

The Nyquist Criterion The Nyquist curve should not encircle the point $-1/k_2$

The Circle Criterion The Nyquist curve should not intersect the disc $D(k_1, k_2)$.

Small Gain Theorem The Nyquist curve has to be contained in a disc centered at the origin, with radius $1/k_2$.

The Passivity Theorem The Nyquist curve has to stay in the right half-plane, $Re(G(i\omega)) \geq 0$.

These conditions are illustrated in Figure 3.3.

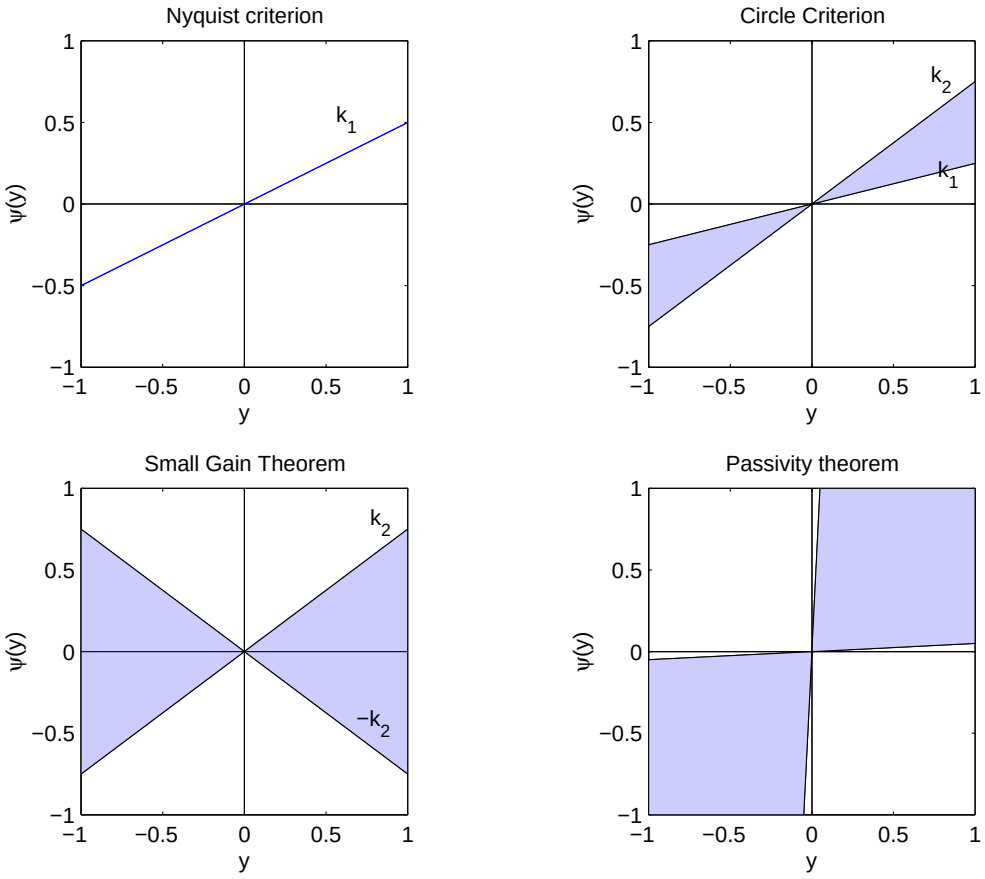


Figure 3.2: Sector conditions on memoryless nonlinearity.

□

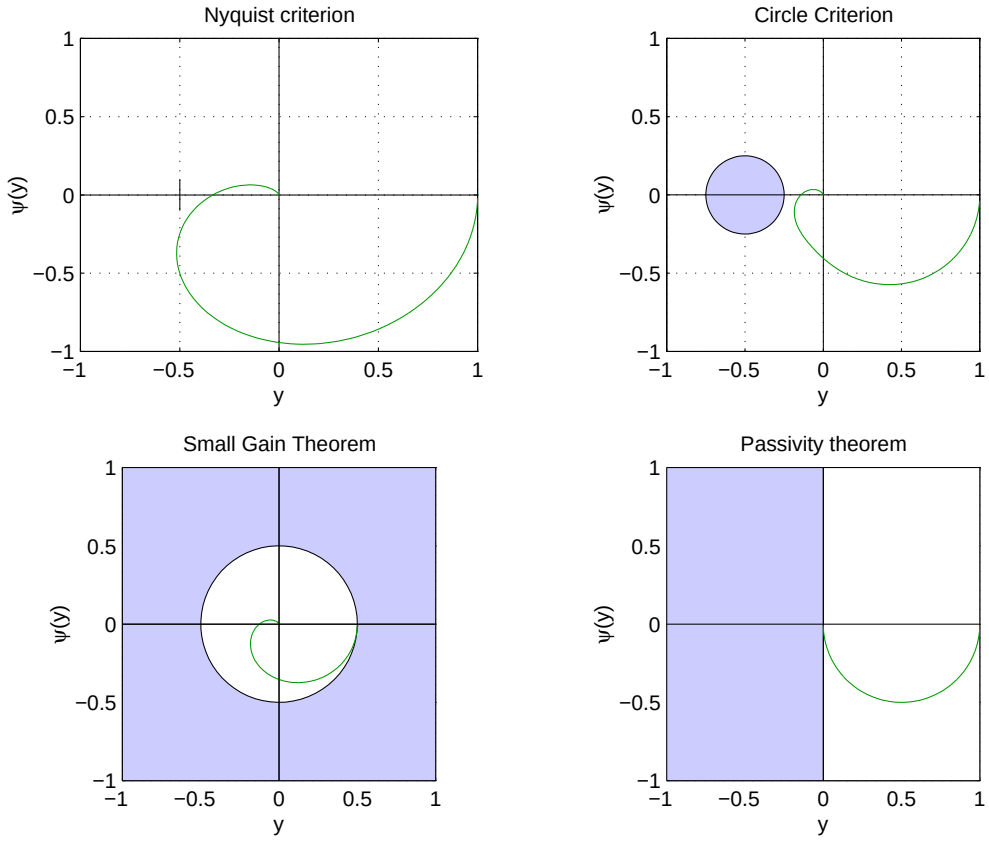


Figure 3.3: Conditions on Nyquist curve matched to sector conditions on memoryless nonlinearity.

SOLUTION 5.5

(a) The systems belong to the sectors $[0, 1]$, $[0, \infty]$ and $[-1, \infty]$ respectively.

(b) Only the saturation nonlinearity (the leftmost nonlinearity) has finite gain, which is equal to one. The other two nonlinearities have infinite gain.

(c) The saturation and the sign nonlinearity are passive. The rightmost nonlinearity is *not* passive.

□

SOLUTION 5.6

Let us first study the complete Circle criterion:

Theorem 1 (4.13 Circle Criterion)**.** *If the system*

$$\dot{x} = Ax - B\psi(y)$$
$$y = Cx$$

satisfies the conditions

1) the matrix A has no eigenvalues on the $j\omega$ axis, and has ρ eigenvalues strictly in the right half-plane;

2) the nonlinearity $\psi(y)$ belongs to the sector $[k_1, k_2]$.

and, in addition, one of the following holds

3a) $0 < k_1 \leq k_2$*, the Nyquist plot of* $G(i\omega)$ *does not enter the disk* $D(k_1, k_2)$ *and encircles it* ρ *times counter-clockwise.*

3b) $0 = k_1 < k_2$*,* $\rho = 0$*, and the Nyquist plot of* $G(i\omega)$ *stays in the half-plane Re*$p > -1/k_2$

3c) $k_1 < 0 < k_2$*,* $\rho = 0$*, and the Nyquist plot of* $G(i\omega)$ *stays in the interior of the disk* $D(k_1, k_2)$

3d) $k_1 < k_2 < 0$*, the Nyquist plot of* $-G(i\omega)$ *does not enter the disk* $D(-k_1, -k_2)$ *and encircles it* ρ *times counter-clockwise*

then the equilibrium point 0 *of the system is globally asymptotically stable.*

The open loop system has one unstable pole, and we are restricted to apply the first or fourth version of the circle criterion. In this example, we can place a disk with center in -3 and with radius 0.8, and apply the first version of the Nyquist criterion to conclude stability for all nonlinearities in the sector $[0.26, 0.45]$.

□

SOLUTION 5.7

(a)
```
>> A=[1 10; 0 1];svd(A)
ans =
   10.0990
    0.0990
```

(b)

$$\sigma_1(AB) = \sup_x \frac{\|ABx\|}{\|x\|} = \sup_x \left(\frac{\|ABx\|}{\|Bx\|} \cdot \frac{\|Bx\|}{\|x} \right)$$
$$\sup_y \left(\frac{\|Ay\|}{\|y\|} \cdot \sup_x \frac{\|Bx\|}{\|x\|} \right) = \sigma_1(A)\sigma_1(B)$$

□

SOLUTION 5.8

(a) The proof follows directly from the definition of passivity, since, according to the definition of a storage function

$$\begin{aligned} \langle u, y \rangle_T &= \int_0^T u^T y \, dt \\ &\geq \int_0^T \dot{V}(x) dt = V(x(T)) - V(x(0)) = V(x(T)) \end{aligned}$$

which is non-negative since $V(x(T)) \geq 0$. Passivity follows.

(b) With $V = \frac{1}{2} x^T P x$ we get

$$\begin{aligned} \dot{V} &= \frac{1}{2} \dot{x}^T P x + \frac{1}{2} x^T P \dot{x} \\ &= \frac{1}{2} x^T (A^T P + PA) x + \frac{1}{2} u^T B^T P x + \frac{1}{2} x^T P B u = -\frac{1}{2} x^T Q x + y^T u \leq y^T u \end{aligned}$$

since $Q > 0$, which proves passivity.

□

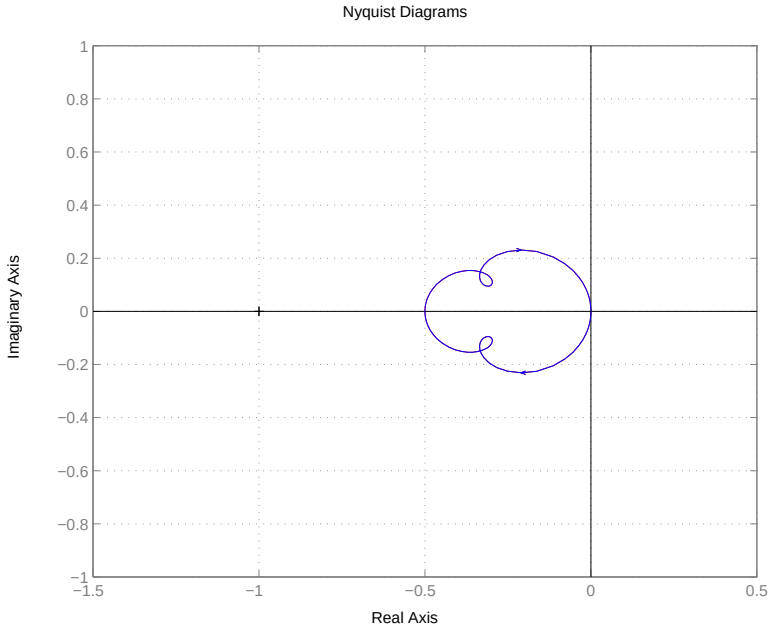


Figure 3.4: Nyquist curve of linear subsystem.

SOLUTION 5.9

The linear system $G(s)$ corresponds to

$$\dot{x} = Ax + Bu, \quad y = B^T Px, \quad x(0) = 0.$$

Let $V = x^T Px$. Then

$$\begin{aligned}\dot{V} &= \dot{x}^T Px + x^T P\dot{x} \\ &= x^T(A^T P + PA)x + 2x^T PBu = -x^T x + 2y^T u\end{aligned}$$

Integrate and use the fact that $V(0) = 0$, then

$$\int_0^T y^2 u dt = V(T) + \int_0^T x^T x dt \geq 0,$$

which proves passivity.

□

SOLUTION 5.10

You can solve this problem in many ways. In the solutions manual, we will base our proof on the circle criterion. With the obvious state vector $x = (\theta, \omega, z)'$, we rewrite the system in the feedback connection form

$$\dot{x} = Ax - B\psi(y) = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -1 & 1 \\ 2 & 0 & -2 \end{bmatrix} x - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{sat}(\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} x)$$

The Nyquist curve of the linear system is illustrated in Figure 3.4. Since the Nyquist curve does not intersect the half plane $Re(G(i\omega)) < -1/2$, we conclude stability for all ψ in the sector $[0, 2]$. Since the saturation element lies in the sector $[0, 1]$, we conclude stability of the closed loop.

□

SOLUTION 5.11

(a) We have

$$\begin{aligned}\langle y, u\rangle &= \int_0^T y(t)u(t)dt = \\ &= \int_0^T \{u(t)u_c(t)\}\{H(u(t)u_c(t))\}dt = \\ &= \int_0^T w(t)H(w(t))dt = \langle w, H(w)\rangle\end{aligned}$$

where $w = u_c u$. Since H is passive, the result follows.

(b) We will only consider the case where $\theta^0 = 0$. The case θ^0 is a little tricker, and the discussion about this case is postponed to the course in Adaptive Control.

If $\theta^0 = 0$, the system equations read

$$\begin{aligned}e(t) &= G(p)\theta u_c(t) \\ \dot{\theta}(t) &= -\gamma u_c(t)e(t)\end{aligned}$$

In light of exercise (a), we can identify the second equation modeling the signal $w(t) = u_c(t)e(t)$ sent into an integrator with gain γ and postmultiplied by u_c (See the lecture slides for an illustration). This system is passive, and interconnected in a negative feedback loop with the strictly passive system G. Stability now follows from the passivity theorem.

□

SOLUTION 5.12

(a) For simplicity we include Δ in the nonlinearity, and thus consider $\hat{G}(s) = 1/[s(s+1)]$ in feedback with $\hat{f}(y) = \Delta K \arctan(y)$. Note that $\mathrm{Re}\, G(\hat{i\omega}) = -1/(1+\omega^2) > -1$ and that $\hat{f}(\cdot)$ is bounded by the sector $[0, \Delta K]$. The Circle Criterion now gives BIBO stability for $\Delta K < 1$.

(b) Small Gain Theorem is not applicable since the gain of $1/s$ is infinite.

□

SOLUTION 5.13

(a) True. Suppose $\hat{x}$ and $\tilde{x}$ are two equilibria. A trajectory that starts in either point, stays there forever since $f(\hat{x}) = f(\tilde{x}) = 0$.

(b) False. As a counter example, take for instance the system from Lecture 3:

$$\begin{aligned}\dot{x}_1 &= x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + x_2 - x_2(x_1^2 + x_2^2).\end{aligned}$$

(c) True. Follows from Lyapunov's linearization method (Lectures 3 and 4).

(d) True. For example,

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} x$$

(e) True. Suppose it was not true. Consider a solution of $\dot{x} = f(x)$ that starts in $x_0 \neq 0$ and ends at $x(T) = x_1 = 0$. Then, $\dot{x} = -f(x)$ has a solution that starts in $x_1 = 0$ and ends at $x_0 \neq 0$. However, $\dot{x} = -f(x)$ also has the solution $x(t) = 0$ for all $t \in (0, T)$, since $\dot{x} = 0$. This is a contradiction to that $\dot{x} = -f(x)$ has a unique solution, which holds because f is $\mathbf{C}^1$ (Lecture 1). Hence, the statement in the problem must be true.

□

SOLUTION 5.14

(a) The linearization is given by

$$A = \frac{df}{dx}(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

with characteristic polynomial $s^2 + s + 1$. Hence, the system is asymptotically stable.

(b)

$$P = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$$

(c)

$$\dot{V} = \frac{dV}{dx} f(x) = 2 \begin{pmatrix} x_1 & x_2 \end{pmatrix} P \begin{pmatrix} -x_2 \\ x_1 + (x_1^2 - 1)x_2 \end{pmatrix}$$

gives the expression.

(d) $c = 4$ corresponds to that largest Ω_c contained in Π.

□

SOLUTION 5.15

1. True
2. False, We get no information about BIBO-stability from that particular methods (if they are not fulfilled). We have to try some other methods.
3. True, only one of them has to be fulfilled.
4. False, One of them has to be strictly passive to get a BIBO-stable system.
5. True
6. True

□

3.6 Solutions to Describing Function Analysis

SOLUTION 6.1

Let the input to the relay be

$$u(t) = A\sin(\omega t) = A\sin(\phi)$$

The ouput of the relay is then

$$y(\phi) = \begin{cases} -H & 0 < \phi < \phi_0 \\ H & \phi_0 < \phi < \pi + \phi_0 \\ -H & \pi + \phi_0 < \phi < 2\pi \end{cases}$$

where $\phi_0 = \arcsin(D/A)$. We get

$$\begin{aligned} a_1 &= \frac{1}{\pi}\int_0^{2\pi} y(\phi)\cos(\phi)d\phi \\ &= \frac{1}{\pi}\int_0^{\phi_0}(-H)\cos(\phi)d\phi + \frac{1}{\pi}\int_{\phi_0}^{\pi+\phi_0} H\cos(\phi)d\phi + \frac{1}{\pi}\int_{\pi+\phi_0}^{2\pi}(-H)\cos(\phi)d\phi \\ &= -\frac{4H}{\pi}\sin(\phi_0) \end{aligned}$$

and

$$\begin{aligned} b_1 &= \frac{1}{\pi}\int_0^{2\pi} y(\phi)\sin(\phi)d\phi \\ &= \frac{1}{\pi}\int_0^{\phi_0}(-H)\sin(\phi)d\phi + \frac{1}{\pi}\int_{\phi_0}^{\pi+\phi_0} H\sin(\phi)d\phi + \frac{1}{\pi}\int_{\pi+\phi_0}^{2\pi}(-H)\sin(\phi)d\phi \\ &= \frac{4H}{\pi}\cos(\phi_0) \end{aligned}$$

We obtain

$$N(A) = \frac{4H}{\pi A}(\cos(\phi_0) - i\sin(\phi_0))$$

The identity $\cos(z) = \sqrt{1-\sin^2(z)}$ gives the desired result.

□

SOLUTION 6.2

Use the interpretation of describing function as "equivalent gain". We have 1-b, 2-c, 3-a, 4-d.

□

SOLUTION 6.3

Denote the nonlinearity by f. For memoryless, static nonlinearities, the describing function does not depend on ω, and the describing function in Slotine and Li reduces to

$$N(A) = \frac{b_1(A) + ia_1(A)}{A}$$

where a_1 and b_1 can be computed as

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(A\sin(\phi))\cos(\phi)\, d\phi \tag{3.7}$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} f(A\sin(\phi))\sin(\phi)\, d\phi. \tag{3.8}$$

(a) First, we notice that the saturation is a odd function, which implies that $a_1 = 0$. In order to simplify the computations of b_1, we set $H = 1$ and note that the saturation can be described as

$$f(A\sin(\phi)) = \begin{cases} A/D\sin(\phi) & 0 \le \phi \le \phi_l \\ 1 & \phi_l < \phi < \pi/2 \end{cases}$$

Here, $\phi_l = \arcsin(D/A)$ denotes the value of ϕ where so that f saturates. Now,

$$\begin{aligned} b_1 &= \frac{1}{\pi}\int_0^{2\pi} u(\phi)\sin(\phi)d\phi = \\ &= \frac{4}{\pi}\int_0^{\pi/2} u(\phi)\sin(\phi)d\phi = \\ &= \frac{4}{\pi}\left(\int_0^{\phi_l} A/D\sin^2(\phi)d\phi + \int_{\phi_l}^{\pi/2}\sin(\phi)d\phi\right) = \\ &= \frac{4}{\pi}\left(\int_0^{\phi_l} A/(2D)(1-\cos(2\phi))d\phi + \int_{\phi_l}^{\pi/2}\sin(\phi)d\phi\right) = \\ &= \frac{4}{\pi}\left(A/(2D)(\phi_l - \sin(\phi_l)\cos(\phi_l)) + \cos(\phi_l)\right) = \\ &= \frac{2A}{D\pi}\left(\phi_l + \frac{D}{A}\cos(\phi_l)\right) \end{aligned}$$

Thus, the describing function for the normalized saturation is

$$N(A) = \frac{2}{D\pi}(\phi_l + \frac{D}{A}\cos(\phi_l))$$

Now, using the calculation rule $N_{\alpha f}(A) = \alpha N_f(A)$, we find that for the saturation under consideration we have

$$N(A) = \frac{2H}{D\pi}(\phi_l + \frac{D}{A}\cos(\phi_l))$$

(b) We see that the nonlinearity is a superposition of a linear function

$$g(e) = \frac{H}{D}e$$

and the nonlinearity $-f(e)$ with $f(e)$ as in (a). Using the fact that a linear function $g(e) = ke$ has describing function $N(A) = k$, and the superposition rule $N_{f+g}(A) = N_f(A) + N_g(A)$, we find

$$N(A) = \frac{H}{D}\left(1 - \frac{2}{\pi}\left\{\phi_l + \frac{D}{A}\cos(\phi_l)\right\}\right)$$

(c) Noting that this nonlinearity can be written as the sum of the two nonlinearities in (a) and (b), we arrive at the describing function

$$N(A) = \frac{2(\alpha-\beta)}{\pi}\left(\phi_l + \frac{D}{A}\cos(\phi_l)\right) + \beta.$$

□

SOLUTION 6.4

Follows from the integration rule

$$\int f(ax)dx = \frac{1}{a}F(ax)$$

where $F(x) = \int f(x)dx$.

□

SOLUTION 6.5

The describing function is

$$N(A) = k_1 + 3A^2k_2/4$$

Note, however, that the output $y(T)$ of the nonlinearity for the input $e(t) = A\sin(\phi)$ is

$$\begin{aligned} y(t) = &A^2k_2/2 + (k_1A + 3A^3k_2/4)\sin(\phi) \\ &- A^2k_2/2 \cdot \cos(2\phi) - A^3k_3/4 \cdot \sin(3\phi) \end{aligned}$$

We conclude that the term $k_2x_2^2$ does not influence $N(A)$. Still, we can not just apply the describing function method, since there is a bias term. If the linear system has integral action, the presence of a constant offset on the input will have a very big influence after some time.

□

SOLUTION 6.6

(a) When the saturation works in the linear range, we have the closed loop dynamics

$$G(s) = \frac{-5s}{s^2 + (1 - 5H)s + 25}$$

which is unstable for $H > 0.2$. Thus, the state can not remain small. In saturation, on the other hand, the nonlinearity generates a constant("step") input to the system. The final value theorem then gives

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} \frac{-5s}{s^2 + s + 25} = 0$$

The observation that $y(t) \to 0$ contradicts the assumption that the nonlinearity remains saturated.

(b) We should investigate intersection of the Nyquist curve and $-1/A$. Since $A \in [H, \infty)$, $-1/N(A)$ lies in the interval $(-\infty, -1/H)$.

The frequency response of the system is

$$G(i\omega) = \frac{-i5\omega}{25 - \omega^2 + i\omega}$$

which intersects the negative real axis for $\omega' = 5$ rad/s. The value of $G(i\omega') = -5$. Thus, there will be an intersection if $H > 0.2$. The frequency of the oscillation is estimated to 5 rad/s, and for fixed H.

(c) The Nyquist curve of the system is shown in Figure 3.5. The function $-1/N(A)$ is also displayed, with an arrow in the direction of increasing A. The Nyquist curve encirlces the points $Re(G(i\omega)) > -5$, indicating increased oscillation amplitude. The points to the left of the intersection are not encircled, indicating stability and a decaying oscillation amplitude. We can thus expect a stable limit cycle.

□

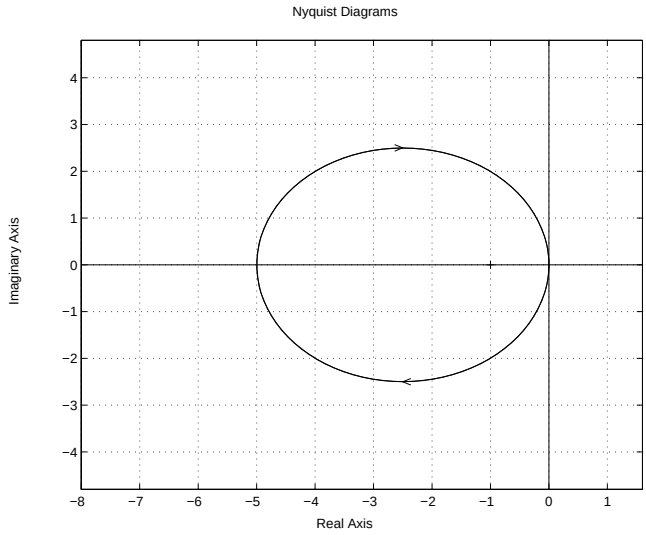


Figure 3.5: Nyquist curve and $-1/N(A)$ for oscillator example.

SOLUTION 6.7

(a) Introduce $\theta_0 = \arcsin(a/A)$ and proceed similarly to the saturation nonlinearity.

(b) The describing function has maximum for

$$A^* = \sqrt{2}a$$

wich gives

$$N(A^*) = \frac{2}{\pi a}$$

The Nyquist curve crosses the negative real axis for $\omega = \sqrt{2}$, for which the gain is $G(i\sqrt{2}) = 2/3$. Thus, we should expect no oscillations if

$$a > \frac{4}{3\pi}.$$

□

SOLUTION 6.8

(a) The descibing function for a relay with amplitude D is given by

$$N(A) = \frac{4D}{\pi A}$$

$-1/N(A)$ lies on the negative real axis. If the Nyquist curve intersects the negative real axis, the describing function methods will predict a sustained oscillation

$$-\frac{4D}{\pi A}|G(i\omega_u)| = -1$$

Thus, given the amplitude A of the oscillation, we estimate the ultimate gain as

$$K_u = 1/|G(i\omega_u)| = \frac{4D}{\pi A}$$

The ultimate period is the period time of the oscillations

$$T_u = 2\pi/\omega$$

(b) From the simulation, we estimate the amplitude $A = 0.6$ which gives $K_u \approx 0.47$. The ultimate period can be estimated directly from the plot to be $T_u \approx 2$. Note that the estimates have good correspondance with the analytical results (which require a full process model)

□

SOLUTION 6.9

Intentionally left blank.

□

SOLUTION 6.10

(a) The system consists of $G(s) = ab\exp(-sL)/s$ in negative feedback with a relay. The curve $-1/N(A)$ for the relay is the negative real axis, which $G(i\omega)$ intersects at points

$$-\frac{\pi}{2} - \omega L = \pi + 2\pi k, \qquad k = 0, 1, 2, \ldots,$$

that is, frequencies

$$\omega_k = \frac{\pi}{2L} + \frac{2\pi}{L}k, \qquad k = 0, 1, 2, \ldots.$$

Only $k = 0$ corresponds to a stable oscillation. The amplitude A of the relay input y follows from the equation $-1/N(A) = G(i\omega)$ or

$$\frac{\pi A}{4} = \frac{ab}{\omega_0},$$

that is,

$$A_0 = \frac{4ab}{\pi\omega_0}.$$

The amplitude of the oscillations in x is thus

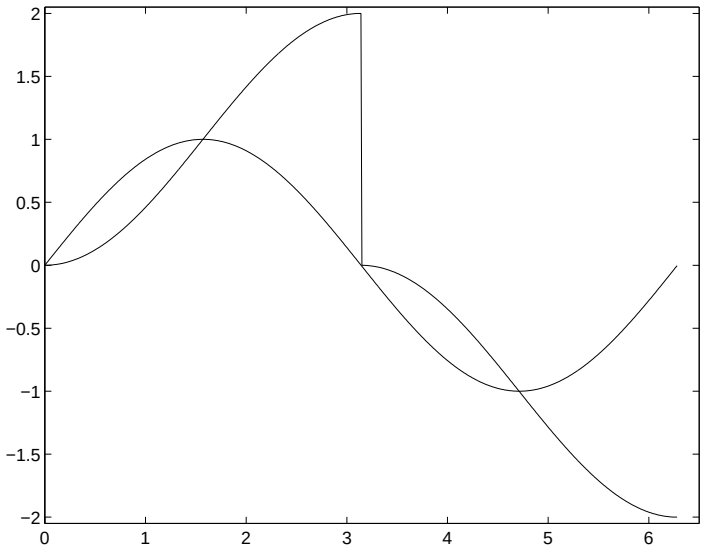
$$\frac{A_0}{b} = \frac{4a}{\pi\omega_0},$$

with ω_0 as given above.

□

SOLUTION 6.11

(a) The output of the Clegg integrator over one period of the input is shown below:



(b) The output of an integrator with sinusoidal input is

$$\int_0^t A\sin\omega t\,dt = \frac{A}{\omega}\big(1-\cos\omega t\big).$$

Using this in the derivation of the describing function $N(A,\omega) = [b_1(\omega) + ia_1(\omega)]/A$ of the Clegg integrator gives

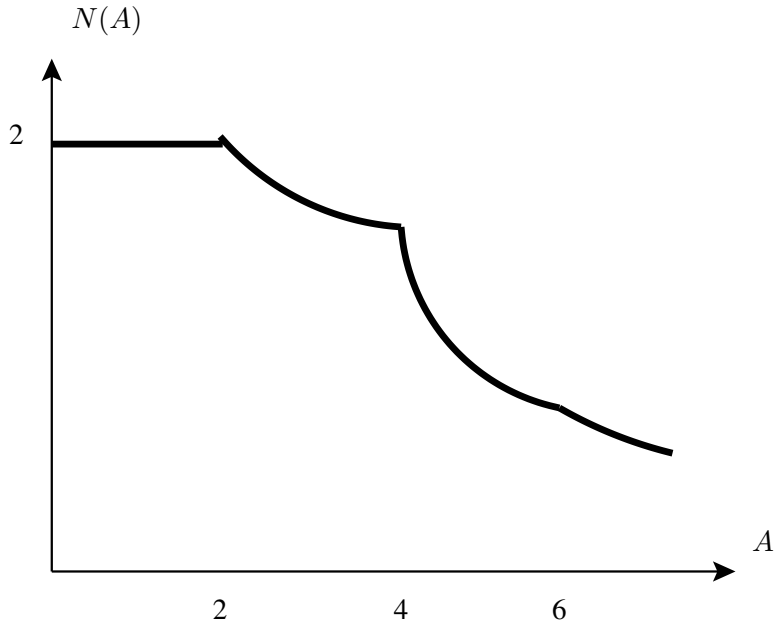
$$a_1(\omega) = \frac{1}{\pi}\int_0^\pi x(\phi)\cos\phi\,d\phi = \frac{2A}{\pi\omega}\int_0^\pi (1-\cos\phi)\cos\phi\,d\phi = \cdots = -\frac{A}{\omega}$$
$$b_1(\omega) = \frac{1}{\pi}\int_0^\pi x(\phi)\sin\phi\,d\phi = \frac{2A}{\pi\omega}\int_0^\pi (1-\cos\phi)\sin\phi\,d\phi = \cdots = -\frac{4A}{\pi\omega}$$

(c) An advantage with the Clegg integrator is that it has a phase lag of 38 degrees, which is better than the 90 degrees for an ordinary integrator. However, a disadvantage is that the Clegg integrator is likely to induce oscillations.

□

SOLUTION 6.12

(a) The describing function represents an amplitude depending gain $N(A)$. Note that $N(A)$ is real-valued (why?). A rough sketch is shown below:

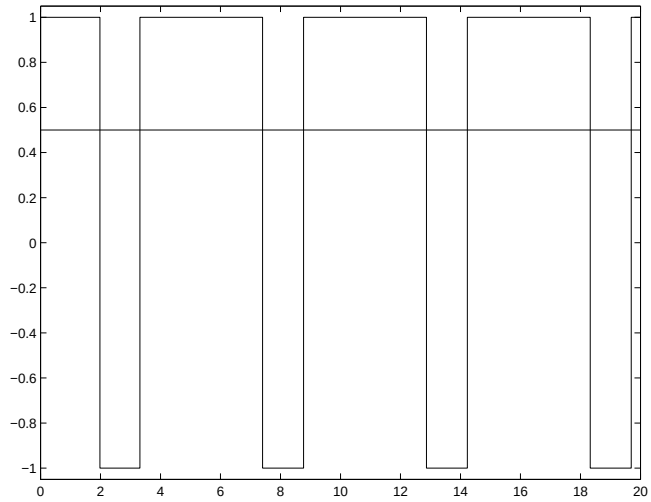


(b) We need to choose G such that its Nyquist curve intersects $-1/N(A)$. Note that $-1/N(A)$ belongs to the negative part of the real axis and that $\min N^{-1}(A) = 1/2$. A suitable candidate is

$$G(s) = \frac{2}{s(s+1)^2}.$$

□

SOLUTION 6.13



(a) See plot above.

(b) The Nyquist curve of e^{-s}/s intersects the negative real axis for the first time at $\omega = \pi/2$. The intersection corresponds to a stable oscillation. The period time estimate is thus $2\pi/\omega = 4$.

(c) Any trajectory starting in $x(0) \neq 0$ will at some instance intersect the switching surface $x = 0$. Suppose that time instance is at $t_0 = 0$ and the intersection is from the positive side. Then at time T the integrator shifts sign from $\dot{x} = -1$ to $\dot{x} = +1$. At $2T$ the state intersects the switching surface from the negative side, and the procedure repeats itself. Hence, the period is $2T + 2T = 4T$.

(d) $-1/N(A)$ for the saturation $\mathrm{sat}(\cdot)$ (without k) starts at $(-1, 0)$ and follows the negative real axis towards $-\infty$. The Nyquist curve of ke^{-s}/s intersects $(-1, 0)$ when $k \sin\omega/\omega = 1$, see (b). Hence, for $k = \pi/2$. So the DF analysis predicts an oscillation for $k > \pi/2$.

□

3.7 Solutions to Anti-windup

SOLUTION 7.1

We would like to write the system equations as

$$\begin{aligned} v &= G(s)(-u) \\ u &= \phi(v) \end{aligned}$$

where $\phi(\cdot)$ denotes the saturation. Block diagram manipulations give

$$\begin{aligned} v &= u - \left(\frac{AR}{A_w A} + \frac{BS}{A_w A} \right) u \\ &= \left(\frac{AR + BS}{AA_w} - 1 \right) (-u) = G(s)(-u) \end{aligned}$$

Since the saturation element belongs to the sector $[0, 1]$, we invoke the circle criterion and conclude stability if the Nyquist curve of $G(i\omega)$ does not enter the half plane $Re(G(i\omega)) < -1$. This gives the desired condition.

□

SOLUTION 7.2

(a) Loopgain given by

$$G_0(s) = F(s)G(s) = \frac{K_c N(sT_i + 1 + s^2 T_i T_d)(s+b)}{T_i^2 s^2 (s+0.1)(s+bN)}$$

We choose $b = 0.1$ for simpler calculations. Note, that in reality this will not necessarily be a good choice. With this choice, however, the imaginary part of $G_0(i\omega)$ is given by

$$-\frac{K_c N(0.1T_i N - 1 + \omega^2 T_i T_d)}{\omega T_i^2 (N + \omega^2)}$$

which is negative for all $\omega > 0$ if $N > 10/T_i$. Thus, the Nyquist curve of the loopgain is never crossing the negative real axis and especially not the real axis to the left of -1, and the describing function method predicts no autooscillations.

(b) It holds that

$$\tilde{U}(s) = W(s)\left[U(s) - \tilde{U}(s)\right] - F(s)G(s)U(s)$$

from which follows that $\tilde{U}(s) = -H(s)U(s)$ with

$$H(s) = \frac{F(s)G(s) - W(s)}{1 + W(s)}$$

The describing function methods predicts no autooscillations if the Nyquist curve for $H(s)$ does not cross the negative real axis to the left of -1. This is equivalent with the Nyquist curve of

$$1 + H(s) = \frac{1 + F(s)G(s)}{1 + W(s)}$$

not crossing the negative real axis. We start by calculating the denominator.

$$\begin{aligned} 1 + F(s)G(s) &= \frac{s^2(s+0.1) + K_c(s + 1/T_i + s^2 T_d)}{s^2(s+0.1)} \\ &= \frac{(s + \alpha\omega_0)(s^2 + 2\zeta\omega_0 s + \omega_0^2)}{s^2(s+0.1)} \\ &= \frac{(s+1)(s^2 + 0.7s + 0.25)}{s^2(s+0.1)} \end{aligned}$$

This leads to

$$1 + H(s) = \frac{(s+1)(s^2 + 0.7s + 0.25)}{s(s+0.1)(s + 1/T_t)}$$

If $T_t = 1$ it holds that

$$1 + H(s) = \frac{s^2 + 0.7s + 0.25}{s(s+0.1)}$$

Let $s = i\omega$. The phase of the denominator is $-\pi/2$ for $\omega = 0$ and decreases to $-\pi$ for $\omega \to \infty$. The phase of the nominator is 0 for $\omega = 0$ and increases to π for $\omega \to \infty$. Therefor it holds that the phase of $1 + H(i\omega)$ is larger than $-\pi$ for all frequencies $\omega > 0$. Thus, the describing function method predicts no autooscillations. Note that even all values for T_t such that $0 < T_t \leq 1$ are allowed, as the phase of $(i\omega + 1)/(i\omega + 1/T_t)$ is positive. Note also that we didn't take into account values $T_t > 1$.

(c) The method in (a) does not require to measure or compute the saturated signal, which is an advantage. A drawback is that the amplification of high frequency components increases, which can be a problem especially in the case of strong measurement noise in the problematic frequency range. The method in (b) does not modify the closed loop systems dynamic if the control signal is not saturated, which is an advantage. A drawback is that the saturated control signal needs to be measured or calculated.

□

SOLUTION 7.4

Large T_t corresponds to low feedback of the error signal $v-u$ (see the lecture slides for notation). Hence, $T_t = 3$ corresponds to the largest overshoot in y, $T_t = 2$ corresponds to the slightly smaller overshoot etc. Similarly for u, $T_t = 3$ corresponds to the curve that is saturated the longest amount of time etc.

□

SOLUTION 7.5

(a) The upper plots show y_{sp} (solid) and y (dashed). The middle plots show P (solid) and I (dashed). The lower plots show u (solid) and v (dashed).

From P in the middle plots we see that $K \approx 1.5$. From the left plots, it follows that the maximum of

$$I(t) = \frac{K}{T_i}\int_0^t [y_{\text{sp}}(s) - y(s)]\, ds$$

is $\max_t I(t) \approx 1$ and attained at $t \approx 2$. From the upper left plot we have $\int_0^2 [y_{\text{sp}}(s) - y(s)]\, ds \approx 1$. Hence, approximately $T_i = K = 1.5$. From the lower plots we see that the saturation level is equal to 0.5.

(b)

$$G(s) = -\frac{(KT_t - 1)T_i s + KT_t}{T_i s(T_t s + 1)}$$

(c) With notation from the lectures, the saturation satisfies $k_1 = 0$ and $k_2 = 1$. Note the negative feedback. The Circle Criterion is thus fulfilled if the Nyquist curve of $-G$ is to the right of the line $\{z \in \mathbb{C} : \text{Re}\, z = -1\}$. Here $\text{Re}\,(-G(i\omega)) = -1/(\omega^2 + 1)$, so the criterion is fulfilled. Hence, the closed-loop system is BIBO stable for all $K > 1$.

□

3.8 Solutions to Friction, Backlash and Quantization

SOLUTION 8.1

The model is given by

$$\frac{dz}{dt} = v - \frac{|v|}{g(v)}z \tag{3.9}$$

$$F = \sigma_0 z + \sigma_1(v)\frac{dz}{dt} + F_v v \tag{3.10}$$

(a) For any constant velocity, v, (3.9) converges to the value

$$z = \frac{g(v)}{|v|}v = g(v)\text{sign}(v)$$

and F therefore converges to

$$F = \sigma_0 g(v)\text{sign}(v) + F_v v$$

(b) Consider the storage function $V(z) = z^2/2$. We have

$$zv = z\frac{dz}{dt} + \frac{|v|}{g(v)}z^2 \geq z\frac{dz}{dt} = \dot{V}(t)$$

and passivity follows from the discussion in (+).

(c) Consider the storage function $V(z) = \sigma_0 z^2$. We have

$$Fv = F_v v^2 + (\sigma_1 \dot{z} + \sigma_0 z)(\dot{z} + \frac{|v|}{g(v)}z) \tag{3.11}$$

$$\geq \sigma_1 \dot{z}^2 + \frac{|v|}{g(v)}\sigma_0 z^2 + (\frac{|v|}{g(v)}\sigma_1 + \sigma_0)z\dot{z} \tag{3.12}$$

Next, we separate out the storage function derivative and make a completion of squares to estimate the additional terms

$$Fv \geq \sigma_0 z\dot{z} + \sigma_1 \dot{z}^2 + \sigma_0 \frac{|v|}{g(v)}z^2 + \sigma_1 \frac{|v|}{g(v)}z\dot{z} \tag{3.13}$$

$$= \dot{V} + \sigma_1 \left(\dot{z} + \frac{|v|}{2g(v)}z\right)^2 + \left(\sigma_0 \frac{|v|}{g(v)} - \sigma_1 \left(\frac{|v|}{2g(v)}\right)^2\right) z^2 \tag{3.14}$$

Since the second term is non-negative, we have

$$Fv \geq \dot{V}$$

and thus passivity if

$$\sigma_0 - \sigma_1 \frac{|v|}{4g(v)} > 0$$

This concludes the proof.

□

SOLUTION 8.2

(a) The describing function for a relay has been derived on Lecture 6 to be

$$N(A) = \frac{4F_0}{\pi A}$$

(b) Using the superposition property of describing functions for static nonlinearities $N_{f+g} = N_f + N_g$, and the fact that for a scalar gain $y = ku$ the describing function is $N(A) = k$, we obtain

$$N(A) = F_v + \frac{4F_0}{\pi A}$$

(c) Stiction is a point-wise phenomenon (occurring for $v = 0$) with finite amplitude, and has no influence on the integral calculations involved in the describing function computation. The describing function is therefore the same as in (b).

□

SOLUTION 8.3

Recall from the lecture slides that the process is given by

$$\begin{aligned}\dot{x} &= v \\ \dot{v} &= -F + u\end{aligned}$$

The velocity is observed through

$$\begin{aligned}\widehat{v} &= z_v + K_v x \\ \dot{z}_v &= -\widehat{F} + u - K_v \widehat{v}\end{aligned}$$

where $\widehat{F}$ denotes the estimated friction force, estimated by the observer

$$\begin{aligned}\widehat{F} &= (z_F + K_F|\widehat{v}|)\text{sign}(\widehat{v}) \\ \dot{z}_F &= -K_F(u - \widehat{F})\text{sign}(\widehat{v})\end{aligned}$$

Defining the observer errors

$$\begin{aligned}e_v &= v - \widehat{v} \\ e_F &= F - \widehat{F}\end{aligned}$$

we obtain the observer dynamics

$$\begin{aligned}\dot{e}_v &= \dot{v} - \dot{\widehat{v}} = -F + u - (-\widehat{F} + u - K_v\widehat{v}) - K_v v = -e_F - K_v e_v \\ \dot{e}_F &= \dot{F} - \dot{\widehat{F}} = \dot{F} - (-K_F(u - \widehat{F}) - K_f(-\hat{F} + u - K_v\widehat{v}) - K_v v) = \\ &= \dot{F} - K_f K_v e_v\end{aligned}$$

The term $\dot{F}$ is zero (except at zero velocity where it is not well defined). Putting $\dot{F} = 0$, we obtain

$$\begin{bmatrix}\dot{e}_v \\ \dot{e}_F\end{bmatrix} = \begin{bmatrix}-K_v & -1 \\ -K_v K_f & 0\end{bmatrix}\begin{bmatrix}e_v \\ e_F\end{bmatrix} \tag{3.15}$$

with the characteristic equation

$$\lambda(s) = s^2 + K_v s - K_v K_f$$

We conclude that the error dynamics are locally asymptotically stable if

$$\begin{aligned} K_v &> 0, \\ -K_v K_f &> 0 \end{aligned}$$

which implies the desired conditions.

□

SOLUTION 8.4

Intentionally left blank.

□

SOLUTION 8.5

We have

$$\langle u, y \rangle_T = \int_0^T uy \, dt = \int_0^T u\text{sat}(u) \, dt \geq 0$$

We conclude passivity from to the definition given in the lecture slides.

□

SOLUTION 8.6

Intentionally left blank.

□

SOLUTION 8.7

Intentionally left blank.

□

SOLUTION 8.8

(a) The system is defined by

$$\begin{aligned} \dot{x} &= \text{sat}(u - x) \\ y &= x + \text{sat}(u - x) \end{aligned}$$

If $|u - x| \leq 1$ then $y = u$. If $u - x > 1$ then

$$\begin{aligned} \dot{x} &= 1 \\ y &= x + 1 \end{aligned}$$

and thus x will increase until $u - x < 1$ so that $y = u$ holds again. Similar for $u - x < -1$.

□

SOLUTION 8.9

a) $\lim_{A\to\infty} Im\ N(A) = 0$ and $\lim_{A\to\infty} Re\ N(A) = 1$ then $-\frac{1}{N(A)} \to -1$ as $A \to \infty$. This means that when the amplitude is large enough the Nyquist curve for G(s) will intersect $-\frac{1}{N(A)}$ in -1. The conclusion is that the backlash can be neglected.

b) If the amplitude is smaller then the size of the backlash, the input signal will have no effect on the output.

□

3.9 Solutions to High-gain and Sliding mode

SOLUTION 9.1

Let the output of the nonlinearity be u, so that $u = f(v)$.

(a) We have

$$u = v^2, \qquad v \geq 0$$

which implies that

$$v = \sqrt{u}, \qquad u \geq 0$$

(b) The piecewise linear characteristic

$$u = \begin{cases} k_1 v, & |v| \leq d \\ (k_1 - k_2)d + k_2 v & |v| > d \end{cases}$$

gives the inverse

$$v = \begin{cases} u/k_1, & |u| \leq k_1 d \\ (u - \text{sign}(u)(k_1 - k_2)d)/k_2 & |u| > k_1 d \end{cases}$$

Consider a (unit) dead-zone with slope $k_1 = \epsilon$ in the interval $|v| \leq d$, and slope $k_2 = 1$ otherwise. We obtain the inverse

$$v = \begin{cases} u/\epsilon, & |u| \leq \epsilon d \\ u + \text{sign}(u)(1 - \epsilon)d, & |u| > \epsilon d \end{cases}$$

The dead-zone nonlinearity and its inverse are shown in Figure 3.6.

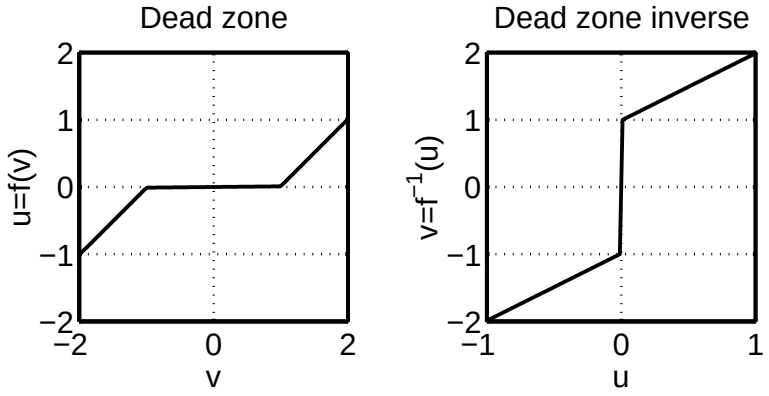


Figure 3.6: Deadzone and its inverse.

(c) See the slides from the lecture of backlash.

□

SOLUTION 9.2

$$\frac{dG_{cl}}{G_{cl}} = S\frac{dG}{G} = \frac{1}{1+KG}\frac{dG}{G} = 0.001 \Rightarrow$$

$$K = 0.298 \approx 0.3$$

□

SOLUTION 9.3

(a) As an example, study

$$\dot{x} = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix} x - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{sgn}(x_2)$$

for $(x_1, x_2) = (2, \varepsilon)$. We get

$$\dot{x} = \begin{pmatrix} -2\varepsilon \\ 2 - 3\varepsilon - 1 \end{pmatrix} \to \begin{matrix} 0 \\ 1 \end{matrix}$$

The phase-portrait close to the x_1 - axis ($\varepsilon \to 0$) is shown in Figure 3.7.

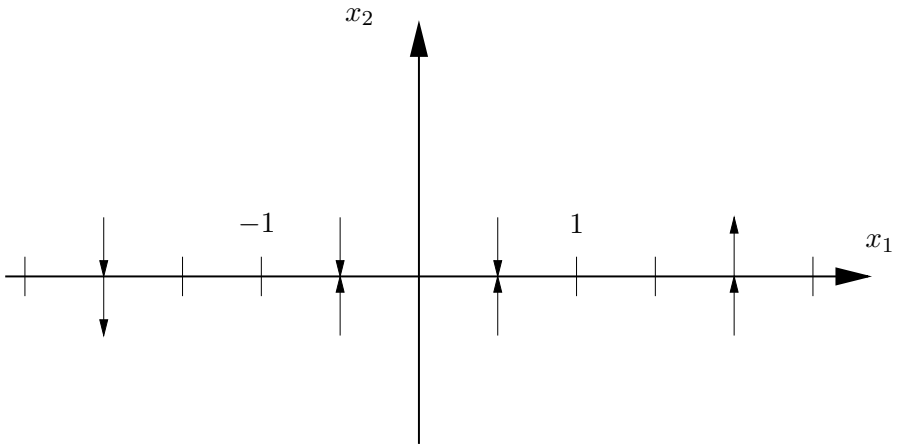


Figure 3.7:

(b) The switching surface is given by $\sigma(x) = x_2$, where $\dot{\sigma}(x) = 0 \Rightarrow 0 = \dot{x}_2 = x_1 - \underbrace{3x_2}_{=0} + u$ which gives $u_{eq} = -x_1$.

□

SOLUTION 9.4

(a) The sliding surface in a sliding mode design should be invariant, *i.e.*, if $x(t_s)$ belongs to the sliding surface $\sigma(x) = 0$, at time t_s, then it belongs to the set $\sigma(x) = 0$ for all future times $t \geq t_s$. Thus, it must hold that

$$\sigma(x) = \dot{\sigma}(x) = 0$$

which yields the dynamics on the sliding surface.

(i) We have

$$\dot{\sigma}(x) = \dot{x}_1 - x_2 = \dot{x}_1 - x_1 = 0$$

The third equality implies that $x_1(t) \to \pm\infty$ on the sliding surface. Thus, forcing this surface to be a sliding mode would give unstable solutions.

(ii) Similarly as above

$$\dot{\sigma}(x) = \dot{x}_1 + 2\dot{x}_2 = \dot{x}_1 + 2x_1 = 0$$

Thus, the equivalent dynamics along this surface satisfy $\dot{x}_1 = -2x_1$ and are hence stable.

(iii) We have

$$\dot{\sigma}(x) = \dot{x}_1 = 0$$

The dynamics on this sliding surface would thus be stable, but not asymptotically stable.

(b) According to the lecture slides, the sliding mode control law is

$$u = -\frac{p^T Ax}{p^T B} - \frac{\mu}{p^T B}\text{sign}(\sigma(x))$$

Where the sliding surface is given by

$$\sigma(x) = p^T x = 0$$

Thus, in this example we have $p^T = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and

$$u = -(x_1 + x_2) - \mu\text{sign}(x_1 + x_2)$$

(c) According to the robustness result of the sliding mode controller presented on the lecture, the above controller will force the system toward the sliding mode if μ is chosen large enough, and if $\text{sign}(p^T\hat{B}) = \text{sign}(p^T B)$, which implies $\text{sign}(\hat{b}) = \text{sign}(b)$. Since the nominal design has $\hat{b} = 1$, we must have

$$b > 0 \tag{3.16}$$

It remains to check that the dynamics of the sliding mode remains stable. (Otherwise, we could have a situation where the controller forces the state onto the sliding mode, but where the sliding mode dynamics are unstable. The state would then tend toward infinity along the sliding mode.) In this case, we can verify that on the sliding mode, we have $\dot{\sigma}(x) = 0$ for all values of the parameter a.

□

SOLUTION 9.5

(a) The pendulum energy is given by

$$E(x) = mgl(1 - \cos(x_1)) + \frac{J_p}{2}x_2^2$$

If the energy of the pendulum hanging downwards is taken to be $E(0) = 0$, the energy for $x_1 = \pi$, $x_2 = 0$ is $E_0 = 2mgl$.

(b) The time derivative of the Lyapunov function candidate reads

$$\begin{aligned}\dot{V}(x) &= 2(E(x) - E_0)\frac{d}{dt}E(x) = \\ &= 2(E(x) - E_0)(-mgl\sin(x_1)\dot{x}_1 + J_p x_2 \dot{x}_2) = \\ &= 2(E(x) - E_0)(-mlx_2\cos(x_1)u)\end{aligned}$$

Applying the suggested control, we obtain

$$\dot{V}(x) = -2kml(E(x) - E_0)^2 x_2 \cos(x_1)\text{sign}(x_2\cos(x_1)) \leq 0$$

with equality attained for $E(x) = E_0$, or $x_2 = 0$ or $x_1 = \pi/2$. The only unwanted invariant manifold is $x_1 = x_2 = 0$.

(c) The phase portrait of the closed loop system is shown in Figure 3.8. We notice how the state is driven to the set $E(x) = E_0$, and that this set contains no stable equilibrium points. Note that the velocity approaches zero as the pendulum approaches the upright position. Since the equilibrium point is unstable, and the control for this state is zero, the pendulum does not remain in the upright position.

Extra. Feel free to design a stabilizing controller for the upright position (using, for example the results from Exercise 12.7). In particular, how should you switch between the two control strategies to make the system stable? (Some Lyapunov theory will help you on this one)

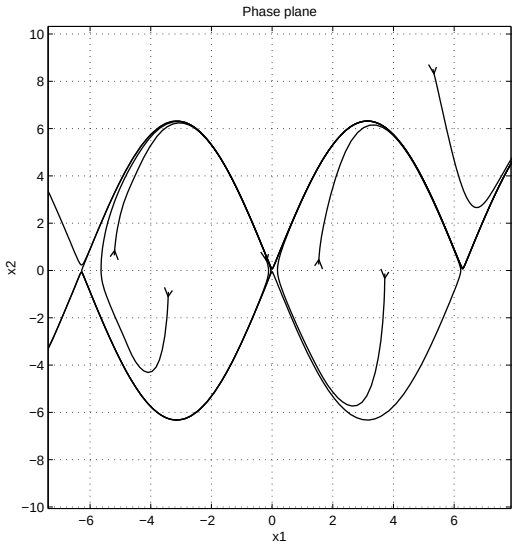


Figure 3.8: Phase plane for pendulum under energy control.

□

SOLUTION 9.6

(a) In steady-state the output reaches $\frac{\pi}{2}$ for $a = 1$ and $\frac{\pi}{6}$ for $a = 2$.

(b) Feedback with $k \gg a$ gives

$$\dot{x} = -a\sin(x) + k(r - x) \approx kr - kx$$

which is a linear system with static gain $= 1$.

(c) The benefit is that we obtain an almost linear system although the control signals may become large and the system will be more sensitive to measurement noise.

□

SOLUTION 9.7

If we use operator notation (such that L, L^{-1} and N) operates on the variable to the right, then we can write P as

$$P = N + L = (NL^{-1} + 1)L$$

the inverse of P can be written as

$$P^{-1} = L^{-1}(NL^{-1} + 1)^{-1}$$

which is the transfer function shown in the block diagram. A few things to be noted

- The inversion only involves the direct inversion of the linear system which is easy to analyze and compute. A causal and stable inverse of L implies that L is minimum phase.
- If P is not partitioned as above, then we can create our own partitioning as

  $$P = n = \underbrace{n - L}_{=N} + L$$

  where we can pick our own choice of L. Of course we pick an easily inverted L such as, e.g., $L = 1$.
- It can be noted that numerical problems might occur, especially in the case of continuous time models where the partitioning is created as in the item above.

□

SOLUTION 9.8

(a) The equilibria in the specified region are $(0, 0)$, $(1, 0)$, and $(3/25, 88/125)$. The linearizations are given by

$$\dot{z} = \begin{pmatrix} 5 & 0 \\ 0 & -3/5 \end{pmatrix} z, \qquad \dot{z} = \begin{pmatrix} -5 & -5/3 \\ 0 & 11/5 \end{pmatrix} z, \qquad \dot{z} = \begin{pmatrix} 21/20 & -3/4 \\ 11/5 & 0 \end{pmatrix} z.$$

(b) There are no equilibria in the specified region.

(c)

$$f(x, u) = \begin{pmatrix} 5x_1(1 - x_1) - \dfrac{20x_1x_2}{2 + 10x_1} \\ \dfrac{16x_1x_2}{2 + 10x_1} - \dfrac{6x_2}{10} - \dfrac{x_2}{2}(1 - u) \end{pmatrix}$$
$$u = -\operatorname{sgn}\sigma(x) = -\operatorname{sgn}(x_2 - 1).$$

(d) The equivalent control $u_{\text{eq}} \in [-1, 1]$ follows from the equation $\dot{\sigma}(x) = 0$, that is,

$$\frac{16x_1}{2 + 10x_1} - \frac{6}{10} - \frac{1}{2}(1 - u_{\text{eq}}) = 0,$$

where we used that $x_2 = 1$ at the sliding mode. Solving for $u_{\text{eq}} = u_{\text{eq}}(x_1)$ and plugging into the system dynamics yields the sliding dynamics

$$\dot{x}_1 = 5x_1(1 - x_1) - \frac{20x_1}{2 + 10x_1}.$$

It follows from the constraint $u_{\text{eq}} \in [-1, 1]$ and Equation (3.9) that the sliding mode takes place at $\{x : x_1 > 3/25, x_2 = 0\}$ (compare the phase plane analysis).

The physical interpretation of the sliding mode is that the predator population x_2 is kept at the level $\alpha = 1$, while only the prey population x_1 may change. If the predator population increases above α, it decreases due to harvest. If the predator population decreases below α, it increases due to the excess of prey.

□

SOLUTION 9.9

The system is on the correct form and we can directly use the "recipe" from the lecture slides. Define

$$f(x) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Choose $\sigma(x) = p^T x$ such that $p^T = \begin{bmatrix} p_1 & p_2 \end{bmatrix}$ are coefficient of a stable polynomial. For example choose $p_1 = p_2 = 1 \Rightarrow \sigma(x) = x_1 + x_2$. The control law is then given by

$$\begin{aligned} u &= -\frac{p^T f(x)}{p^T g(x)} - \frac{\mu}{p^T g(x)} \text{sign}(\sigma(x)) = -\frac{(2x_1 - x_2 + x_1)}{1} - \frac{\mu}{1} \text{sign}(x_1 + x_2) \\ &= [\mu = 1] = -3x_1 + x_2 - \text{sign}(x_1 + x_2) \end{aligned}$$

The equivalent control on the sliding surface is when $\sigma(x) = \dot{\sigma}(x) = 0$.

$$\begin{aligned} \dot{\sigma}(x) = \dot{x}_1 + \dot{x}_2 = 2x_1 - x_2 + u_{eq} + x_1 = 0 \Rightarrow \\ u_{eq} = -3x_1 + x_2 = -4x_1 \end{aligned}$$

Dynamics on the sliding surface are

$$\begin{aligned} \dot{x}_1 &= 2x_1 - x_2 + u_{eq} = -3x_1 \\ \dot{x}_2 &= x_1 = -x_2 \end{aligned}$$

Hence on the sliding surface the solution approaches the origin.

□

SOLUTION 9.10

(a) On the manifold S we have $x_1 = ax_2 = 0 \Leftrightarrow x_1 = -ax_2$. Insert into second state equation: $\dot{x}_2 = \frac{-ax_2}{1+a^2x_2^2}$. Hence if $a < 0$ then unstable, trajectories will go to infinity. If $a = 0$ then every point on the manifold is a equilibrium. If $a > 0$ then stable and trajectories approaches the origin.

(b) Use Lyapunov candidate $V(\sigma) = \frac{1}{2}\sigma^2$ and design u such that $V(\sigma)$ is a Lyapunov function.

$$\begin{aligned} \dot{V}(\sigma) &= \sigma\dot{\sigma} = \sigma(\dot{x}_1 + a\dot{x}_2) \\ &= \sigma(-2x_1 - \frac{x_2}{1 + x_2^1} + u + \frac{ax_1}{1 + x_1^1}) < 0, \quad \forall \sigma \neq 0 \end{aligned}$$

Choose for example

$$\begin{aligned} u = 2x_1 + \frac{x_2}{1 + x_2^1} - \frac{ax_1}{1 + x_1^1} - \text{sign}(\sigma) \Rightarrow \\ \dot{V}(\sigma) = -\sigma \text{sign}(\sigma) < 0, \quad \forall \sigma \neq 0 \end{aligned}$$

Hence the manifold $\sigma = 0$ is globally asymptotically stable.

(c) On the sliding manifold we have

$$\begin{aligned}
&\sigma(x) = \dot{\sigma}(x) = 0 \Rightarrow \\
&\dot{\sigma}(x) = \dot{x}_1 + a\dot{x}_2 = -2x_1 - \frac{x_2}{1+x_2^1} + u_{eq} + \frac{ax_1}{1+x_1^1} = 0 \Rightarrow \\
&u_{eq} = 2x_1 + \frac{x_2}{1+x_2^1} - \frac{ax_1}{1+x_1^1}
\end{aligned}$$

□

3.10 Solutions to Computer Exercises: Sliding Mode, IMC and Gain scheduling

SOLUTION 10.1

(a) Straightforward manipulations give

$$G(s) = \frac{K}{sT+1}e^{-sL} = \frac{1}{sV_m/q(t)+1}e^{-sV_d/q(t)}$$

(b) The step response gives parameters $a = 0.9$, $L = 1$. Using the results from (a) and $a = KL/T$ we obtain

$$\begin{aligned}
a &= V_d/V_m \\
L &= V_d/q(t)
\end{aligned}$$

Since the experiment was performed for $q(t) = 1$, we see that $L = 1/q(t)$. Now, a gain scheduled PI controller can be constructed using Ziegler-Nichols recommendations as

$$\begin{aligned}
K_p &= 0.9/a = 1 \\
T_i &= 3L = 3/q(t)
\end{aligned}$$

(c) One way of implementing the controlled system is shown in Figure 3.9.

□

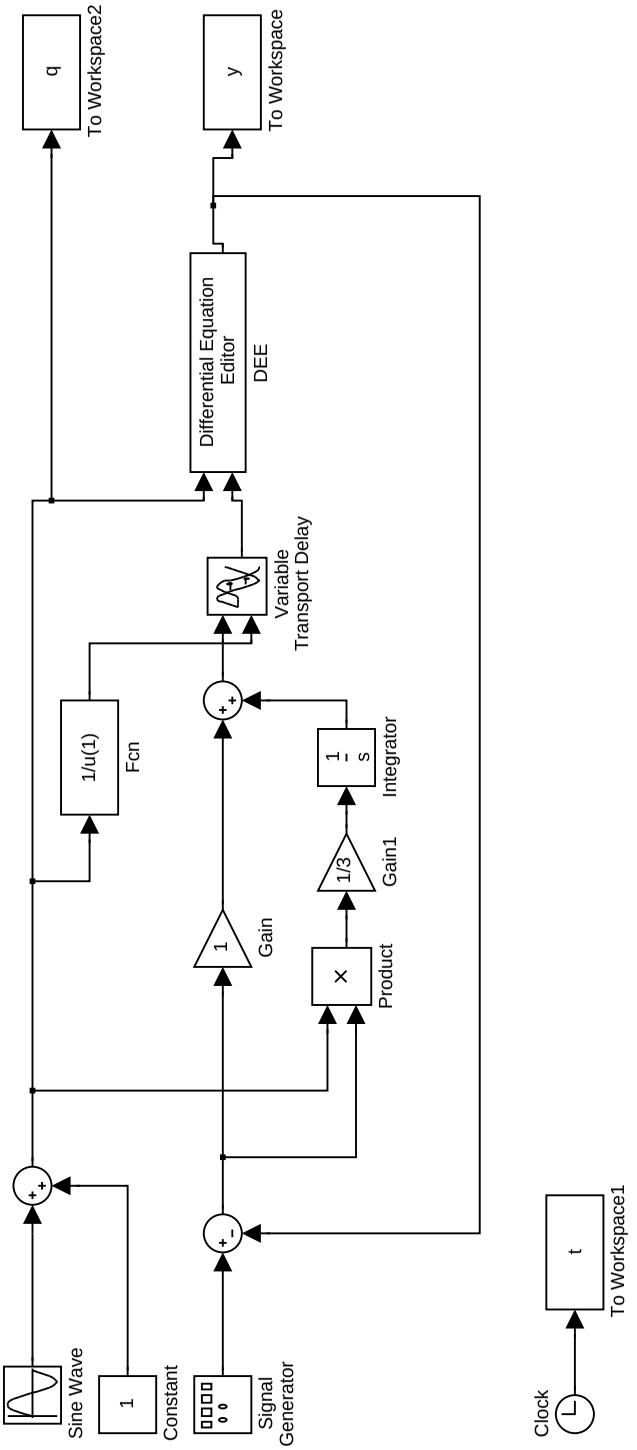


Figure 3.9: Possible implementation of the gain scheduling controller in Exercise 10.1.

SOLUTION 10.2

As you can see in Figure 3.10 the control signal will be hard on the actuators.

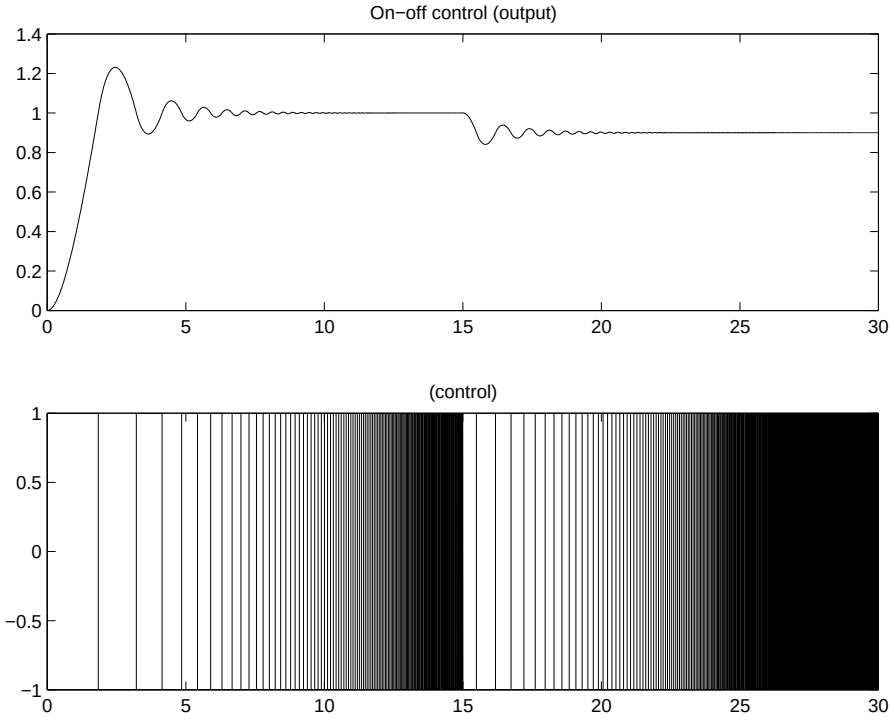


Figure 3.10: Bang-bang control

□

SOLUTION 10.3

The time it will take to reach the sliding surface will depend on μ.

□

SOLUTION 10.4

Hint: Stop the simulation after a while since it seems like Matlab/Simulink has a problem with small values of x_1 and x_2.

□

3.11 Solutions to Gain scheduling and Lyapunov based methods

SOLUTION 11.1

Intentionally left blank.

□

SOLUTION 11.2

(a) With $u = -z$ the $x(t)$ is given by

$$x(t) = \frac{2x_0}{(2 - x_0 z_{20})e^t + x_0 z_{20} e^{-t}}$$

which will have finite escape time if $x_0 z_{20} > 2$.

(b) With $K = (-1 \ -1)$ we get $u = -z_1 - z_2 - x^3$

□

SOLUTION 11.3

For the nominal system we use $\psi(x) = -(x_1^3 + x_1 + 2x_2^3 + x_2)$ and with $\rho(x) = \sqrt{x_1^2(1 + x_1)^2 + x_2^2(1 + x_2)^2}$ and $k = 0.5$ we can use

$$\begin{aligned} u &= \psi(x) - \frac{\rho(x)}{1 - k} \text{ sqn } \left(\frac{dV}{dx} g(x) \right) \\ &= -x_1^3 - x_1 - 2x_2^3 - x_2 - 2\rho(x) \text{ sqn } (x_1 + x_2) \end{aligned}$$

□

SOLUTION 11.4

Left blank

□

SOLUTION 11.5

Left blank

□

SOLUTION 11.6

$$u = (-2x_1 - 1)(x_1^2 - x_1^3 + x_2) - x_1 - (x_2 + x_1^2 + x_1) \tag{3.17}$$

□

SOLUTION 11.7

Left blank

□

SOLUTION 11.8

Left blank

□

SOLUTION 11.9

$$u = -2x_1^2 x_2 - x_1^2 - (x_2 + x_1^2) - x_1$$

□

SOLUTION 11.12

(a) Plugging in the transformation and the control law gives

$$\dot{z} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} z$$

Since both eigenvalues of this matrix are -1, the system is asymptotically stable.

(b) Let

$$k(x) = u(z(x)) = -x_1 - 2x_2 - 2f(x_1) - [x_2 + f(x_1)]f'(x_1).$$

From the linear system in (a), we have the Lyapunov function $V(z) = z^T P z$, where $P = P^T > 0$ is the solution to the Lyapunov equation $PA + A^T P = -Q$. Choosing $Q = 2I$ (the identity matrix) gives

$$P = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

Hence,

$$V(x) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^T P \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + f(x_1) \end{pmatrix}^T \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 + f(x_1) \end{pmatrix},$$

which is easily checked to fulfill the conditions for the Lyapunov theorem for global stability (including $V(x) \to \infty$ as $\|x\| \to \infty$).

□

SOLUTION 11.13

(a) $x = (r, \dot{r}, \theta, \dot{\theta})^T$ gives

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ -g \sin x_3 - \beta x_2 + x_1 x_4^2 \\ x_4 \\ \dfrac{-2x_1 x_2 x_4 - g x_1 \cos x_3 + u}{x_1^2 + 1} \end{pmatrix} = f(x, u)$$

(b) $f(x_0, u_0) = 0$ gives $x_0 = (x_{10}, 0, k\pi, 0)^T$ with x_{10} and u_0 being the solutions to the equation $g x_{10} \cos k\pi = u_0$ and k being an integer.

(c) Take $x_0 = (0, 0, 0, 0)^T$ and $u_0 = 0$. Then,

$$\frac{\partial f}{\partial x}(x_0, u_0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\beta & -g & 0 \\ 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \end{pmatrix}, \qquad \frac{\partial f}{\partial u}(x_0, u_0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(d) Denote the linearization in (c) by

$$\delta\dot{x} = A\delta x + B\delta u.$$

Choosing $L = (\ell_1, \ell_2, \ell_3, \ell_4)$ such that the eigenvalues of $A - BL$ are in the left half-plane gives a control law $u = -Lx$, which is stabilizing also for $\dot{x} = f(x, u)$.

(e) The system $\ddot{r} + \beta\dot{r} = 0$ together with the suggested $V(r, \dot{r})$ gives

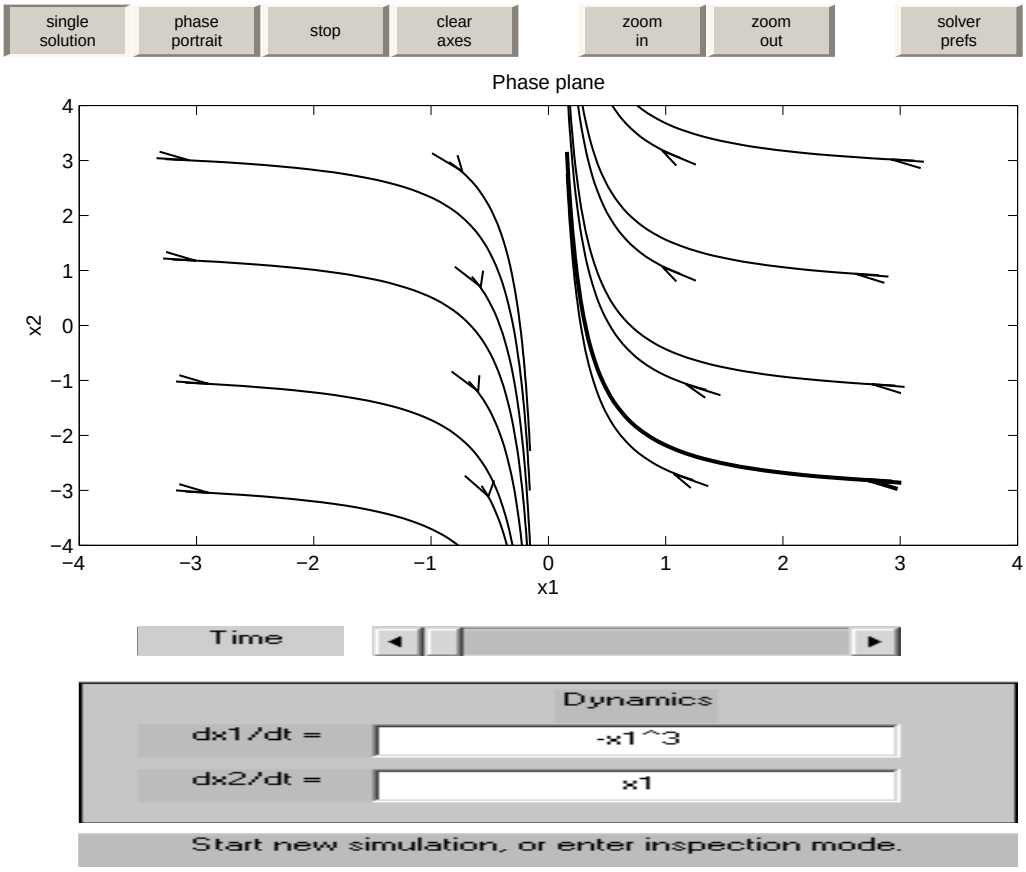
$$\begin{aligned} &V(0,0) = 0 \\ &V(r,\dot{r}) > 0, \qquad (r,\dot{r}) \neq (0,0) \\ &\dot{V}(r,\dot{r}) = -\beta\dot{r}^2 \leq 0 \\ &V(r,\dot{r}) \to \infty, \qquad \|(r,\dot{r})\| \to \infty, \end{aligned}$$

and, hence, global stability. Asymptotic stability does not follow because $\dot{V}(r, \dot{r}) \not< 0$ for all $(r, \dot{r}) \neq (0, 0)$. (What's the physical interpretation?)

□

SOLUTION 11.14

(a) ICTools gives the phase portrait below:



(b)

$$\dot{V} = \frac{dV}{dx} f(x) = \begin{pmatrix} 2x_1 & 2x_2 \end{pmatrix} \begin{pmatrix} -x_1^3 + u \\ x_1 \end{pmatrix} = -2x_1^4 + 2x_1 u + 2x_1 x_2$$

If we choose $u = -x_2$, then $\dot{V} = -2x_1^4$. Hence, x_1 will tend to zero. It follows from the equation $\dot{x}_2 = x_1$ that x_2 will tend to a constant $\bar{x}_2$, say. Suppose $\bar{x}_2 \neq 0$. Then, $\dot{x}_1 = -x_1^3 - x_2$ implies that $\dot{x}_1 \to \bar{x}_2 \neq 0$, which contradicts that $x_1 \to 0$. Hence, $\bar{x}_2 = 0$. Global stability follows from that V is radially unbounded.

(c) Choosing $u = x_1^3 + v$ gives the system

$$\begin{aligned}\dot{x}_1 &= v\\ \dot{x}_2 &= x_1,\end{aligned}$$

which is globally stabilized, for example, by $v = -2x_1 - x_2$. Hence,

$$u = x_1^3 - 2x_1 - x_2$$

is a possible control law.

□

3.12 Solutions to Feedback linearization

SOLUTION 12.1

The control is given by

$$u = \phi(x_1) + \psi(x_2) - a_1 x_1 - a_2 x_2 + r.$$

□

SOLUTION 12.2

The system is defined by

$$(*) \quad \begin{cases} \dot{x}_1 = x_1^2 + x_2 \\ \dot{x}_2 = u \\ y = x_1 \end{cases}$$

Change of variables

$$\begin{cases} z_1 = y \\ z_2 = \dot{y} \end{cases}$$

giving

$$\dot{z}_1 = \dot{y} = z_2$$

$$\begin{aligned}
\dot{z}_2 &= \ddot{y} = \frac{d}{dt}(\dot{x}_1) \\
&= \frac{d}{dt}\left(x_1^2 + x_2\right) \\
&= 2x_1\dot{x}_1 + \dot{x}_2 = [\text{from } (*)] \\
&= 2x_1\left(x_1^2 + x_2\right) + u \\
&= \begin{bmatrix} (*) \Rightarrow x_2 = \dot{x}_1 - x_1^2 \\ x_1 = y = z_1 \\ \dot{x}_1 = \dot{y} = z_2 \end{bmatrix} \\
&= 2z_1\left(z_1^2 + z_2 - z_1^2\right) + u \\
&= 2z_1 z_2 + u = \alpha(z) + \beta(z)u
\end{aligned}$$

An exact linearization is then

$$u = \frac{-\alpha(z) + \bar{u}}{\beta(z)} = -2z_1 z_2 + \bar{u}.$$

□

SOLUTION 12.3

(a) With $y = x_2$.

$$\begin{aligned}
\dot{y} &= \dot{x}_2 = \sqrt{1+x_1} - \sqrt{1+x_2} \\
\ddot{y} &= \ddot{x}_2 = \frac{d}{dt}\left(\sqrt{1+x_1} - \sqrt{1+x_2}\right) = \\
&= \frac{1}{2\sqrt{1+x_1}}\dot{x}_1 - \frac{1}{2\sqrt{1+x_2}}\dot{x}_2 = \cdots = \\
&= \frac{1}{2}\left(\frac{1}{\sqrt{1+x_1}} - \frac{\sqrt{1+x_1}}{\sqrt{1+x_2}}\right) + \frac{u}{2\sqrt{1+x_1}}
\end{aligned}$$

It follows that the relative degree is 2.

(b,c) Select the states $z_1 = y, z_2 = \dot{y} \Rightarrow$

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= /\text{as above}/ = \frac{1}{2}\left(\frac{1}{\sqrt{1+x_1}} - \frac{\sqrt{1+x_1}}{\sqrt{1+x_2}}\right) + \frac{u}{2\sqrt{1+x_1}} = \\
&= \left| \begin{array}{l} z_1 = y = x_2 \\ z_2 = \dot{y} = \dot{x}_2 = \sqrt{1+x_1} - \sqrt{1+x_2} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} x_2 = z_1 \\ \sqrt{1+x_1} = z_2 + \sqrt{1+z_1} \end{array} \right| = \\
&= \frac{1}{2}\left(\frac{1}{z_2+\sqrt{1+z_1}} - \frac{z_2+\sqrt{1+z_1}}{\sqrt{1+z_1}}\right) + \frac{1}{2}\frac{1}{z_2+\sqrt{1+z_1}}u = \\
&= \alpha(z) + \beta(z)u
\end{aligned}$$

Now choose $u = \frac{1}{\beta(z)}\left(\bar{u} - \alpha(z)\right)$ to obtain a feedback law based achieving an exact linearization.

□

SOLUTION 12.4

$$G: \left\{ \begin{array}{l} \dot{x}_1 = -x_1 + 7x_2 \\ \dot{x}_2 = -x_2 + \cos x_1 + \omega \end{array} \right.$$

Desired: u exactly linearizing the system.

Obtained by: $u = r - \cos x_1$

□

SOLUTION 12.5

$u = -y^4 + y^2 + r = -x_1^4 + x_1^2 + r$

□

SOLUTION 12.7

(a) We notice that all state equation but the last one are linear. The last state equation reads

$$\dot{x}_n = f(x) + g(x)u$$

If we assume that $g(x) \neq 0$ for all x, we can apply the control

$$u = h(x, v) = \frac{1}{g(x)}\left(-f(x) + Lx + v\right)$$

renders the last state equation linear

$$\dot{x}_n = Lx + v$$

The response from v to x is linear, and the closed loop dynamics is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots \\ & & & & \\ 0 & 0 & 0 & \ddots & 0 \\ l_1 & l_2 & l_3 & \dots & l_n \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} v$$

(You may recognize this as the controller form from the basic control course). For the control to be well defined, we must require that $g(x) \neq 0$ for all x.

(b) The above procedure suggest the control

$$u = \frac{1}{b\cos(x_1)} \left(-a\sin(x_1) + l_1 x_1 + l_2 x_2 + v\right)$$

which results in the closed loop system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ l_1 & l_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

The system matrix has a double eigenvalue in $s = -1$ if we let

$$l_1 = -1, l_2 = -2$$

The control law is well defined for $x_1 \neq \pi/2$. This corresponds to the pendulum being horizontal. For $x_1 = \pi/2$, u has no influence on the system. Notice how the control "blows up" nearby this singularity. *Extra.* You may want to verify by simulations the behavior of the modified control

$$u = \text{sat}(h(x, v))$$

for different values of the saturation level.

(c) The above procedure suggest the control

$$u = -x^2 - x + v$$

Letting $v = 0$, we apply the control to the perturbed system

$$\dot{x} = (1 + \epsilon)x^2 - x^2 - x = \epsilon x^2 - x$$

and note that for $x > 1/\epsilon$, we have $\dot{x} > 0$, which implies that the trajectories tend to infinity. Thus, global cancellation is non-robust in the sense that it may require a very precise mathematical model.

□

3.13 Solutions to Optimal Control

SOLUTION 13.3

Note that $L = 0$. We have

$$H = \lambda_1 x_3 + \lambda_2 x_4 + \lambda_3 \alpha \cos u + \lambda_4(\alpha \sin u - g)$$

Minimizing H wrt u gives

$$\tan u = \frac{\lambda_4}{\lambda_3}$$

The adjoint equations are

$$\begin{aligned}\dot{\lambda}_1 &= 0\\ \dot{\lambda}_2 &= 0\\ \dot{\lambda}_3 &= -\lambda_1\\ \dot{\lambda}_4 &= -\lambda_2\end{aligned}$$

This gives $\lambda_3(t) = Ct + D$ and $\lambda_4 = At + B$ for some constants A, B, C, D.

How could one determine A, B, C, D? They of course depend on ϕ and ψ. The optimal trajectories can be generated by $\dot{x} = f(x, u^*)$, and one chooses the trajectory that satisfies the boundary conditions on x and λ at $t = t_f$. It is generally a hard problem.

□

SOLUTION 13.6

a) We have $L = 1$, $\phi = 0$, $\psi(t_f) = (x_1(t_f), x_2(t_f))^T = (0, 0)^T$ and t_f free. We get

$$H = n_0 + \lambda_1 x_2 + \lambda_2 u$$

Hence

$$u(t) = \begin{cases} 1 & \lambda_2(t) < 0\\ ? & \lambda_2(t) = 0\\ -1 & \lambda_2(t) > 0\end{cases}$$

The adjoint equations are

$$\begin{aligned}\dot{\lambda}_1 &= 0\\ \dot{\lambda}_2 &= \lambda_1\end{aligned}$$

This gives $\lambda_1 = \mu_1$, $\lambda_2(t) = \mu_1 t + B$. Now $\mu_1 = B = 0$ is impossible since this leads to $n_0 = 0$ (since we should have $H \equiv 0$), and we can not have $[n_0 \ \ \mu_1 \ \ \mu_2] = 0$. Therefore $\lambda_2(t)$ is linear and nonzero. This shows a).

b) We know that $u = \pm 1$ with at most one switch. The simplest way to get $\sigma(x)$ (note that σ should be a function of x not of t) is to solve the equations for such input signals. For $u(t) = 1$ we get

$$x_1 + C_1 = x_2^2/2$$

This gives the phase plane in the figure For $u = -1$ we get

$$x_1 + C_2 = -x_2^2/2$$

This gives the phase plane in the figure Now we know that we get to the origin with $u = \pm 1$ with at

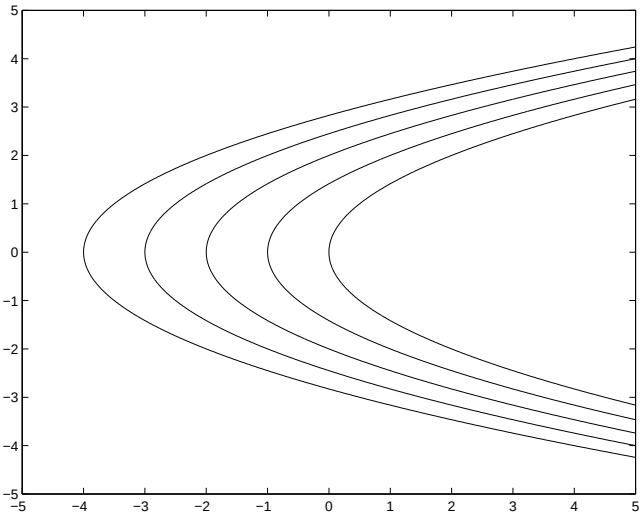


Figure 3.11: Phase plane for $u = 1$

most one switch. Consider especially the two curves for $u = \pm 1$ that pass through the origin. From the plots we see that the control law is given by

$$u(t) = -\text{sign}\left\{x_1(t) + \frac{1}{2}\text{sign}\{x_2(t)\}x_2^2(t)\right\}$$

since then $u(t) = -1$ above the switch curve and $u(t) = 1$ below it.

□

SOLUTION 13.11

$$H = n_0 P + \lambda P - \lambda T$$

Adjoint equation $\dot{\lambda}^T = -H_x$

$$\dot{\lambda}_1 = \lambda_1, \quad \lambda_1(1) = \mu_1$$

Solution

$$\lambda(t) = \mu_1 e^{t-1}$$

$$H = \underbrace{(n_0 + \mu_1 e^{t-1})}_{\sigma(t)} P - \lambda^T$$

$$P^*(t) = \begin{cases} 0, & \sigma(t) > 0 \\ P_{max}, & \sigma(t) < 0 \end{cases}$$

$$\mu_1 < 0 \quad \Rightarrow \quad \sigma(t) \text{ decreasing}$$

Hence

$$P^*(t) = \begin{cases} 0, & 0 \le t \le t_1 \\ P_{max}, & t_1 < t \le 1 \end{cases}$$

Time t_1 is given by

$$1 = T(1) = \int_{t_1}^{T} e^{-(t-\tau)} P_{max}\, d\tau$$

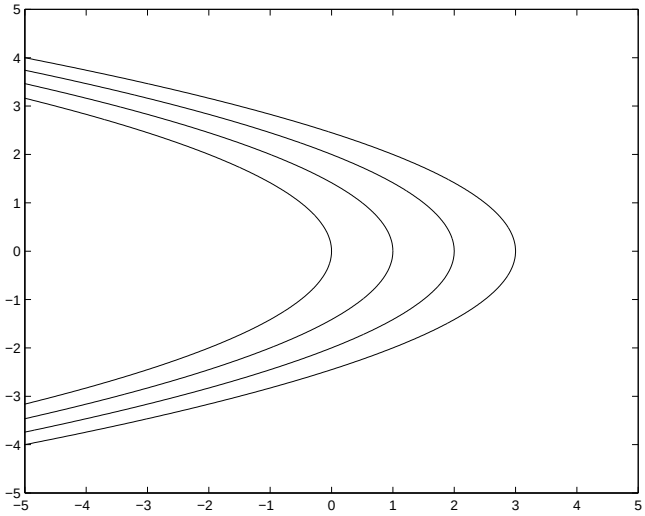


Figure 3.12: Phase plane for $u = -1$

Has solution $t_1 \leq 1$ if

$$P_{max} \geq \frac{1}{1 - e^{-1}}$$

□

SOLUTION 13.12

(a)

$$H = L + \lambda^T f = \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u + \lambda^T (Ax + Bu)$$

(b)

$$\dot{\lambda}(t) = -\frac{\partial H^T}{\partial x}(x^*(t), u^*(t), \lambda(t)) = -Qx^*(t) - A^T \lambda(t)$$

with

$$\lambda(t_f) = \frac{\partial \phi^T}{\partial x}(x^*(t_f)) = 0$$

(c) Since H is a quadratic form in u and there is no constraints on u, the minimum is given by the solution to

$$0 = \frac{\partial H}{\partial u} = Ru + B^T \lambda,$$

which hence gives $u = R^{-1} B^T \lambda$.

(d) Combining

$$\dot{\lambda}(t) = \dot{S}(t) x(t) + S(t)[Ax^*(t) - BR^{-1} B^T S(t) x^*(t)]$$

and

$$\dot{\lambda}(t) = -Qx^*(t) - A^T \lambda(t) = -Qx^*(t) - A^T S(t) x^*(t)$$

gives the result.

(e) For $t_f = \infty$ we may set $\dot{S} = 0$. Then, the differential equation in $S(t)$ becomes an algebraic equation in the (constant) matrix S:

$$A^T S + SA - SBR^{-1} B^T S + Q = 0$$

See Lecture 13.

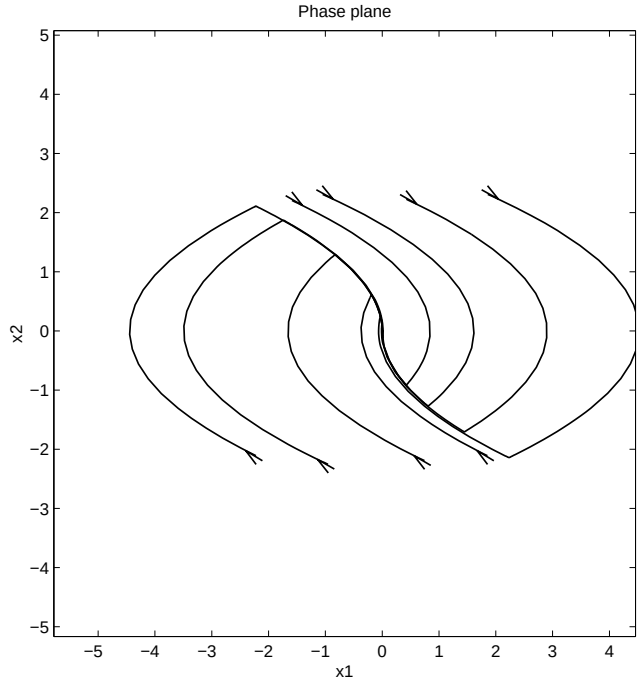


Figure 3.13: Phase plane with switching curve

(f)

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial u}\dot{u} + \frac{\partial H}{\partial \lambda}\dot{\lambda} = \frac{\partial H}{\partial x}f - f^T\frac{\partial H}{\partial x} = 0$$

□

SOLUTION 13.13

See the minimum-time control example in Lecture 13. From that it follows that the optimal control is given by $u^*(t) = -C\,\mathrm{sgn}\,\lambda_2(t) = C\,\mathrm{sgn}(c_1 t - c_2)$ Hence, $p(t) = c_1 t - c_2$ is a first-order polynomial.

□

SOLUTION 13.14

The system is minimal, so a unique positive definite solution of the Algebraic Riccatti Equation (ARE) will exist. With

$$Q = \begin{bmatrix} \frac{5}{2} & 0 \\ 0 & 0 \end{bmatrix}, \quad R = 2$$

the ARE

$$-A^T P - PA + PBR^{-1}B^T P - Q = 0$$

gives the system

$$\begin{cases} 0 &= -\frac{5}{2} - 2P_{12} + \frac{1}{2}P_{12}^2 \\ 0 &= -P_{22} - P_{11} + P_{12} + \frac{1}{2}P_{12}P_{22} \\ 0 &= -2P_{12} + 2P_{22} + \frac{1}{2}P_{22}^2 \end{cases}$$

with the positive definite solution

$$P = \begin{bmatrix} 2+3\sqrt{6} & 5 \\ 5 & -2+2\sqrt{6} \end{bmatrix}$$

and the optimal control $u = -\frac{5}{2}x_1 + (1 - \sqrt{6})x_2$

□

3.14 Solutions to Fuzzy control

SOLUTION 14.1

	NL	NS	ZE	PS	PL
NL	PL	PL	PM	PS	ZE
NS	PL	PM	PS	ZE	NS
ZE	PM	PS	ZE	NS	NM
PS	PS	ZE	NS	NM	NL
PL	ZE	NS	NM	NL	NL

□