Variance Analysis in SISO Linear Systems Identification

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Abstract

Causal single input single output linear time invariant systems are considered. Expressions for the asymptotic (co)variance of system properties estimated using the prediction error method are derived. These expressions delineate the impacts of model structure, model order, true system dynamics, and experimental conditions. A connection to recent results on frequency function estimation is established. Also, simple model structure independent upper bounds are established. These bounds are shown to be significantly more accurate than what is obtained using the now classic asymptotic (in model order) variance formula $m\,\Phi_v(\omega)/\Phi_u(\omega)$ (with m being model order, Φ_u input spectrum and Φ_v noise spectrum) for frequency function estimates. Explicit variance expressions and bounds are provided for common system properties such as impulse response coefficients and non-minimum phase zeros. As an illustration of the insights the expressions provide, they are used to derive conditions on the input spectrum which makes the asymptotic variance of non-minimum zero estimates independent of the model order and model structure.

I. Introduction

In system identification, as in all types of modeling, it is important to be able to assess the model error. Different assumptions on the system and the noise lead to different ways to quantify the model error, see [24]. Assuming the noise to be stochastic and that the system can be described by a model within the used model set leads to error quantification using confidence ellipsoids based on the asymptotic covariance matrix of the parameter estimates [19]. Recently

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also techniques for non-asymptotic confidence regions have been developed [3, 4, 2]. In this contribution, though, we will focus on the traditional asymptotic covariance matrix which in many cases give reliable information of the model error [6]. We will consider prediction error identification of causal single-input single-output (SISO) finite dimensional linear time invariant (LTI) systems. The unknown system parameters will be denoted by $\theta = [\theta_1 \ \cdots \ \theta_n] \in \mathbb{R}^{1\times n}$ (vectors will be taken as row vectors), with θ^o denoting the true value (we will assume that the true system is in the model class). We will assume that (see [19] for exact conditions) the parameter estimate $\hat{\theta}_N \in \mathbb{R}^{1\times n}$ has the property that the (normalized) model error $\sqrt{N}(\hat{\theta}_N - \theta^o)$ becomes normally distributed as the sample size N of the data set grows to infinity

$$\sqrt{N}\left(\hat{\theta}_N - \theta^o\right) \in \mathrm{As}\mathcal{N}\left(0, \mathrm{AsCov}\,\hat{\theta}_N\right) \tag{1}$$

The asymptotic covariance matrix $\mathrm{AsCov}\,\hat{\theta}_N$ of the limit distribution is a measure of the model accuracy. This is reinforced by that, under mild additional conditions [19],

$$\lim_{N\to\infty} N \cdot \mathbf{E}\left[(\hat{\theta}_N - \mathbf{E}\hat{\theta}_N)^{\mathrm{T}}(\hat{\theta}_N - \mathbf{E}\hat{\theta}_N)\right] = \mathrm{AsCov}\,\hat{\theta}_N$$

Under the assumptions above

$$\mathrm{AsCov}\,\hat{\theta}_N = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi(\mathrm{e}^{\mathrm{j}\omega})\Psi^*(\mathrm{e}^{\mathrm{j}\omega})\mathrm{d}\omega\right]^{-1} \tag{2}$$

where $\Psi : \mathbb{C} \to \mathbb{C}^{n\times 2}$ is the gradient of the one-step ahead output predictor and where superscript * denotes complex conjugate transpose. We will use $\langle\Psi,\Psi\rangle$ to denote the integral on the right-hand side of (2) in the following. However, our interest will not be the model parameters θ themselves but some "system theoretic" quantity. We will let such a quantity be represented by a differentiable function $J : \mathbb{R}^{1\times n} \to \mathbb{C}^{1\times p}$ for some integer $p \geq 1$. Given an estimate $\hat{\theta}_N$ of θ^o, a natural estimate of $J(\theta^o)$ is $J(\hat{\theta}_N)$. Using Gauss' approximation formula and (2), it can be shown [19], that the asymptotic covariance of $J(\hat{\theta}_N)$ is given by

$$\mathrm{AsCov}\ J(\hat{\theta}_N) = \Lambda^{\mathrm{T}}\left[\langle\Psi,\Psi\rangle\right]^{-1}\overline{\Lambda}$$

where Λ is the derivative $\Lambda := J'(\theta^o) \in \mathbb{C}^{n\times p}$. We shall be slightly more general and allow cases where $\langle\Psi,\Psi\rangle$ is singular and *define*

$$\mathrm{AsCov}\ J(\hat{\theta}_N) = \Lambda^{\mathrm{T}}\left[\langle\Psi,\Psi\rangle\right]^{\dagger}\overline{\Lambda} \tag{3}$$

The motivation for using the Moore-Penrose pseudo-inverse $[\langle \Psi, \Psi \rangle]^{\dagger}$ stems from that this gives the correct variance for properties that are uniquely defined by the data even if the parameter estimate is non-unique. We refer to [11, 32] for details.

When the model structure, the true system and the experimental conditions (such as whether the system is operating in open or closed loop) are known, it is straightforward to compute (3) numerically. However, such a procedure typically reveals little in terms of how system properties and design variables (model order, model structure, experimental conditions etc.), influence the asymptotic covariance. In [12] (see also [13]), a geometric approach is used to re-express (3) in a form more tangible for interpretation. The use of the technique is illustrated by analyzing the impact system complexity, additional inputs and additional sensors have on the asymptotic covariance. Our work is based on this idea and we will derive expressions for (3) for a class of system properties including frequency responses, impulse response coefficients, poles and zeros, and system norms.

A case that has attracted significant interest in the past is the variance of frequency function estimates $G(\mathrm{e}^{\mathrm{j}\omega}, \hat{\theta}_N)$. For the prediction error method it was shown in [18] that[1]

$$\lim_{m \to \infty} \frac{1}{m} \mathrm{AsVar}\, G(\mathrm{e}^{\mathrm{j}\omega}, \hat{\theta}_N) = \frac{\Phi_v(\mathrm{e}^{\mathrm{j}\omega})}{\Phi_u(\mathrm{e}^{\mathrm{j}\omega})} \tag{4}$$

where m is the model order and Φ_u and Φ_v are the spectral densities of the input signal and noise, respectively. This simple and elegant expression, which is valid for open loop identification, revealed that for large model orders, the accuracy of the frequency function estimate does not depend on the model structure or the number of the estimated parameters, but only the model order m (which may be different from the number of estimated parameters). Furthermore, it shows that the accuracy of the frequency function estimate at a particular frequency only depends on the input and noise spectrum at that particular frequency. Various refinements can be found in [35, 36, 26, 15, 9].

The frequency function result in [18] also covered closed loop identification using input and output measurements as data, and was extended to some alternative closed loop identification methods in [8]. Quantifications exact for finite model orders were presented in [27]. The asymptotic covariance of the parameter estimates for Box-Jenkins models were studied in [5] for a range

[1]When J is a scalar we use asymptotic variance, AsVar, as terminology for (3).

of different closed loop identification methods and from this study it is known that knowledge of the noise model maximizes the accuracy of the system parameters. Quantification of frequency response errors still receives significant interest and some additional contributions are [31, 1, 3].

Parallel to the interest in the accuracy of frequency response estimates, there has been a series of results regarding the accuracy of estimated *non-minimum phase* (NMP) zeros and unstable poles, the interest arising due to the importance of such zeros and poles in control. For poles and zeros of magnitude larger than one, the main conclusion is that the asymptotic variance approaches a *finite* limit as the model order tends to infinity [17, 23].

A related and very interesting contribution is the recent paper [7] where conditions are established for the minimum degree of richness of the excitation required for the information matrix $\langle \Psi, \Psi \rangle$ to be non-singular.

Contributions and outline

As pointed out in [12], the geometric approach has its origin in [26], where exact expressions for the asymptotic variance of frequency function estimates for LTI models were derived using the theory of reproducing kernels; a theory which is based on orthogonal projections. Our contribution to the characterization of the variance error for estimates of LTI systems can be seen as an extension of the work in [26] to general system properties J, using new techniques which deepens the geometrical interpretation of (3) initiated in [26]. As a result, our contributions provides an alternative system theoretic interpretation of the results in [26], see Section IV-A, thus furthering the understanding of frequency function estimation.

More precisely, the contributions of this paper are:

i) Section II: *Re-parametrization formulae.* We provide formulae for re-expressing (3) when the quantity of interest is parametrized in other parameters than those used in the system identification. These expressions are useful when comparing different model parametrizations in terms of the asymptotic variance they yield for the estimate of a specific system property.

ii) Section III: *A general characterization of (3) for Linear Time Invariant systems.* Here we provide general formulas, and bounds, for (3), valid for different experimental conditions and model parametrizations.

iii) Section IV: *Expressions for the asymptotic variance for some properties of LTI systems.* We provide expressions and upper bounds for the asymptotic variance of estimated frequency functions, NMP-zeros, $\mathcal{L}_2$-gains and impulse response coefficients.

iv) *Model structure independent upper bounds for (3).* At present there is surprisingly little in terms of rules of thumbs available regarding model quality in system identification; the expression (4) for the variance of the frequency function estimate and some similar variance expressions for pole/zero estimates, being exceptions. Thus determining suitable experiment length and excitation in order to achieve a certain accuracy of, for example, an impulse response coefficient or an estimate of the $\mathcal{L}_2$ gain of the system, requires extensive calculations based on (3). A spin-off of our new expression for (3) is that it is easy to provide simple model structure independent upper bounds for (3). We hope this to be of value to practitioners.

A preliminary version of this paper has appeared as [21].

Notation

For functions $f : \mathbb{C} \to \mathbb{C}^{n\times m}$, $f^*(z) = (f(z))^*$, the complex conjugate transpose of $f(z)$, $f^*(z^{-*})$ denotes $f^*(z)|_{z^*=z^{-1}}$ and $f^{-*}(z)$ and $f^{-*}(z^{-*})$ denote $(f^{-1}(z))^*$ and $(f^{-1}(z))^*|_{z^*=z^{-1}}$, respectively. On the unit circle $f^*(z^{-*}) = f^*(z)$ and when the elements of f are real rational, it holds that $f^*(z^{-*}) = f^T(z^{-1})$. We will consider vector valued complex functions as row vectors and the inner product of two such functions $f, g : \mathbb{C} \to \mathbb{C}^{1\times m}$ is defined as $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{j}\omega})g^*(\mathrm{e}^{\mathrm{j}\omega})\mathrm{d}\omega$. When f and g are matrix-valued functions, we will still use the notation $\langle f, g\rangle$ to denote $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{j}\omega})g^*(\mathrm{e}^{\mathrm{j}\omega})\mathrm{d}\omega$ whenever the dimensions of f and g are compatible. When $W(\omega)$ $m \times m$ is a positive definite hermitian matrix, the $\mathcal{L}_2^W$-norm of $f : \mathbb{C} \to \mathbb{C}^{n\times m}$ is given by $\|f\|_W = \sqrt{\mathrm{Tr}\,\langle fW, f\rangle}$ where Tr denotes the Trace operator. When $W = I$ (the identity), we write $\|f\|$ and denote this the $\mathcal{L}_2$-norm of f. The space $\mathcal{L}_2^{n\times m}$ consists of all functions $f : \mathbb{C} \to \mathbb{C}^{n\times m}$ such that $\|f\| < \infty$ and when $n = 1$, the notation is simplified to $\mathcal{L}_2^m$. For $f : \mathbb{C} \to \mathbb{C}^{n\times m}$, $f_i : \mathbb{C} \to \mathbb{C}^{1\times m}$ denotes the ith row of f. If $\Psi \in \mathcal{L}_2^{n\times m}$ for some positive integers n and m, then $\mathcal{S}_\Psi$ denotes the subspace to $\mathcal{L}_2^m$ generated by the span of the rows of Ψ. $\mathcal{H}_2^m$ is defined as the subspace of $\mathcal{L}_2^m$ that consists of all $\mathcal{L}_2^m$-functions that are analytic outside the unit circle. Suppose that $f \in \mathcal{L}_2^{p\times m}$ and that $\mathcal{S} \subseteq \mathcal{L}_2^m$, then the rows of $\tilde{f} := \mathbf{P}_{\mathcal{S}}\{f\}$ consists of the orthogonal projection on $\mathcal{S}$ of the corresponding rows of f.

For a differentiable function $f:\ \mathbb{R}^{1\times n} \to \mathbb{C}^{1\times p}$, $f'(x_o)$ is a $n\times p$ matrix with $\frac{\partial f_j(x)}{\partial x_i}\big|_{x=x_o}$ as ijth entry, the partial derivative $\frac{\partial f(\bar{x})}{\partial x_i}$ is defined analogously.

The Moore-Penrose pseudo-inverse of a matrix A is denoted $A^{\dagger}$.

II. Technical preliminaries

Often the quantity J of interest can be expressed in terms of some generic system parameters, such as the impulse response coefficients in case of LTI systems. In this section we will provide a lemma which facilitates the comparison of the asymptotic variance (3) for such quantities for different model structures.

A. Re-parametrization

Our results are based on the following theorem.

Theorem II.1 (Theorem II.5 in [12])**.** *Suppose that* $J:\mathbb{R}^{1\times n}\to\mathbb{C}^{1\times p}$ *is differentiable and let the asymptotic covariance matrix* $\mathrm{AsCov}\ J(\hat{\theta}_N)$ *be defined by (3) where* $\Psi\in\mathcal{L}_2^{n\times m}$*. Suppose that* $\gamma\in\mathcal{L}_2^{p\times m}$ *is such that*

$$\Lambda = \langle \Psi, \gamma\rangle \tag{5}$$

then

$$\mathrm{AsCov}\ J(\hat{\theta}_N) = \langle \mathbf{P}_{\mathcal{S}_\Psi}\{\gamma\}, \mathbf{P}_{\mathcal{S}_\Psi}\{\gamma\}\rangle^{\mathrm{T}} \tag{6}$$

In particular, when J *is scalar,*

$$\mathrm{AsVar}\ J(\hat{\theta}_N) = \|\mathbf{P}_{\mathcal{S}_\Psi}\{\gamma\}\|^2 \tag{7}$$

There is a large freedom in the choice of γ. In Lemma II.8 in [12] it is shown that all solutions $\gamma\in\mathcal{L}_2^{p\times m}$ to the equation $\Lambda=\langle\Psi,\gamma\rangle$ are given by

$$\gamma = \Lambda^{*}\langle\Psi,\Psi\rangle^{\dagger}\Psi + s^{\perp}, \tag{8}$$

where $s^{\perp}$ is any $\mathcal{L}_2^{p\times m}$-function orthogonal to $\mathcal{S}_\Psi$. We will explore this degree of freedom in the next lemma, where a reparametrization of $J(\theta)$ is used to find an expression for a γ that fulfills the condition (5).

Lemma II.2. *Let* $\Psi \in \mathcal{L}_2^{n\times m}$ *and let* $\tau : \mathbb{R}^{1\times n} \to \mathbb{C}^{1\times n_\tau}$ *for some* $n_\tau \in \{1, 2, \ldots, \infty\}$*, where the elements* $\tau_k(\theta)$*,* $k = 1, \ldots, n_\tau$ *are differentiable. Suppose that there exists some function* $W : \mathbb{C} \to \mathbb{C}^{m\times m}$ *such that*

$$\Psi(z)W(z) = \sum_{k=1}^{n_\tau} \tau_k'(\theta^o)\ \mathcal{T}_k(z) \in \mathcal{L}_2^{n\times m} \tag{9}$$

for some orthonormal $\mathcal{L}_2^m$*-functions* $\{\mathcal{T}_k\}_{k=1}^{n_\tau}$.

Let $J : \mathbb{R}^{1\times n} \to \mathbb{C}^{1\times p}$ *be differentiable, with* $J(\theta)$ *defined by*

$$J(\theta) = J_\tau(\tau(\theta)) \tag{10}$$

for some function J_τ *for which the partial derivatives with respect to* τ_k*,* $k = 1, \ldots, n_\tau$ *exists at* θ^o *and satisfy*

$$\nabla J_\tau(z) := \sum_{k=1}^{n_\tau} \left(\frac{\partial J_\tau(\tau(\theta^o))}{\partial \tau_k}\right)^* \mathcal{T}_k(z) \in \mathcal{L}_2^{p\times m} \tag{11}$$

Suppose that J_τ *and* τ *are such that the chain rule applies:*

$$J'(\theta^o) = \sum_{k=1}^{n_\tau} \tau_k'(\theta^o)\ \frac{\partial J_\tau(\tau(\theta^o))}{\partial \tau_k} \tag{12}$$

and assume that W *and* ∇J_τ *are such that*

$$\gamma(z) := \nabla J_\tau(z) W^*(z^{-*}) \in \mathcal{L}_2^{p\times m} \tag{13}$$

Then (5)–(6) *hold with this* γ.

Proof: All that has to be proven is that (5) holds with γ as in (13). First notice that

$$\langle \Psi(z), \gamma(z)\rangle = \langle \Psi(z), \nabla J_\tau(z) W^*(z^{-*})\rangle = \langle \Psi W, \nabla J_\tau\rangle$$

and from (9)–(11) and the orthonormality of $\{\mathcal{T}_k\}$ it follows that

$$\langle \Psi W, \nabla J_\tau\rangle = \sum_{k=1}^{n_\tau} \tau_k'(\theta^o)\ \frac{\partial J_\tau(\tau(\theta^o))}{\partial \tau_k}$$

which according to the assumption (12), equals $J'(\theta^o) =: \Lambda$. ∎

We remark that (12) is satisfied if J_τ and τ both are Fréchet differentiable, see, e.g., [20]. However, it is often straightforward to verify (12) directly. We will defer further discussion of this result to Section III-B where Lemma II.2 is used to derive an asymptotic variance expression applicable when the underlying system is single-input single-output linear time invariant.

The next result is Lemma II.9 in [12] adapted to the re-parametrization in Lemma II.2.

Lemma II.3. *Consider the assumptions in Lemma II.2 and let* $\{\mathcal{B}_k\}_{k=1}^r$, $r \leq n$, *be an orthonormal basis for* $\mathcal{S}_\Psi$. *Assume also that*

$$\frac{\partial J_\tau(\tau(\theta^o))}{\partial \tau_k} = \mathcal{T}_k(z_o)\alpha \tag{14}$$

for some $\alpha \in \mathbb{C}^{m\times p}$ *and* $z_o \in \mathbb{C}$, *and let* γ *be defined by* (13).

Then

$$\mathbf{P}_{\mathcal{S}_\Psi}\{\gamma\} = \alpha^* W^*(z_o) \sum_{k=1}^{r} \mathcal{B}_k^*(z_o)\ \mathcal{B}_k \tag{15}$$

and

$$\begin{aligned}&\langle \mathbf{P}_{\mathcal{S}_\Psi}\{\gamma\}, \mathbf{P}_{\mathcal{S}_\Psi}\{\gamma\}\rangle \\ &\qquad = \alpha^* W^*(z_o) \sum_{k=1}^{r} \mathcal{B}_k^*(z_o)\mathcal{B}_k(z_o)W(z_o)\alpha\end{aligned} \tag{16}$$

Furthermore

$$\langle \Psi, \gamma\rangle = \Psi(z_o)W(z_o)\alpha \tag{17}$$

Proof: We start with proving (17). Since $\Psi = \Omega\Gamma$ with $\Gamma = [\mathcal{B}_1^{\mathrm{T}}, \ldots, \mathcal{B}_r^{\mathrm{T}}]^{\mathrm{T}}$ for some matrix Ω, it is sufficient to prove that

$$\langle \mathcal{B}_k, \gamma\rangle = \mathcal{B}_k(z_o)W(z_o)\alpha, \quad k = 1, \ldots, r$$

With γ as in (13) with (14) it holds

$$\begin{aligned}\langle \mathcal{B}_k, \gamma\rangle &= \sum_{l=1}^{n_\tau} \langle \mathcal{B}_k, \mathcal{T}_l W^*\rangle\ \mathcal{T}_l(z_o)\ \alpha \\ &= \sum_{l=1}^{n_\tau} \langle \mathcal{B}_k W, \mathcal{T}_l\rangle\ \mathcal{T}_l(z_o)\ \alpha = \mathbf{P}_{\mathcal{Y}}\{\mathcal{B}_k W\}(z_o)\ \alpha\end{aligned}$$

where $\mathcal{Y}$ is the space spanned by $\{\mathcal{T}_l\}_{l=1}^{n_\tau}$. However, due to (9), $\mathcal{B}_k W \in \mathcal{Y}$ for $k = 1, \ldots, n$ so the projection can be removed giving (17).

Inserting (17) (which we just proved) in

$$\mathbf{P}_{\mathcal{S}_\Psi}\{\gamma\} = \sum_{k=1}^{r} \langle \gamma, \mathcal{B}_k\rangle \mathcal{B}_k$$

proves (15).

Finally, the result (16) is immediate from (15) using the orthonormality of the $\mathcal{B}_k$'s. ■

Remark: Notice that under the conditions in Lemma II.3, the condition that $\nabla J_\tau W^*(z^{-*}) \in \mathcal{L}_2^{p\times m}$ in (13) can be written as

$$\alpha^* \sum_{k=1}^{n_\tau} \mathcal{T}_k^*(z_o)\mathcal{T}_k(z)\; W^*(z^{-*}) \in \mathcal{L}_2^{p\times m}$$

This may restrict the set of points $z_o \in \mathbb{C}$ for which Lemma II.3 is applicable when $n_\tau = +\infty$.

In Sections IV-A and IV-B we will provide examples when (14) holds.

B. Orthonormal basis functions

Lemma II.3 shows that when an orthonormal basis for $\mathcal{S}_\Psi$ is available, it is sometimes possible to express the asymptotic variance explicitly (without orthonormalization using, e.g., Gram-Schmidt). A well known case [25] is when

$$\mathcal{S}_\Psi = \mathrm{Span}\left\{\frac{z^{-1}}{L_n(z)}, \frac{z^{-2}}{L_n(z)}, \ldots, \frac{z^{-m}}{L_n(z)}\right\} \tag{18}$$

where $L_n(z) = \prod_{k=1}^{n}(1-\xi_k z^{-1})$, $|\xi_k| < 1$ for some set of specified poles $\{\xi_1, \ldots, \xi_n\}$ and where $m \geq n$. Then, it holds that

$$\mathcal{S}_\Psi = \mathrm{Span}\{\mathcal{B}_1, \ldots, \mathcal{B}_m\}$$

where $\{\mathcal{B}_k\}$ are the Takenaka-Malmquist functions given by

$$\mathcal{B}_k(z) := \frac{\sqrt{1-|\xi_k|^2}}{z-\xi_k}\;\cdot\;\Phi_{k-1}(z),\;\; k = 1, \ldots, m \tag{19}$$

$$\Phi_k(z) := \prod_{l=1}^{k} \frac{1-\overline{\xi_l} z}{z-\xi_l},\;\;\Phi_0(z) := 1 \tag{20}$$

and with $\xi_k = 0$ for $k = n+1, \ldots, m$. In [26] it is shown that the structure (18) holds for common model structures such as Output-Error and Box-Jenkins provided the input spectrum has no zeros and sufficiently many numerator coefficients are estimated. It is worth noticing that the system zeros do not affect the basis functions above.

III. SISO LTI SYSTEMS

In this section we will apply the results in Section II to the modeling of causal finite dimensional SISO linear time invariant (LTI) systems.

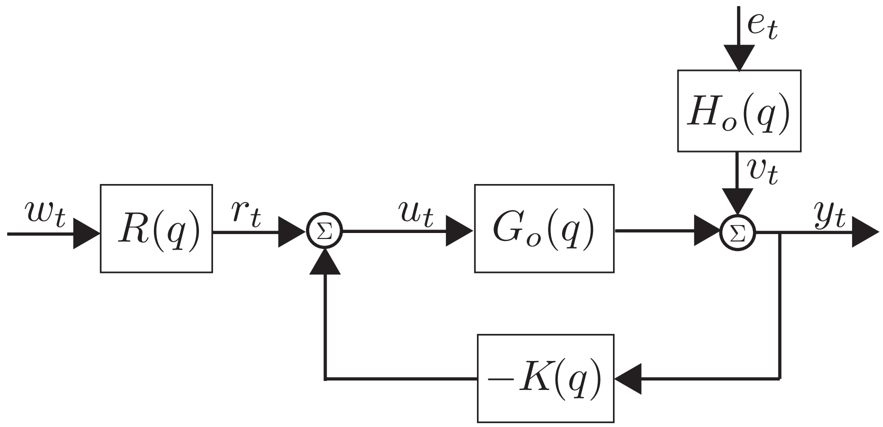

Figure 1. Block diagram of SISO LTI system with output feedback

A. System and model assumptions

Throughout the paper we will assume that the true system is given by a causal finite dimensional SISO LTI system $G_o(\mathrm{q})$ (q is the forward shift operator) depicted in Figure 1 where u_t and y_t represent the measured input and output, respectively, where e_t and w_t are zero mean white noise sequences with variance λ_o and 1, respectively. The causal finite dimensional LTI filter R represents a stable minimum phase spectral factor of the reference signal r_t, and H_o is an inversely stable finite dimensional LTI filter that is normalized to be monic, i.e., $\lim_{z\to\infty} H_o(z) = 1$. The system G_o includes at least one unit time delay, so that the feedback loop is well defined, and we also assume the entire system to be internally stabilized by the causal finite dimensional LTI controller K. Furthermore, we will assume that neither G_o nor K have poles on the unit circle. The system is said to be operated in open loop when $K = 0$. Next, we introduce a quite general family of model structures that will be covered.

The system is modeled by

$$y_t = T(\mathrm{q}, \theta)\chi_t, \qquad \chi_t = [u_t,\ e_t]^{\mathrm{T}} \tag{21}$$

where $T(\mathrm{q}, \theta) = [G(\mathrm{q}, \theta), H(\mathrm{q}, \theta)]$ is a causal finite dimensional LTI model of the system and the noise dynamics, parametrized by the vector $\theta \in \mathbb{R}^{1\times n}$. The noise model may also be independently parametrized by a separate vector η, and then we write $H(\mathrm{q}, \eta)$. This distinction is only used when it has important implications and for the general treatment we can consider the noise model $H(\mathrm{q}, \theta)$.

The model parametrization is such that the *true* system is in the model set, that is, there is a, not necessarily unique, parameter θ^o such that

$$G_o(\mathrm{q}) = G(\mathrm{q}, \theta^o), \quad H_o(\mathrm{q}) = H(\mathrm{q}, \theta^o)$$

The model $T(z, \theta)$ is continuously differentiable with respect to θ in a neighborhood of θ^o. The type of model described above includes all standard black-box model structures such as ARMAX, output error and Box-Jenkins.

Now, introduce the spectral factor of the signal-to-noise ratio

$$R_{\text{SNR}}(z) = R_\chi(z) R_v^{-1}(z)$$

where $R_v = \sqrt{\lambda_o} H_o$ is a stable minimum phase spectral factor of the noise spectrum Φ_v and where R_χ is a stable spectral factor of the spectrum Φ_χ of χ, i.e.

$$R_\chi := \begin{bmatrix} S_o R & -K S_o H_o \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda_o} \end{bmatrix} \tag{22}$$

where $S_o(\mathrm{q}) = 1/(1 + K(\mathrm{q}) G_o(\mathrm{q}))$ is the closed loop sensitivity function. It is straightforward to show that the predictor gradient, normalized by $1/\sqrt{\lambda_o}$, is given by

$$\Psi(z) = T'(z, \theta^o) R_{\text{SNR}}(z) \tag{23}$$

where $T'(z, \theta) = \begin{bmatrix} \frac{\partial G(z,\theta)}{\partial \theta} & \frac{\partial H(z,\theta)}{\partial \theta} \end{bmatrix}$.

We will assume that the model parametrization is such that Ψ is stable. The stability assumption on the closed loop system and the assumptions on G_o and K imply that $R_{\text{SNR}}(z)$ and its inverse are real rational functions without poles on the unit circle and hence are $\mathcal{L}_2^{2\times 2}$ functions, as well as bounded on the unit circle.

Our main assumption is that prediction error identification results in an asymptotic covariance $\mathrm{AsCov}\ J(\hat{\theta}_N)$ of the quantity of interest J given by (3). We refer to [19] for exact conditions and to [11] for a discussion of the case when θ^o is non-unique and $\langle \Psi, \Psi \rangle$ singular.

B. *Asymptotic covariance of LTI system properties*

In this section we will derive an expression for the asymptotic covariance (3) of the estimate $J(\hat{\theta}_N)$ of an arbitrary differentiable quantity $J : \mathbb{R}^{1\times n} \to \mathbb{C}^{1\times p}$ when Ψ in (3) is given by (23). While this can be done on a case by case basis for different model structures using Theorem

II.1, we will instead use (a generalization of) impulse response coefficients as an intermediate parametrization in order to, through Lemma II.2, obtain an expression that is valid regardless of the model structure.

Take $\{\mathcal{G}_k(z)\}_{k=1}^{\infty}$ and $\{\mathcal{H}_k(z)\}_{k=1}^{\infty}$ to be two sequences of orthonormal $\mathcal{L}_2$-functions and for $k = 1, 2, \ldots$ define the orthonormal functions

$$\mathcal{T}_{2k-1}(z) = [\mathcal{G}_k(z) \;\; 0],\; \mathcal{T}_{2k}(z) = [0 \;\; \mathcal{H}_k(z)],\; k = 1, 2, \ldots \tag{24}$$

With $\tau = [\tau_1 \;\; \tau_2 \;\; \cdots]$, any transfer function $T = [G \;\; H]$ satisfying the assumptions in Section III-A can be represented by

$$T(z) = [G(z) \;\; H(z)] = \sum_{k=1}^{\infty} \tau_k \; \mathcal{T}_k(z) \tag{25}$$

on the unit circle, for suitable choices of $\{\mathcal{G}_k(z)\}_{k=1}^{\infty}$ and $\{\mathcal{H}_k(z)\}_{k=1}^{\infty}$. It is worth noticing that also (casual) unstable G can be represented by (25) on the unit circle and that with $\mathcal{G}_k(z) = \mathcal{H}_k(z) = z^{-(k-1)}$, (25) corresponds to the usual impulse response representation.

Also the original model (21), which is parametrized by the vector θ, can be expressed through the parametrization (25):

$$T(z, \theta) = \sum_{k=1}^{\infty} \tau_k(\theta)\mathcal{T}_k(z) \tag{26}$$

or

$$G(z, \theta) = \sum_{k=1}^{\infty} g_k(\theta)\mathcal{G}_k(z), \quad H(z, \theta) = \sum_{k=1}^{\infty} h_k(\theta)\mathcal{H}_k(z)$$

where $g_k = \tau_{2k-1},\; h_k = \tau_{2k}$. We will denote by τ^o the model parameters corresponding to θ^o, i.e. $\tau^o = \tau(\theta^o)$.

We will first establish some properties of the maps $\tau_k : \mathbb{R}^{1\times n} \to \mathbb{C}$, $k = 1, \ldots$.

Lemma III.1. *Under the assumptions in Section III-A,* $\tau_k(\theta),\; k = 1, \ldots$ *are differentiable at* θ^o *and*

$$T'(z, \theta^o) = \sum_{k=1}^{\infty} \tau_k'(\theta^o)\mathcal{T}_k(z) \in \mathcal{L}_2^{n\times 2} \tag{27}$$

Proof: By assumption, the elements of $T(z, \theta)$ are finite dimensional real rational functions with no poles on the unit circle, i.e. they can be written as $B_i(z, \theta)/A_i(z, \theta)$, $i = 1, 2$ for some polynomials B_i and A_i with real coefficients where $A_i(z, \theta^o)$, $i = 1, 2$, does not have any roots on the unit circle. Thus $T(z, \theta^o)$ belongs to $\mathcal{L}_2^2$ and hence τ_k can be expressed through the inverse transformation

$$\tau_k(\theta^o) = \langle T(z, \theta^o), \mathcal{T}_k(z) \rangle \tag{28}$$

By assumption $T(z, \theta)$ is continuously differentiable with respect to θ in a neighborhood of θ^o and hence the right hand side of (28) is differentiable at θ^o with derivative given by differentiation under the integral sign [30, Theorem 9.42]. Thus

$$\tau_k'(\theta^o) = \langle T'(z, \theta^o), \mathcal{T}_k(z) \rangle, \; k = 1, \ldots \tag{29}$$

Now the elements of $T'(z, \theta^o)$ are given by

$$B_i'(z, \theta^o)/A_i(z, \theta^o) - B_i(z, \theta^o) A_i'(z, \theta^o)/A_i^2(z, \theta^o), \; i = 1, 2$$

and since by assumption $A_i(z, \theta^o)$, $i = 1, 2$ does not have any poles on the unit circle, $T'(z, \theta^o) \in \mathcal{L}_2^2$. But then (29) are the Fourier coefficients of $T'(z, \theta^o)$ and (27) follows. We remark that $\{\mathcal{T}_k\}$ does not necessarily have to be an orthonormal basis for $\mathcal{L}_2^2$. If this is not the case we can adjoin orthonormal functions to $\{\mathcal{T}_k\}$ so that it becomes a basis and reason as above. However since the τ_k's corresponding to the added basis functions are identically zero their derivatives will be zero and the corresponding terms will not show up in (27). ■

Theorem III.2. *Suppose that* $J_\tau(\tau^o) \in \mathbb{C}^{1 \times p}$ *is estimated by* $J(\hat{\theta}_N) = J_\tau(\tau(\hat{\theta}_N))$.

Assume that

i) *The system and model assumptions in Section III-A hold.*
ii) *The partial derivatives of* J_τ *with respect to* τ_k, $k = 1, \ldots, n_\tau$ *exists at* θ^o *and satisfy*

$$\nabla J_\tau(z) := \sum_{k=1}^{n_\tau} \left(\frac{\partial J_\tau(\tau^o)}{\partial \tau_k} \right)^* \mathcal{T}_k(z) \in \mathcal{L}_2^{p \times m} \tag{30}$$

iii) *The chain rule* (12) *applies.*

Then

$$\begin{aligned} &\operatorname{AsCov} J(\hat{\theta}_N) \\ &= \left\langle \mathbf{P}_{\mathcal{S}_\Psi} \left\{ \nabla J_\tau \; R_{\text{SNR}}^{-*} \right\}, \mathbf{P}_{\mathcal{S}_\Psi} \left\{ \nabla J_\tau \; R_{\text{SNR}}^{-*} \right\} \right\rangle^{\mathrm{T}} \end{aligned} \tag{31}$$

When J *is scalar,* (31) *becomes*

$$\operatorname{AsVar} J(\hat{\theta}_N) = \left\| \mathbf{P}_{\mathcal{S}_\Psi} \left\{ \nabla J_\tau \; R_{\text{SNR}}^{-*} \right\} \right\|^2$$

Proof: The result follows from Lemma II.2 with $W = R_{\text{SNR}}^{-1}(z)$ if the conditions of this lemma can be verified. First it follows from Lemma III.1 that $\tau_k'(\theta^o)$ exists for $k = 1, \ldots$. Next, with Ψ as in (23) we get

$$\Psi(z)\; R_{\text{SNR}}^{-1}(z) = T'(z, \theta^o)$$

and from Lemma III.1 we have that

$$T'(z, \theta^o) = \sum_{k=1}^{\infty} \tau_k'(\theta^o)\; \mathcal{T}_k(z) \in \mathcal{L}_2^{n \times m}$$

so (9) is verified. Furthermore, (11) and (12) follows directly from Assumptions ii) and iii) in the theorem. Finally (13) follows from Assumption ii) in the theorem and that $R_{\text{SNR}}^{-*}(z^{-*})$ is bounded on the unit circle by the assumptions in Section III-A. All conditions of Lemma II.2 have now been verified and the result follows. ■

The result in Theorem III.2 is basically applicable whenever the predictor gradient is given by (23) and thus very general. Thus the expression (31) is an exact representation of the asymptotic variance (3) which is valid for a wide range of LTI model structures, including commonly used structures such as ARMAX, output-error and Box-Jenkins, and it can be used for both open loop and closed loop identification. Furthermore it expresses the variance of any property of the estimated model, (provided this property can be expressed as a differentiable function of the (impulse response) coefficients τ_k satisfying the conditions in the theorem).

Remarks:

1) The property of interest enters the expression only through the function ∇J_τ which in some sense describes the sensitivity of the property J to changes in the transfer function T. One could interpret ∇J_τ as something similar to a derivative $\nabla J_\tau \sim \Delta J / \Delta T$.
2) ∇J_τ is weighted by $R_{\text{SNR}}^{-*}(z^{-*})$ which is a spectral factor of the ratio $\Phi_v(z)\Phi_\chi^{-1}(z)$. This ratio is known from the expression
$$\lim_{m \to \infty} \frac{1}{m} \operatorname{AsCov} T(\mathrm{e}^{\mathrm{j}\omega}, \hat{\theta}_N) = \Phi_v(\mathrm{e}^{\mathrm{j}\omega}) \Phi_\chi^{-\mathrm{T}}(\mathrm{e}^{\mathrm{j}\omega}) \tag{32}$$
derived in [18] and can be interpreted as the frequency-wise noise to signal ratio.
3) The space $\mathcal{S}_\Psi$ is the span of the rows of
$$\Psi(z) = T'(z, \theta^o)\; R_{\text{SNR}}^{-1}(z) = T'(z, \theta^o) R_\chi(z) R_v^{-1}(z)$$

The structure of this space is thus to a large extent determined by the model structure (through T'). However, the true system also determines $T'(z, \theta^o)$ (through θ^o) and together with the experimental conditions also acts as translation through the factor $R_\chi(z) R_v^{-1}(z)$.

From the observations above we see that there is a certain decoupling between different quantities: The function of interest influences only the function that is projected and the model structure influences only the subspace on which the function is projected. However, the experimental conditions are present both in the function to be projected and the subspace.

We also remark that Theorem III.2 illustrates the flexibility offered by (8). The function $\nabla J_\tau(z)\ R_{\text{SNR}}^{-*}(z^{-*})$ is a function in the set (8) of functions γ that can be used in Theorem II.1 such that $\langle \Psi, \gamma \rangle$ is the sensitivity of the quantity of interest with respect to the model parameters (this is the essence of Lemma II.2 and Theorem III.2). However, this function is chosen with care so that it can be used regardless of the model structure (which determines $T'(z, \theta^o)$). It is due to this that the decoupling between the function of interest and the model structure, discussed above, is obtained. As we will see in Section III-E, this also opens up the possibility to derive upper bounds for the asymptotic variance that are model structure independent. This is one of the features offered by the geometric approach employed in this paper. For further discussion on the geometric approach we refer to the companion paper [12].

C. Some special cases when J is scalar

Below we will consider some special cases that lead to simplifications of (31). First we need to separate the two columns of the matrix ∇J_τ as

$$\nabla J_\tau(z) = \begin{bmatrix} \nabla J_\tau^g(z) & \nabla J_\tau^h(z) \end{bmatrix}$$

$$\nabla J_\tau^g(z) = \sum_{k=1}^{\infty} \left(\frac{\partial J_\tau(\tau^o)}{\partial g_k} \right)^* \mathcal{G}_k(z)$$

$$\nabla J_\tau^h(z) = \sum_{k=1}^{\infty} \left(\frac{\partial J_\tau(\tau^o)}{\partial h_k} \right)^* \mathcal{H}_k(z)$$

and for simplicity we consider the case when J and J_τ are scalars for which case also ∇J_τ^g and ∇J_τ^h are scalars.

Corollary III.3 (Simplifications of Theorem III.2)**.** *Under the assumptions in Theorem III.2 we have the following special cases for scalar functions J:*

1) J ***independent of noise model*** H***:***

When J *does not depend on the noise model* H*, the asymptotic variance of* $J(\hat{\theta}_N)$ *is given by*

$$\operatorname{AsVar} J(\hat{\theta}_N) = \left\| \mathbf{P}_{\mathcal{S}_\Psi} \left\{ \nabla J_\tau^g \begin{bmatrix} \frac{\sqrt{\lambda_0} H_o^*}{S_o^* R^*} & 0 \end{bmatrix} \right\} \right\|^2 \tag{33}$$

2) Open loop operation:

When $K = 0$ *the asymptotic variance of* $J(\hat{\theta}_N)$ *is given by*

$$\operatorname{AsVar} J(\hat{\theta}_N) = \left\| \mathbf{P}_{\mathcal{S}_\Psi} \left\{ \begin{bmatrix} \nabla J_\tau^g \frac{\sqrt{\lambda_0} H_o^*}{R^*} & \nabla J_\tau^h H_o^* \end{bmatrix} \right\} \right\|^2 \tag{34}$$

Furthermore $\mathcal{S}_\Psi$ *is the span of the rows of*

$$\begin{bmatrix} \frac{G'(z,\theta^o) R(z)}{\sqrt{\lambda_o} H_o(z)}, & \frac{H'(z,\theta^o)}{H_o(z)} \end{bmatrix}$$

3) Open loop and independent parametrization:

When $K = 0$ *and the model* $G(z,\theta)$ *and the noise model* $H(z,\eta)$ *are independently parametrized, the asymptotic variance of* $J(\hat{\theta}_N, \hat{\eta}_N)$ *is given by*

$$\begin{aligned} &\operatorname{AsVar} J(\hat{\theta}_N, \hat{\eta}_N) = \\ &= \left\| \mathbf{P}_{\mathcal{S}_{\Psi_G}} \left\{ \nabla J_\tau^g \frac{\sqrt{\lambda_0} H_o^*}{R^*} \right\} \right\|^2 + \left\| \mathbf{P}_{\mathcal{S}_{\Psi_H}} \left\{ \nabla J_\tau^h H_o^* \right\} \right\|^2 \end{aligned} \tag{35}$$

where $\Psi_G = R(z) G'(z,\theta^o)/H_o(z)$ *and* $\Psi_H = H'(z,\eta_o)/H_o(z)$*, respectively.*

Proof: The proofs are straightforward and therefore omitted. ∎

Remarks:

1) We noted earlier that the weighting $R_{\text{SNR}}^{-1}(z)$ appearing in (31) is a spectral factor of $\Phi_v(z)\Phi_\chi^{-1}(z)$. The upper left corner of this ratio is given by Φ_v/Φ_u^r where Φ_u^r denotes the spectrum of the contribution of r to u. When we only consider properties of the model G (and not H) only this ratio matters and $H_o^*/(S_o^* R^*)$ in (33) is a spectral factor of this ratio.
2) Notice that scalings in the input amplitude only affects the weighting $\sqrt{\lambda_o} H_o/R$ in (35) and not the space $\mathcal{S}_\Psi$ since the span of Ψ is unaffected by frequency independent scalings factors.

D. An explicit expression of the asymptotic variance

The most complicated step of evaluating the variance expression (31) is the projection onto the space $\mathcal{S}_\Psi$. If an orthonormal basis $\{\mathcal{B}_k(z)\}_{k=1}^n$ of $\mathcal{S}_\Psi$ is known, then as shown in [12] the projection can be computed from

$$\mathbf{P}_{\mathcal{S}_\Psi}\{f\} := \sum_{k=1}^{n} \langle f, \mathcal{B}_k \rangle \mathcal{B}_k$$

For some properties J, the asymptotic variance (31) can be expressed directly as a function of an orthonormal basis for the space $\mathcal{S}_\Psi$.

Theorem III.4. *Let the assumptions in Section III-A be in force. Suppose that*

$$\Lambda = T'(z_o, \theta^o)\alpha \tag{36}$$

holds for some $z_o \in \mathbb{C}$ *and some* $\alpha \in \mathbb{C}^{2\times p}$*, then the asymptotic covariance can be expressed as*

$$\begin{aligned}&\mathrm{AsCov}\; J(\hat{\theta}_N) = \\ &\alpha^{\mathrm{T}} R_{\mathrm{SNR}}^{-\mathrm{T}}(z_o) \sum_{k=1}^{n} \mathcal{B}_k^{\mathrm{T}}(z_o)\overline{\mathcal{B}_k(z_o)}\; \overline{R_{\mathrm{SNR}}^{-1}(z_o)}\; \overline{\alpha}\end{aligned} \tag{37}$$

where $\{\mathcal{B}_k\}_{k=1}^n$ *is any orthonormal basis for the space* $\mathcal{S}_\Psi$*.*

The result (37) *also holds under the conditions of Theorem III.2 if, in addition, the condition*

$$\left.\frac{\partial J_\tau(\tau)}{\partial \tau_k}\right|_{\tau=\tau(\theta^o)} = \mathcal{T}_k(z_o)\alpha \tag{38}$$

holds for some $\alpha \in \mathbb{C}^{2\times p}$ *and* $z_o \in \mathbb{C}$*.*

Proof: The relation (36) implies that $\Lambda = \Psi(z_o)R_{\mathrm{SNR}}^{-1}(z_o)\alpha$ and then Lemma II.9 in [12] gives (37).

Next, from (38) follows that

$$\nabla J_\tau\; R_{\mathrm{SNR}}^{-*}(z_o) = \alpha^* \sum_{k=1}^{\infty} \mathcal{T}_k^*(z_o)\; \mathcal{T}_k\; R_{\mathrm{SNR}}^{-*}(z_o)$$

and now Lemma II.3 gives (37). ■

Conditions (38) and (36) for Theorem III.4 are closely related. Condition (38) can be formulated as

$$\frac{\partial J_\tau(\tau)}{\partial \tau_k} = \frac{\partial T(z_o,\tau)}{\partial \tau_k}\alpha$$

while condition (36) can be formulated as

$$\frac{\partial J(\theta)}{\partial \theta_k} = \frac{\partial T(z_o,\theta)}{\partial \theta_k}\alpha$$

and we see that the only difference lies in the parametrization and that it does not matter if the system and the property J are described by the model parameters θ or the by the coefficients τ of the rational orthonormal basis representation.

E. Upper bounds

One advantage with the new expression (31) for the asymptotic covariance (3) is that it is easy to provide simple, model structure independent, bounds for (3). These bounds are obtained by replacing the projection onto $\mathcal{S}_\Psi$ with projections onto the spaces $\mathcal{H}_2^2$ or $\mathcal{L}_2^2$, giving upper bounds for (3). When we are projecting on $\mathcal{L}_2^2$, i.e. when the projection is removed, the bounds derived below are typically conservative even as the model order increases since $\mathcal{S}_\Psi \subseteq \mathcal{H}_2^m$ regardless of the model order and model structure, while the function that is projected, $\nabla J_\tau\ R_{\text{SNR}}^{-*}$, typically has a term that belongs to the complement of $\mathcal{H}_2^m$.

Theorem III.5. *Let the conditions of Theorem III.2 be fulfilled. An upper bound of the asymptotic covariance of $J(\hat{\theta}_N)$ is then given by*

$$\operatorname{AsCov} J(\hat{\theta}_N) \leq \langle \nabla J_\tau \Phi_v \Phi_\chi^{-1}, \nabla J_\tau \rangle^{\mathrm{T}} \tag{39}$$

When J is scalar we get

$$\operatorname{AsVar} J(\hat{\theta}_N) \leq \|\nabla J_\tau\|^2_{\Phi_v \Phi_\chi^{-1}} \tag{40}$$

Proof: By removing the projection in (31) of Theorem III.2 we get an upper bound, c.f. Lemma II.6 in [12]. ∎

We remark that the bounds in Theorem III.5 typically (but not always) through ∇J_τ and $R_{\text{SNR}}^{-1}(z)$ depend on the true underlying system.

Before discussing the expression (40) we also present two special cases that simplify the expression further.

Corollary III.6 (Simplifications of Theorem III.5)**.** *Under the assumptions in Theorem III.5 we have the following simplifying special cases:*

1) Independent of noise model H:

When the property J does not depend on the noise model H, the upper bound (40) *is given by*

$$\mathrm{AsVar}\, J(\hat{\theta}_N) \leq \|\nabla J_\tau^g\|^2_{\Phi_v/\Phi_u^r} \tag{41}$$

where Φ_u^r is the part of the input spectrum that is due to r_t.

2) Open loop:

When the system is identified in open loop, the upper bound (40) *is given by*

$$\mathrm{AsVar}\, J(\hat{\theta}_N) \leq \|\nabla J_\tau^g\|^2_{\Phi_v/\Phi_u} + \left\|\nabla J_\tau^h\right\|^2_{\Phi_v/\lambda_0} \tag{42}$$

Proof: (41) follows from (40) by letting $\nabla J_\tau^h(z) \equiv 0$ and noting that the upper left block of Φ_χ^{-1} is given by $(\Phi_u - \frac{1}{\lambda_0}\Phi_{ue}\Phi_{eu})^{-1} = 1/\Phi_u^r$. In open loop the spectrum $\Phi_\chi = \left[\begin{smallmatrix}\Phi_u & 0\\ 0 & \lambda_o\end{smallmatrix}\right]$ is diagonal, which gives (42) when used in (40). ∎

Notice that the upper bounds above are valid for *any* model structure, which also means that they apply to any model order.

It is obvious that the inverse of the signal to noise ratio, i.e. $\Phi_v\Phi_\chi^{-1}$, plays an important role for the variance. For simplicity we will here consider the variance bound from Corollary III.6 in the open loop case for a function J_τ that is independent of the noise model H. Rewriting the upper bound (42) we get

$$\mathrm{AsVar}\, J(\hat{\theta}) \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\Phi_v(\mathrm{e}^{\mathrm{j}\omega})}{\Phi_u(\mathrm{e}^{\mathrm{j}\omega})} \left|\nabla J_\tau^g(\mathrm{e}^{\mathrm{j}\omega})\right|^2 \mathrm{d}\omega$$

Thus, if the signal to noise ratio is high at frequencies where $|\nabla J_\tau^g(\mathrm{e}^{\mathrm{j}\omega})|$ is large, the model will be accurate.

The next theorem describes a case when a simple bound can be found by projecting onto the subspace $\mathcal{H}_2^2$. This gives a lower (tighter) bound than Theorem III.5 where projection on $\mathcal{L}_2^2$ was considered.

Theorem III.7. *Let the conditions for Theorem III.4 be fulfilled for a z_o such that $|z_o| > 1$. Then an upper bound of the asymptotic covariance is given by*

$$\mathrm{AsCov}\, J(\hat{\theta}_N) \leq \frac{1}{|z_o|^2 - 1}\, \alpha^{\mathrm{T}}\, \Phi_v(z_o)\Phi_\chi^{-T}(z_o)\, \overline{\alpha} \tag{43}$$

Proof: An upper bound of the expression (31) is obtained by projecting onto $\mathcal{H}_2^2$ instead of $\mathcal{S}_\Psi \subseteq \mathcal{H}_2^2$, c.f. Lemma II.6 in [12]. Since all elements of both $G'(z,\theta^o)$ and $H'(z,\theta^o)$ have at least one time delay we will exclude constant functions. Thus, following Theorem III.4, we can use Lemma II.9 in [12] to express the upper bound as

$$\alpha^{\mathrm{T}}\ R_{\mathrm{SNR}}^{-\mathrm{T}}(z_o)\ \sum_{k=1}^{\infty} \mathcal{B}_k^{\mathrm{T}}(z_o)\overline{\mathcal{B}_k(z_o)}\ \ \overline{R_{\mathrm{SNR}}^{-1}(z_o)}\ \overline{\alpha} \tag{44}$$

where $\{\mathcal{B}_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}_2^2$ excluding constant terms.

One such orthonormal basis is given by $\{\mathcal{B}_k(z)\}_{k=1}^{\infty}$ where $\mathcal{B}_k(z) = [z^{-(k+1)/2}\ 0]$ when k is odd, and $\mathcal{B}_k(z) = [0\ z^{-k/2}]$ when k is even. For $|z_o| > 1$ we then get

$$\sum_{k=1}^{\infty} \mathcal{B}_k^{\mathrm{T}}(z_o)\overline{\mathcal{B}_k(z_o)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sum_{k=1}^{\infty} |z_o|^{-2k} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{|z_o|^2 - 1}$$

which when inserted in (44) gives the upper bound in (43). ■

F. Comparison with an existing result

Using the orthonormality of $\{\mathcal{T}_k\}$ and (25) gives that $\tau_l = \langle T, \mathcal{T}_l \rangle$ and hence (with m being the model order)

$$\begin{aligned}
&\lim_{m\to\infty} \frac{1}{m}\ \mathbf{E}\left[(\tau_k(\hat{\theta}_N)^T - \tau_k^o)^{\mathrm{T}}\overline{(\tau_l(\hat{\theta}_N)^T - \tau_l^o)}\right] \\
&= \lim_{m\to\infty} \frac{1}{m}\ \mathbf{E}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{T}_k^*(\mathrm{e}^{\mathrm{j}\omega})(T(\mathrm{e}^{\mathrm{j}\omega},\hat{\theta}_N) - T_o(\mathrm{e}^{\mathrm{j}\omega}))^T d\omega \right.\\
&\left.\frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{(T(\mathrm{e}^{\mathrm{j}\mu},\hat{\theta}_N) - T_o(\mathrm{e}^{\mathrm{j}\mu}))}\mathcal{T}_l(\mathrm{e}^{\mathrm{j}\mu})d\mu\right] = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \\
&\mathcal{T}_k^*(\mathrm{e}^{\mathrm{j}\omega}) \lim_{m\to\infty}\frac{1}{m}\ \mathbf{E}\left[(T(\mathrm{e}^{\mathrm{j}\omega},\hat{\theta}_N) - T_o(\mathrm{e}^{\mathrm{j}\omega}))^T\right. \\
&\left.\overline{(T(\mathrm{e}^{\mathrm{j}\mu},\hat{\theta}_N) - T_o(\mathrm{e}^{\mathrm{j}\mu}))}\right]\ \mathcal{T}_l(\mathrm{e}^{\mathrm{j}\mu})d\omega d\mu
\end{aligned}$$

assuming that the limit operation and the integration commute. If we now use the asymptotic result (32) and another result from [18], namely that frequency function estimates at different frequencies become uncorrelated as the model order $m \to \infty$, the expression above collapses to

$$\lim_{m\to\infty} \frac{1}{m}\ \mathbf{E}\left[(\tau_k(\hat{\theta}_N)^T - \tau_k^o)^{\mathrm{T}}\overline{(\tau_l(\hat{\theta}_N)^T - \tau_l^o)}\right] = 0$$

which in turn suggests that for any J of the type (10) for which the chain rule (12) holds

$$\lim_{m\to\infty} \frac{1}{m}\ \mathrm{AsCov}\ J(\hat{\theta}_N) = 0 \tag{45}$$

We have thus obtained that direct application of the model order asymptotic results in [18] gives that the asymptotic variance is of the order $o(m)$. This is a considerably weaker result than the upper bound (39) derived in this paper. Thus we have shown that, when the conditions of Theorem III.2 hold, the upper bounds derived in this paper are significantly more accurate expressions for the asymptotic covariance than the asymptotic covariance expressions implied by the results in [18]. The fact that the scaling factor m is not present is especially important as it shows that certain properties, even of highly complex systems, are not subject to what is known as the "curse of complexity", i.e. there are system properties that can be accurately identified using full order models also when the system is highly complex. In the next section we will see some examples of such properties. For more details on this important topic we refer the reader to [10, 28, 29].

G. *Geometrical insights*

In this section we will discuss briefly some insights that can be obtained almost directly from the asymptotic variance expressions we have derived above.

Recall the covariance expression (31)

$$\operatorname{AsCov} J(\hat{\theta}_N) = \left\langle \mathbf{P}_{\mathcal{S}_\Psi}\left\{\nabla J_\tau R_{\text{SNR}}^{-*}\right\}, \mathbf{P}_{\mathcal{S}_\Psi}\left\{\nabla J_\tau R_{\text{SNR}}^{-*}\right\}\right\rangle^{\mathrm{T}} \tag{46}$$

and the expression (23) for the prediction error gradient

$$\Psi(z) = T'(z,\theta^o) R_{\text{SNR}}(z)$$

We will start with giving geometric interpretations to some quite well known results. Below Theorem III.2 it was observed that the model structure (recall that the model structure is represented by $T'(z,\theta^o)$) influences only the subspace $\mathcal{S}_\Psi$ in (46). Furthermore, the projection only depends on the span of Ψ, i.e. the subspace $\mathcal{S}_\Psi$. From these two observations it follows that all model structures whose predictor gradients span the same space will have exactly the same asymptotic covariance. For example, order n Laguerre models [33] with poles in ξ will have the same asymptotic variance as fixed denominator models of order n with a pole of multiple n at ξ. It also follows that scaling the model structure, i.e. replacing $T(z,\theta)$ with $\tilde{T}(z,\theta) = \alpha T(z,\theta)$ will not change the asymptotic variance, again since both the function to be projected and the subspace does not change. On the other hand if the experimental conditions are changed so that the signal to noise ratio R_{SNR} is scaled by a factor β, then since $\mathcal{S}_\Psi$ will remain the same (even though Ψ is rescaled), the asymptotic variance is scaled by $1/\beta^2$.

Now, we turn to a less obvious insight that does not seem to be generally known. Suppose that the explicit expression (37) holds for some z_o strictly outside the unit circle and that Ψ contains a pole at z_o^{-1}. Suppose further that the orthonormal basis used in (37) is of the form (19). If we then order the poles in Ψ such that $\xi_1 = z_o^{-1}$, we obtain from (19) that $\mathcal{B}_1(z_o) = \sqrt{1-|z_o|^{-2}}/(z_o - z_o^{-1})$ and $\mathcal{B}_k(z_o) = 0$, $k = 2, \ldots, n$, resulting in that

$$\operatorname{AsCov} J(\hat{\theta}_N) = \frac{1-|z_o|^{-2}}{|z_o - z_o^{-1}|^2}\, \alpha^{\mathrm{T}} \Phi_v(z_o) \Phi_\chi^{-T}(z_o)\, \overline{\alpha} \tag{47}$$

This expression is remarkable in that it is independent of the model structure and model order. Now recall that $\Psi(z) = T'(z, \theta^o) R_{\text{SNR}}(z_o)$. Thus when the assumptions in Theorem III.4 apply and when the experimental conditions can be chosen such that $R_{\text{SNR}}(z)$ has a pole at z_o^{-1}, this choice makes the asymptotic covariance the same for different model structures and arbitrary model order. This insight is important in order to come to terms with the so called "curse of dimensionality" discussed in Section III-F. We will illustrate this idea in Section IV-B where the objective is to identify NMP-zeros. The geometric approach has been used in [22, 14] to generalize this result as well as to show that certain optimality properties also hold from an experiment design perspective.

IV. Analysis of some LTI system properties

In this section we apply the results from Section III to some specific examples of the function $J(\theta)$.

A. Frequency response

We will first look at the covariance of the frequency response estimate, i.e. $J(\theta) = T(\mathrm{e}^{\mathrm{j}\omega_o}, \theta)$ for a fix frequency ω_o when T_o is stable (so that the frequency response is well defined). Then we get

$$\Lambda = T'(\mathrm{e}^{\mathrm{j}\omega_o}, \theta^o) = \Psi(\mathrm{e}^{\mathrm{j}\omega_o}) R_{\text{SNR}}^{-1}(\mathrm{e}^{\mathrm{j}\omega_o})$$

where Ψ is given by (23). Theorem III.4 can be applied to get the covariance expression

$$\begin{aligned}&\operatorname{AsCov} T(\mathrm{e}^{\mathrm{j}\omega_o}, \hat{\theta}_N)\\&= R_{\text{SNR}}^{-\mathrm{T}}(\mathrm{e}^{\mathrm{j}\omega_o}) \sum_{k=1}^{n} \mathcal{B}_k^{\mathrm{T}}(\mathrm{e}^{\mathrm{j}\omega_o}) \overline{\mathcal{B}_k(\mathrm{e}^{\mathrm{j}\omega_o})}\ \overline{R_{\text{SNR}}^{-1}(\mathrm{e}^{\mathrm{j}\omega_o})} \qquad (48)\end{aligned}$$

where $\{\mathcal{B}_k\}_{k=1}^n$ is any orthonormal basis for the space $\mathcal{S}_\Psi$.

It is instructive to also consider the formulation in Theorem III.2 for expressing $\operatorname{AsCov} T(\mathrm{e}^{\mathrm{j}\omega_o}, \hat{\theta}_N)$. We will use basis functions $\mathcal{T}_k$ (24) defined by $\mathcal{G}_k(z) = \mathcal{H}_k(z) = z^{-(k-1)}$ so that

$$T(z,\theta) = \sum_{k=1}^{\infty} \tau_k(\theta)\mathcal{T}_k(z)$$

is parametrized in terms of the impulse responses of $G(z,\theta)$ and $H(z,\theta)$. For this problem $J_\tau(\tau(\theta)) = \sum_{k=1}^{\infty} \tau_k(\theta)\mathcal{T}_k(\mathrm{e}^{\mathrm{j}\omega_o})$ and hence $\partial J_\tau(\tau)/\partial\tau_k = \mathcal{T}_k(\mathrm{e}^{\mathrm{j}\omega_o})$ which implies that ∇J_τ, defined in (30), is not an $\mathcal{L}_2$-function. Thus we instead look at $J(\theta) = T(z_o,\theta)$, $z_o = r\mathrm{e}^{\mathrm{j}\omega_o}$, $r > 1$ and later we let $r \to 1$. The function ∇J_τ is now given by

$$\nabla J_\tau(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sum_{k=1}^{\infty} \bar{z}_o^{-k} z^{-k} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\bar{z}_o^{-1} z^{-1}}{1 - \bar{z}_o^{-1} z^{-1}}$$

which is a function in $\mathcal{L}_2$ so that Assumption ii) in Theorem III.2 holds. Furthermore, Assumption iii) in the same lemma follows directly from Lemma III.1 with $z = \mathrm{e}^{\mathrm{j}\omega_o}$ for this problem. Thus Theorem III.2 applies under the assumptions in Section III-A.

Let $\{\mathcal{B}_k\}_{k=1}^n$ be an orthonormal basis for $\mathcal{S}_\Psi$ and we get

$$\begin{aligned}&\langle \nabla J_\tau(z) R_{\text{SNR}}^{-*}(z^{-*}), \mathcal{B}_k(z)\rangle\\&= \frac{1}{2\pi j} \oint_{|z|=1} \frac{\bar{z}_o^{-1} z^{-1}}{1 - \bar{z}_o^{-1} z^{-1}} R_{\text{SNR}}^{-*}(z^{-*}) \mathcal{B}_k^*(z^{-*}) \frac{\mathrm{d}z}{z}\\&= R_{\text{SNR}}^{-*}(z_o)\mathcal{B}_k^*(z_o) \qquad (49)\end{aligned}$$

In the second equality we use that $z^{-1} R_{\text{SNR}}^{-*}(z^{-*})\mathcal{B}_k^*(z^{-*})$ has all poles outside the unit circle (since all $\mathcal{B}_k$ contain at least one unit time delay which cancels the factor z^{-1}) and residue calculus, see e.g., [34], gives the result (49). The projection $\mathbf{P}_{\mathcal{S}_\Psi}\{\nabla J_\tau R_{\text{SNR}}^{-*}\}$ can now be computed as

$$\begin{aligned}&\mathbf{P}_{\mathcal{S}_\Psi}\{\nabla J_\tau R_{\text{SNR}}^{-*}\}\\&= \sum_{k=1}^{n} \langle \nabla J_\tau R_{\text{SNR}}^{-*}, \mathcal{B}_k\rangle \mathcal{B}_k = R_{\text{SNR}}^{-*}(z_o) \sum_{k=1}^{n} \mathcal{B}_k^*(z_o)\mathcal{B}_k\end{aligned}$$

With $z_o = r\mathrm{e}^{\mathrm{j}\omega_o}$ and $r \to 1$, we get the asymptotic covariance

$$\begin{aligned}
&\mathrm{AsCov}\, T(\mathrm{e}^{\mathrm{j}\omega_o}, \hat{\theta}_N) \\
&= R_{\mathrm{SNR}}^{-\mathrm{T}}(\mathrm{e}^{\mathrm{j}\omega_o}) \sum_{k=1}^{n}\sum_{l=1}^{n} \mathcal{B}_k^{\mathrm{T}}(\mathrm{e}^{\mathrm{j}\omega_o}) \langle \mathcal{B}_k, \mathcal{B}_p \rangle \overline{\mathcal{B}_p(\mathrm{e}^{\mathrm{j}\omega_o})}\; \overline{R_{\mathrm{SNR}}^{-1}(\mathrm{e}^{\mathrm{j}\omega_o})} \\
&= R_{\mathrm{SNR}}^{-\mathrm{T}}(\mathrm{e}^{\mathrm{j}\omega_o}) \sum_{k=1}^{n} \mathcal{B}_k^{\mathrm{T}}(\mathrm{e}^{\mathrm{j}\omega_o}) \overline{\mathcal{B}_k(\mathrm{e}^{\mathrm{j}\omega_o})}\; \overline{R_{\mathrm{SNR}}^{-1}(\mathrm{e}^{\mathrm{j}\omega_o})}
\end{aligned} \tag{50}$$

which, of course, is the same as (48).

The covariance expression (48) was first established in [26] by employing the theory of reproducing kernels, see also [27]. This approach is closely related to Theorem III.4. Above we have shown that the results in [26] can be given an alternative system theoretic interpretation as resulting from a projection of the weighted z-transform of the sensitivity of the system frequency function with respect to the impulse response on a subspace determined by the model structure, the true system and the experimental conditions. The weighting function depends on the noise to signal ratio during the experiment (which in turn depends on the experimental conditions and the true system). Our paper can also be seen as an extension of the work in [26] regarding variance analysis in frequency function estimation to general quantities J.

B. *Non-minimum phase zeros*

Next we consider estimation of NMP-zeros of a stable system G_o. The zeros of the system are defined as the solutions z to the equation $G(z,\theta) = \sum_{k=1}^{\infty} g_k(\theta) z^{-k} = 0$ and we assume that the zero of interest, z_o, is non-minimum phase, i.e. $|z_o| > 1$. The quantity of interest is thus $J(\hat{\theta}_N) = z_o(\hat{\theta}_N)$. Corollary III.3 can be used since J is independent of the noise model H. Similar to [23] we obtain

$$\frac{\partial J_\tau^g(\tau^o)}{\partial g_k} = -\frac{z_o}{\widetilde{G}_o(z_o)} z_o^{-k}$$

where $\widetilde{G}_o(z) = G_o(z)/(1 - z_o z^{-1})$, which gives, for $|z| > 1/|z_o|$, that

$$\nabla J_\tau^g(z) = -\frac{\bar{z}_o}{\overline{\widetilde{G}_o(z_o)}} \sum_{k=1}^{\infty} \bar{z}_o^{-k} z^{-k} = -\frac{\bar{z}_o}{\overline{\widetilde{G}_o(z_o)}} \frac{\bar{z}_o^{-1} z^{-1}}{(1 - \bar{z}_o^{-1} z^{-1})}$$

which is in $\mathcal{L}_2$. It is straightforward to verify that the chain rule (Assumption ii) in Theorem III.2) applies: Suppose, for simplicity, that the numerator polynomial, $B(\mathrm{q},\theta)$, in $G(\mathrm{q},\theta) =$

$B(\mathrm{q},\theta)/A(\mathrm{q},\theta)$ is independently parametrized. Lemma III.1 gives that $g'_k(\theta^o) = \langle G'(z,\theta^o), z^{-k}\rangle$ so that

$$\begin{aligned}\sum_{k=1}^{\infty} g'_k(\theta^o)\,\frac{\partial J^g_\tau(\tau^o)}{\partial g_k} &= -\frac{z_o}{\widetilde{G}_o(z_o)}\,\sum_{k=1}^{\infty}\langle G'(z,\theta^o), z^{-k}\rangle\, z_o^{-k}\\ &= -\frac{z_o}{\widetilde{G}_o(z_o)}\,G'(z_o,\theta^o) = -\frac{z_o}{\widetilde{G}_o(z_o)}\,\begin{bmatrix}\frac{B'(z_o(\theta^o),\theta^o)}{A(z_o(\theta^o),\theta^o)}\\ \frac{-B(z_o(\theta^o),\theta^o)A'(z_o(\theta^o),\theta^o)}{A^2(z_o(\theta^o),\theta^o)}\end{bmatrix}\\ &= -\frac{z_o}{\widetilde{B}_o(z_o)}\,\begin{bmatrix}B'(z_o(\theta^o),\theta^o)\\ 0\end{bmatrix}\end{aligned}$$

where $\widetilde{B}_o(z) = B(z,\theta^o)/(1-z_oz^{-1})$. But the last expression equals $J'(\theta^o)$ [23].

We have thus verified the conditions in Theorem III.2 and the asymptotic variance can be calculated in the same way as in (49)-(50), alternatively Theorem III.4 could be applied, and, without giving all details, we get

$$\mathrm{AsVar}\, z_o(\hat{\theta}_N) = \frac{\lambda_0|z_o|^2}{|\widetilde{G}_o(z_o)|^2}\frac{|H_o(z_o)|^2}{|R(z_o)|^2}\sum_{k=1}^{n}|\mathcal{B}^1_k(z_o)|^2 \tag{51}$$

where $\mathcal{B}_k := [\mathcal{B}^1_k, \mathcal{B}^2_k]$ and $\{\mathcal{B}_k\}_{k=1}^n$ is an orthonormal basis for $\mathcal{S}_\Psi$. The expression (51) is derived in [23] using other techniques. Also explicit expressions and bounds for $\sum_{k=1}^n |\mathcal{B}^1_k(z_o)|^2$ are derived in [23] for certain model structures. When an orthonormal basis of the type given in Section II-B can be used, we see from (51) that the asymptotic variance for a NMP-zero will be large if there is another NMP-zero nearby. This follows from that the factor $|\widetilde{G}_o(z_o)|^2$ in the denominator will be small in this case and that the basis functions are independent of the system zeros (see Section II-B).

Bounds on the asymptotic variance can be derived using Theorem III.7:

$$\mathrm{AsVar}\, z_o(\hat{\theta}_N) \leq \frac{\lambda_0}{(1-|z_o|^{-2})|\widetilde{G}_o(z_o)|^2}\frac{|H_o(z_o)|^2}{|S_o(z_o)R(z_o)|^2} \tag{52}$$

or using Corollary III.6:

$$\begin{aligned}\mathrm{AsVar}\, z_o(\hat{\theta}_N) &\leq \frac{1}{|\widetilde{G}_o(z_o)|^2}\left\|\frac{1}{(1-\bar{z}_o^{-1}z^{-1})}\right\|^2_{\Phi_v/\Phi_u^r}\\ &= \frac{\lambda_0}{(1-|z_o|^{-2})|\widetilde{G}_o(z_o)|^2}\left\|\frac{H_o}{S_oR}\right\|^2_{wZ}\end{aligned} \tag{53}$$

where $w_Z(\omega) = (1-|z_o|^{-2})/|1-z_o^{-1}\mathrm{e}^{\mathrm{j}\omega}|^2$ can be seen as a weighting function with $\frac{1}{2\pi}\int_{-\pi}^{\pi} w_Z(\omega)\mathrm{d}\omega = 1$. The weighting function $w_Z(\omega)$ focuses on frequencies around $\omega = \arg(z_o)$. Good accuracy of a NMP-zero estimate is thus guaranteed if $|H_o/(S_o R)|$ is small in this frequency range.

The bound (53) is always larger than the bound (52) which is a tight bound in the sense that equality holds (in many cases) when the model order n goes to infinity, see [23], which provides a quite complete asymptotic variance analysis of both zero and pole estimates.

Before closing this section, we illustrate the idea outlined at the end of Section III-G, i.e., that the input can be used to make the asymptotic variance the same for different model structures and arbitrary model orders. For simplicity we will assume that the NMP zero is real and that the system is operating in open loop so that the prediction error gradient (23) is given by

$$\Psi(z) = T'(z, \theta^o) \begin{bmatrix} \frac{R(z)}{\sqrt{\lambda_o}\, H_o(z)} & 0 \\ 0 & \frac{1}{H_o(z)} \end{bmatrix}$$

Assume first that an output error model is used and that the spectral factor of the input is chosen as $R(z) = 1/(z - z_o^{-1})$. From [26] it follows that (19), with $\xi_1 = z_o^{-1}$ and the other poles being the poles of the true system dynamics counted twice, is an orthonormal basis when the number of parameters in the numerator polynomial is at least the number of parameters in the denominator polynomial + 1. Similar to (47), (51) then collapses to

$$\begin{aligned} &\mathrm{AsVar}\, z_o(\hat{\theta}_N) \\ &= \frac{\lambda_0 |z_o|^2}{|\widetilde{G}_o(z_o)|^2} \frac{|H_o(z_o)|^2}{|R(z_o)|^2} \sum_{k=1}^{n} |\mathcal{B}_k^1(z_o)|^2 = \frac{\lambda_0\ (|z_o|^2 - 1)}{|\widetilde{G}_o(z_o)|^2} \end{aligned}$$

Due to the presence of $R(z)$ in Ψ it can easily be shown that exactly the same result is obtained for Box-Jenkins models (under the same order condition). Thus an input with spectral factor $R(z) = 1/(z - z_o^{-1})$ ensures that the asymptotic variance of an estimate of an NMP-zero at z_o becomes independent both of the model and system order and also the model structure. One may argue that this choice of input is infeasible since it depends on the to be estimated zero. However, this insight is of independent value and it has been shown that using an estimate of the NMP-zero in R instead may also reduce the sensitivity of the asymptotic variance with respect to model order and model structure, see [16].

C. $\mathcal{L}_2$-norm

Now we consider the asymptotic variance of the $\mathcal{L}_2$-norm of the estimated model $G(z,\theta) = \sum_{k=1}^{\infty} g_k(\theta)\mathcal{G}_k(z)$, cf. (25). The $\mathcal{L}_2$-norm is given by

$$\|G(\cdot,\theta)\| = \sqrt{\langle G(\cdot,\theta), G(\cdot,\theta)\rangle} = \sqrt{\sum_{k=1}^{\infty} g_k^2(\theta)}$$

and the function $\nabla J_\tau^g(z)$ is

$$\nabla J_\tau^g(z) = \sum_{k=1}^{\infty} \frac{g_k(\theta^o)\mathcal{G}_k(z)}{\sqrt{\sum_{k=1}^{\infty} g_k^2(\theta^o)}} = \frac{G_o(z)}{\|G_o\|} \in \mathcal{L}_2$$

It is straightforward to verify the chain rule, Assumption ii) in Theorem III.2. Thus we can use Corollary III.3 to express the asymptotic variance as

$$\operatorname{AsVar}\|G(\cdot,\hat{\theta}_N)\| = \frac{\left\|\mathbf{P}_{\mathcal{S}_\Psi}\left\{\begin{bmatrix}\frac{\sqrt{\lambda_o}G_o H_o^*}{S_o^* R^*} & 0\end{bmatrix}\right\}\right\|^2}{\|G_o\|^2} \tag{54}$$

The projection in (54) may be cumbersome to calculate, but we can use Corollary III.6 to get an upper bound of the asymptotic variance:

$$\operatorname{AsVar}\|G(\cdot,\hat{\theta}_N)\| \leq \frac{\|G_o\|^2_{\Phi_v/\Phi_u^r}}{\|G_o\|^2}$$

The bound can also be written in the form

$$\operatorname{AsVar}\|G(\cdot,\hat{\theta}_N)\| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\Phi_v(\omega)}{\Phi_u^r(\omega)} w_G(\omega)\mathrm{d}\omega$$

where $w_G(\omega) = |G_o(\mathrm{e}^{\mathrm{j}\omega})|^2/\|G_o\|^2$ is a weighting function with $\frac{1}{2\pi}\int_{-\pi}^{\pi} w_G(\omega)\mathrm{d}\omega = 1$. The weighting function $w_G(\omega)$ gives more weight to frequencies where the gain is large.

D. Impulse response

In this example we look at the asymptotic variance of the coefficients τ_k of the estimated model $T(z,\theta) = \sum_{k=1}^{\infty} \tau_k(\theta)\mathcal{T}_k(z)$, cf. (26), but we assume that only the first n_τ coefficients are of interest and we let

$$J_\tau(\tau) = \tau^{\mathrm{T}} = \begin{bmatrix}\tau_1 & \cdots & \tau_{n_\tau}\end{bmatrix}$$

Now $\frac{\mathrm{d}J_\tau}{\mathrm{d}\tau} = I$ (the identity matrix) and hence

$$\nabla J_\tau(z) = \begin{bmatrix} \mathcal{T}_1(z) \\ \vdots \\ \mathcal{T}_{n_\tau}(z) \end{bmatrix}$$

It is straightforward to verify the chain rule, Assumption ii) in Theorem III.2. Thus the asymptotic covariance can be expressed as

$$\begin{aligned} \mathrm{AsCov}\, \tau^{\mathrm{T}}(\hat{\theta}_N) &= \langle \mathbf{P}_{\mathcal{S}_\Psi}\{\nabla J_\tau R_{\mathrm{SNR}}^{-*}\}, \mathbf{P}_{\mathcal{S}_\Psi}\{\nabla J_\tau R_{\mathrm{SNR}}^{-*}\}\rangle^{\mathrm{T}} \\ &\leq \langle \nabla J_\tau \Phi_v \Phi_\chi^{-1}, \nabla J_\tau \rangle^{\mathrm{T}} \end{aligned} \tag{55}$$

where the inequality comes from Theorem III.5. If we consider the impulse response coefficients g_k and h_k corresponding to $\mathcal{G}_k(z) = \mathcal{H}_k(z) = z^{-k}$ in (24) we get for the 2×2 diagonal blocks of (55) that

$$\mathrm{AsCov} \begin{bmatrix} g_k(\hat{\theta}_N) & h_k(\hat{\theta}_N) \end{bmatrix} \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_v(\mathrm{e}^{\mathrm{j}\omega}) \Phi_\chi^{-\mathrm{T}}(\mathrm{e}^{\mathrm{j}\omega}) \mathrm{d}\omega$$

V. Conclusions

The main results in this paper are the formulae (31) and (37) which express the asymptotic covariance as defined by (3). We have shown that these geometric expressions provide insights into how various quantities affect the asymptotic covariance. In particular we demonstrated that one can use the experimental conditions to make the asymptotic variance independent of model order and model structure in some cases.

We have also used these expressions to derive novel model structure independent upper bounds of the asymptotic covariance, in particular for a number of commonly estimated quantities such as system zeros and gains and impulse response coefficients. We have shown that these bounds are significantly less conservative as compared to the variance expressions that result from using the (asymptotic in model order) variance formulae for frequency function estimation in [18].

Our work has its foundation in [26], where the significance of the subspace spanned by the prediction error gradient was acknowledged and we have shown that the results in [26] are recovered from the results in this paper.

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References

[1] Bombois, X., Anderson, B. and Gevers, M. [2005]. Quantification of frequency domain error bounds with guaranteed confidence level in prediction error identification, *Systems and Control Letters* **54**(5).

[2] Campi, M., Ooi, S. and Weyer, E. [2004]. Non-asymptotic quality assessment of generalised FIR models with periodic inputs, *Automatica* **40**(12): 2029–2041.

[3] Campi, M., Ooi, S. and Weyer, E. [2005]. Guaranteed non-asymptotic confidence regions in system identification, *Automatica* **41**(10): 1751–1764.

[4] Douma, S. and Van den Hof, P. [2006]. Probabilistic model uncertainty bounding: An approach with finite-time perspectives, *14th IFAC Symposium on System Identification*, Newcastle, Australia, pp. 1021–1026.

[5] Forssell, U. and Ljung, L. [1999]. Closed-loop identification revisited, *Automatica* **35**(7): 1215–1241.

[6] Garatti, S., Campi, M. and Bittanti, S. [2004]. Assessing the quality of identified models through the asymptotic theory - when is the result reliable?, *Automatica* **40**(8): 1319–1332.

[7] Gevers, M., Bazanella, A. and Miskovic, L. [2008]. Informative experiments, or how to get just sufficiently rich?, *IEEE Transactions on Automatic Control* . Provisionally accepted.

[8] Gevers, M., Ljung, L. and Van den Hof, P. [2001]. Asymptotic variance expressions for closed-loop identification, *Automatica* **37**(5): 781–786.

[9] Hildebrand, R. and Gevers, M. [2004]. Quantification of the variance of estimated transfer functions in the presence of undermodeling, *IEEE Transactions on Automatic Control* **49**(8): 1345–1350.

[10] Hjalmarsson, H. [2005]. From experiment design to closed loop control, *Automatica* **41**(3): 393–438.

[11] Hjalmarsson, H. [2009]. System identification of complex and structured systems, *European Journal of Control / Plenary address European Control Conference* . To appear.

[12] Hjalmarsson, H. and Mårtensson, J. [2007a]. A geometric approach to variance analysis in system identification, *IEEE Transactions on Automatic Control* . Submitted.

[13] Hjalmarsson, H. and Mårtensson, J. [2007b]. A geometric approach to variance analysis in system identification: Theory and nonlinear systems, *Proceedings of the 46th IEEE*

Conference on Decision and Control.

[14] Hjalmarsson, H., Mårtensson, J. and Wahlberg, B. [2006]. On some robustness issues in input design, *Proceedings of the 14th IFAC Symposium on System Identification*.

[15] Hjalmarsson, H. and Ninness, B. [2006]. Least-squares estimation of a class of frequency functions: A finite sample variance expression, *Automatica* **42**(4): 589–600.

[16] Jansson, H. [2004]. *Experiment Design with Applications in Identification for Control*, PhD thesis, S3–Automatic Control, Royal Institute of Technology, Stockholm, Sweden.

[17] Lindqvist, K. [2001]. *On Experiment Design in Identification of Smooth Linear Systems*, Licentiate thesis, Royal Institute of Technology, Stockholm, Sweden.

[18] Ljung, L. [1985]. Asymptotic variance expressions for identified black-box transfer function models, *IEEE Transactions Automatic Control* **30**(9): 834–844.

[19] Ljung, L. [1999]. *System Identification: Theory for the user*, second edn, Prentice Hall.

[20] Luenberger, D. [1969]. *Optimization by vector space methods*, John Wiley & Sons, Inc, New York.

[21] Mårtensson, J. and Hjalmarsson, H. [2007]. A geometric approach to variance analysis in system identification: Linear time-invariant systems, *Proceedings of the 46th IEEE Conference on Decision and Control*.

[22] Mårtensson, J. and Hjalmarsson, H. [2007]. Robustness issues in experiment design for system identification, *IEEE Transactions on Automatic Control* . Provisionally accepted.

[23] Mårtensson, J. and Hjalmarsson, H. [2009]. Variance error quantifications for identified poles and zeros, *Automatica* . To appear.

[24] Ninness, B. and Goodwin, G. [1995]. Estimation of model quality, *Automatica* **31**(12): 32–74.

[25] Ninness, B. and Gustafsson, F. [1997]. A unifying construction of orthonormal bases for system identification, *IEEE Transactions on Automatic Control* **42**(4): 515–521.

[26] Ninness, B. and Hjalmarsson, H. [2004]. Variance error quantifications that are exact for finite model order, *IEEE Transactions on Automatic Control* **49**(8): 1275–1291.

[27] Ninness, B. and Hjalmarsson, H. [2005]. On the frequency domain accuracy of closed-loop estimates, *Automatica* **41**(7): 1109–1122.

[28] Rojas, C., Barenthin, M. and Hjalmarsson, H. [2008]. The cost of complexity in identification of fir systems, *16th IFAC World Congress*, Seoul, South Korea.

[29] Rojas, C., Barenthin, M., Welsh, J. and Hjalmarsson, H. [2008]. The cost of complexity in system identification: The FIR case, *IEEE Transactions on Automatic Control* . Submitted.

[30] Rudin, W. [1976]. *Principles of Mathematical Analysis*, McGraw-Hill, London.

[31] Schoukens, J., Van den Hof, P. and Pintelon, R. [2006]. Reliability of parametric variance estimates for identified transfer functions, *14th IFAC Symposium on System Identification*, Newcastle, Australia.

[32] Stoica, P. and Marzetta, T. L. [2001]. Parameter estimation problems with singular information matrices, *IEEE Transactions on Signal Processing* **49**(1): 87–90.

[33] Wahlberg, B. [1991]. System identification using Laguerre models, *IEEE Trans. on Automatic Control* **36**(5): 551–562.

[34] Wunsch, D. [1993]. *Complex variables with applications*, second edn, Addison Wesley Publishing Company.

[35] Xie, L. and Ljung, L. [2001]. Asymptotic variance expressions for estimated frequency functions, *IEEE Trans. Automatic Control* **46**(12): 1887–1899.

[36] Xie, L. and Ljung, L. [2004]. Variance expressions for spectra estimated using auto-regressions, *Journal of Econometrics* **118**(1-2): 247–256.