Repetition:

MIMO:

signals - time varying vectors

$F(s), G(s)$ - Matrices $\Rightarrow$ $GF \neq FG$
order matters!

State-space: Same as for SISO, but with larger matrices $\Rightarrow$

System poles = eigenvalues of the A matrix.

Singular values:

Square root of eigenvalues of G^*G.

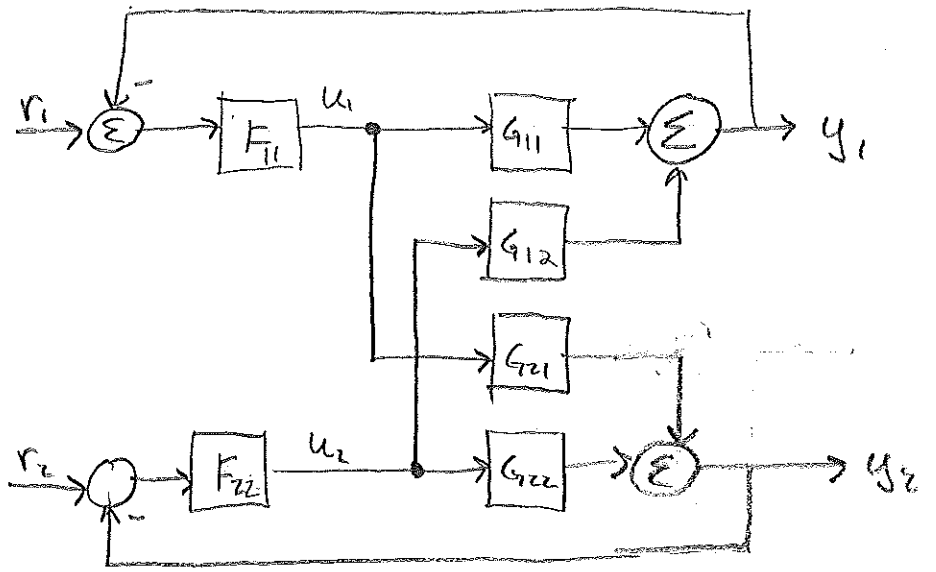
Calculate by $\det(\sigma I - G^*(i\omega)G(i\omega)) = 0$

Theory:

Decentralized control:

For each output, pick an input and use a SISO-controller for that loop.

Ex: $G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$ $u_1 \to y_1$ & $u_2 \to y_2$ $\Rightarrow$ $F = \begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix}$



How to chose the pairings?

RGA-matrix is a useful tool

$$RGA = G \times (G^{-1})^T$$

multiply elementwise (Hadamard product)

Pairing rules:
1) Avoid pairings involving negative elements in $RGA(0)$.
2) Chose pairing with elements close to 1 in $RGA(i\omega_c)$
3) Avoid pairings that limit bandwidth (RHP-zeros & time delays in the $G(s)$ elements)

<u>Pairing</u>: To get a plant with correct pairing, we use a permutation matrix P.

<u>Ex</u>: $RGA(0) = \begin{pmatrix} -0{,}1 & 1{,}1 \\ 1{,}1 & -0{,}1 \end{pmatrix}$

$\Rightarrow \quad u_1 \to y_2 \quad u_2 \to y_1 \quad \Rightarrow$

$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad P \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_1 \end{pmatrix}$

$\Rightarrow$ $\tilde{G} = GP$ is reordered such that

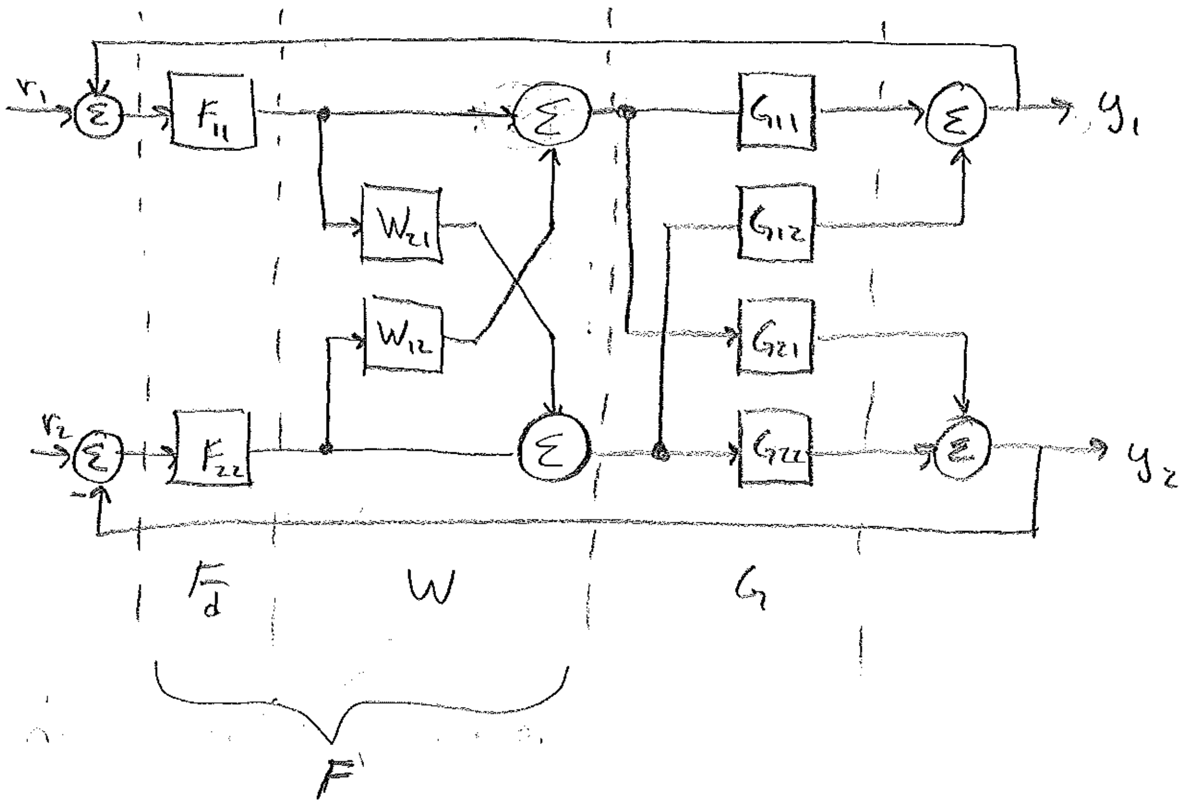
$\tilde{u}_1 \to y_1 \quad \tilde{u}_2 \to y_2$

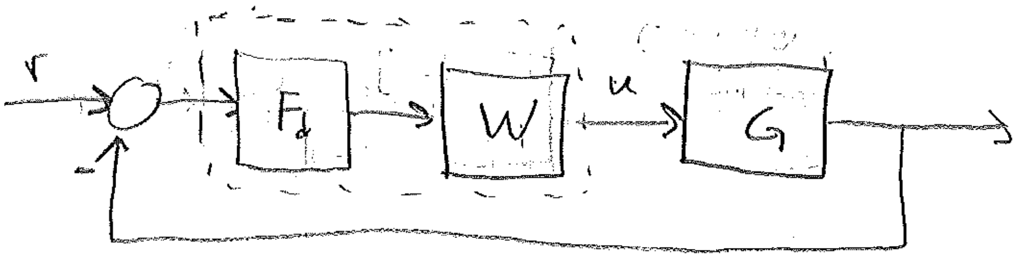
Decoupling: Assume we have ordered our inputs and outputs such that $u_1 \to y_1$, $u_2 \to y_2 \cdots$

If the off diagonal elements in G are non-zero we will have couplings.

Ex: A step in r_1 will generate som u_1 which will affect not only y_1 but also y_2 through G_{21}. This will cause the controller F_2 to generate u_2 to counteract this. This u_2 will affect y_1 through G_{12} etc...

If we think of the interactions as disturbances it seems natural to use feedforward to counteract them.





Note:

1, Design regulator F_d

2, Decouple (Diagonalize GW)

$\Rightarrow$ $W = \begin{bmatrix} 1 & W_{12} \\ W_{21} & 1 \end{bmatrix} \Rightarrow F = W F_d$

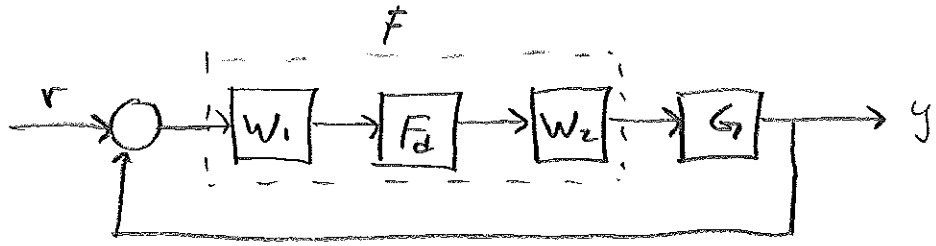
Alternatively we could try

1, Decouple (Diagonalize plant with $\tilde{G} = W_1 G W_2$)

2, Design regulator F_d $\Rightarrow$

$F = W_2 F_d W_1$

It then seems natural to chose (if possible) $W_2 = G^{-1}$, $W_1 = I$ $\Rightarrow$ $\tilde{G} = I$ $\Rightarrow$ $L(s) = F_d(s)$



8.2 Given the system

$$G(s) = \begin{pmatrix} \frac{1}{10s+1} & \frac{-2}{2s+1} \\ \frac{1}{10s+1} & \frac{s-1}{2s+1} \end{pmatrix}$$

a) Use RGA-analysis to determine a pairing for decentralized control.

We are not given any desired bandwidth
$\Rightarrow$ Check RGA(0).

$$RGA(0) = G(0) \times (G(0)^{-1})^T$$

$$G(0) = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \Rightarrow$$

$$G(0)^{-1} = \underbrace{\frac{1}{-1+2}}_{1} \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \Rightarrow$$

$$(G(0)^{-1})^T = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \Rightarrow$$

$$RGA(0) = \begin{bmatrix} 1\cdot(-1) & (-2)\cdot(-1) \\ 1\cdot 2 & (-1)\cdot 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

Note: The row & columnsums are always equal to 1 for RGA-matrices;

It's a good way to check calculations.

We want to avoid negative elements $\Rightarrow$
We use the pairing $u_2 \rightarrow y_1$ & $u_1 \rightarrow y_2$

$\Rightarrow P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

b) Given a controller $F_d = \begin{bmatrix} F_{d11} & 0 \\ 0 & F_{d22} \end{bmatrix}$

Create a controller such that

1) The stationary errors in the idividual loops don't affect each other

2) The controller F is expressed in terms of F_d.

The stationary errors are related by

$$\begin{pmatrix} E_1(s) \\ E_2(s) \end{pmatrix} = \begin{matrix} R_1(s) \\ R_2(s) \end{matrix} - \underbrace{G(s)F(s)}_{L(s)} \begin{pmatrix} E_1(s) \\ E_2(s) \end{pmatrix}$$

$E_1(0)$ & $E_2(0)$ independent $\Rightarrow$ $L(0)$ diagonal

Let $F = PWF_d$, $W = \begin{bmatrix} 1 & W_{12} \\ W_{21} & 1 \end{bmatrix}$ $\Rightarrow$

we had to reorder inputs!

$L(0) = G(0)PWF_d$ ← already diagonal

$$G(0)PW = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & W_{12} \\ W_{21} & 1 \end{bmatrix} = \begin{bmatrix} -2+W_{21} & -2W_{12}+1 \\ -1+W_{21} & -W_{12}+1 \end{bmatrix}$$

$$\Rightarrow W_{12} = 0{,}5 \qquad W_{21} = 1 \Rightarrow W = \begin{bmatrix} 1 & 0{,}5 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow F = PWD = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0{,}5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_{d11} & 0 \\ 0 & F_{d22} \end{bmatrix} = \begin{bmatrix} F_{11d} & F_{d22} \\ F_{11d} & 0{,}5F_{d22} \end{bmatrix}$$

8,5 Given the system $\frac{20}{s+20}\begin{pmatrix} \frac{9}{s+1} & 2 \\ 6 & 4 \end{pmatrix}$

Calculate the RGA matrix for $s = i\omega_c = i20$ and use this information to find a pairing for decentralised control.

$$RGA = G \times (G^{-1})^T$$

$(G^{-1})^T$:

$$G^{-1} = \frac{s+20}{20} \cdot \frac{1}{\frac{36}{s+1} - 12} \begin{pmatrix} 4 & -2 \\ -6 & \frac{9}{s+1} \end{pmatrix}$$

$$\Rightarrow (G^{-1})^T = \frac{s+20}{20} \, \frac{s+1}{12(2-s)} \begin{pmatrix} 4 & -6 \\ -2 & 9/(s+1) \end{pmatrix} \Rightarrow$$

$$RGA = \begin{pmatrix} \frac{3}{2-s} & \frac{-(s+1)}{2-s} \\ \frac{-(s+1)}{2-s} & \frac{3}{2-s} \end{pmatrix}$$

$RGA(i\omega_c)$: $s = i20 \Rightarrow$ Use matlab $\Rightarrow$

$$RGA(i20) = \begin{pmatrix} 0{,}015 + 0{,}15i & 0{,}985 - 0{,}15i \\ 0{,}985 - 0{,}15i & 0{,}015 + 0{,}15i \end{pmatrix}$$

We want to select a pairing with $RGA(i\omega_c)$ elements close to 1 $\Rightarrow$

$u_1 \rightarrow y_2 \quad u_2 \rightarrow y_1$

Note: $RGA(0) = \begin{pmatrix} 1{,}5 & -0{,}5 \\ -0{,}5 & 1{,}5 \end{pmatrix}$ which implies the opposite pairing.

8.17: We are given a State-space model of a tank-system.

$$\dot{x} = \underbrace{\begin{pmatrix} -1{,}5 & 0{,}5 \\ 0{,}5 & -1{,}5 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{B=I} u$$

$$y = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{C=I} x$$

a) Pair inputs & outputs
b) Compute a decoupling
c) Find closed-loop poles

a) Which pairing of the signals is preferable according to RGA(0)?

We start by finding G(s) from the State-space model.

$$G(s) = C[sI - A]^{-1}B = [sI - A]^{-1} = [G^{-1}]^{-1} \Rightarrow$$

$$G^{-1} = sI - A$$

$$G^{-1} = \begin{bmatrix} s+1{,}5 & -0{,}5 \\ -0{,}5 & s+1{,}5 \end{bmatrix} = (G^{-1})^T \quad \text{(symmetric!)}$$

$$\Rightarrow G = \frac{1}{\underbrace{(s+1{,}5)^2 - 0{,}5^2}_{(s+1)(s+2)}} \begin{bmatrix} s+1{,}5 & 0{,}5 \\ 0{,}5 & s+1{,}5 \end{bmatrix}$$

$$RGA = G \times (G^{-1})^T = \frac{1}{(s+1)(s+2)} \begin{bmatrix} (s+1,5)^2 & -0,5^2 \\ -0,5^2 & (s+1,5)^2 \end{bmatrix} \Rightarrow$$

$$RGA(0) = \begin{bmatrix} \frac{2,25}{2} & \frac{-0,25}{2} \\ \frac{-0,25}{2} & \frac{2,25}{2} \end{bmatrix}$$ Rows & Cols sum to 1. OK!

The rule that we should <u>avoid negative elements</u> in RGA(0) $\Rightarrow$ $u_1 \to y_1$ & $u_2 \to y_2$.

b) Compute a decoupling matrix W such that the system $\tilde{G}(s) = G(s)W$ is decoupled at stationarity ($\omega=0$).

We want $\tilde{G}(0) = G(0)W$ diagonal.

(1) let $W_1 = G(0)^{-1} \Rightarrow \tilde{G}(s) = G(s)\,G(0)^{-1} \Rightarrow \tilde{G}(0) = I$

diagonal

(2) let $W_2 = \begin{bmatrix} 1 & W_{12} \\ W_{21} & 1 \end{bmatrix}$

$$\tilde{G}(0) = G(0)W_2 = \frac{1}{2}\begin{bmatrix} 1,5 & 0,5 \\ 0,5 & 1,5 \end{bmatrix}\begin{bmatrix} 1 & W_{12} \\ W_{21} & 1 \end{bmatrix} =$$

$$= \frac{1}{2} \begin{bmatrix} 1,5 + 0,5W_{21} & 1,5W_{12} + 0,5 \\ 0,5 + 1,5W_{21} & 0,5W_{12} + 1,5 \end{bmatrix}$$ diagonal $\Rightarrow$

$$\Rightarrow W_{12} = -1/3 \quad , \quad W_{21} = -1/3 \Rightarrow$$

$$W_2 = \begin{bmatrix} 1 & -1/3 \\ -1/3 & 1 \end{bmatrix}$$

c) Find closed-loop poles with and without decoupling for $F = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} = KI$ (proportional controller with gain K).

The closed-loop is given by

$$\dot{x} = Ax + Bu \quad , \quad u = W \cdot K \cdot I(r-y)$$
$$y = Cx$$

$$\Rightarrow \quad \dot{x} = Ax + \underbrace{B}_{I} WKI(r - \underbrace{C}_{I} x) = (A - KW)x + KWr$$

Poles are given by the roots of

$$\det(sI - A + KW) = 0$$

<u>No decoupling:</u> $W = I \Rightarrow$

$$sI - A + KW = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1{,}5 & 0{,}5 \\ 0{,}5 & -1{,}5 \end{bmatrix} + \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} =$$

$$= \begin{bmatrix} s + 1{,}5 + K & -0{,}5 \\ -0{,}5 & s + 1{,}5 + K \end{bmatrix}$$

$$\det(sI - A + KW) = (s + 1{,}5 + K)^2 - 0{,}5^2 = 0 \Rightarrow$$

$$s = -K - 2 \quad \& \quad s = -K + 1$$

$K = 10 \Rightarrow$ closed loop poles for

$$s \in \{-11, -12\}$$

With decoupling: $W = G(0)^{-1} \Rightarrow$

$$sI - A + kW = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1{,}5 & 0{,}5 \\ 0.5 & -1{,}5 \end{bmatrix} + \begin{bmatrix} 1{,}5K & -0{,}5K \\ -0{,}5K & 1{,}5K \end{bmatrix} =$$

$$= \begin{bmatrix} s + 1{,}5(1+K) & -0{,}5(1+K) \\ -0{,}5(1+K) & s + 1{,}5(1+K) \end{bmatrix} \Rightarrow$$

$$\det(sI - A + kW) = \left(s + 1{,}5(1+K)\right)^2 - \left(0{,}5(1+K)\right)^2 = 0$$

$$\Rightarrow \quad s = -1{,}5(1+K) \pm 0{,}5(1+K)$$

$K = 10 \Rightarrow$ closed loop poles for

$$s = -16{,}5 \pm 5{,}5 \Rightarrow s \in \{-11, -22\}$$

With decoupling: $W = \begin{pmatrix} 1 & -1/3 \\ -1/3 & 1 \end{pmatrix}$

$$sI - A + kW = \begin{bmatrix} s + 1{,}5 + k & -0{,}5 - \frac{K}{3} \\ -0{,}5 - \frac{K}{3} & s + 1{,}5 + K \end{bmatrix}$$

$$\det(sI - A + kW) = (s + 1{,}5 + K)^2 - \left(0{,}5 + \frac{K}{3}\right)^2 = 0$$

$$\Rightarrow s = 1{,}5 + K \pm \left(0{,}5 + \frac{K}{3}\right) = 11{,}5 \pm \frac{11{,}5}{3}$$