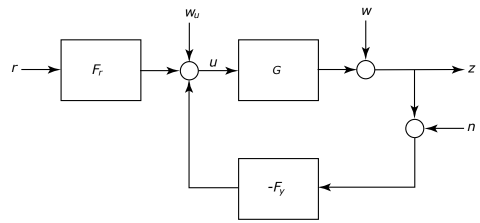
EL2520
Control Theory and Practice

Lecture 5: Multivariable systems

Mikael Johansson
School of Electrical Engineering
KTH, Stockholm, Sweden

So far…



SISO control revisited:

- Signal norms, system gains and the small gain theorem
- The closed-loop system and the design problem
  - Characterized by six transfer functions: need to look at all!
  - Fundamental limitations and waterbed effect.
- Loop shaping to satisfy sensitivity function specifications.

From now and on: MIMO

Linear systems with multiple inputs and multiple outputs

- Basic properties of multivariable systems
- State-space theory, state feedback and observers
- H_2 and H_1 optimal control
- Decentralized and decoupled control
- Robust loop shaping

The final part of the course considers systems with constraints

Today's lecture

Basic properties of multivariable systems

- Transfer matrices
- Block diagram calculations
- Gains and directions
- The multivariable frequency response
- Poles and zeros

Chapters 2-3 in the textbook.

Transfer matrices

The **Laplace transform** X(s) of a signal x(t) is defined by

$$X(s) = \int_{t=0}^{\infty} x(t)e^{-st}\,dt$$

Given a linear system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

and assuming u(t)=0 for t<0 and x(0)=0,

$$Y(s) = \{C(sI - A)^{-1}B + D\}U(s) = G(s)U(s)$$

If system has multiple inputs and outputs, Y and U are vector-valued and **G(s) is a matrix** (i.e. a matrix-valued function of s).

Quiz: transfer matrices

1. What is the transfer matrix G(s) for the system

$$\dot{x}_1 = -x_1 + u_1$$
$$\dot{x}_2 = -x_2 + u_2$$
$$y_1 = x_1$$
$$y_2 = x_2$$

How does G(s) change when

2. Input two also affects the first state: $\dot{x}_1 = -x_1 + u_1 + u_2$

3. The second state also affects output one: $y_1 = x_1 + x_2$

4. The second state influences the first: $\dot{x}_1 = -x_1 - x_2 + u_1$

Answer and observations

1. $$G(s) = \begin{pmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+1} \end{pmatrix}$$

Independent subsystems → (block)diagonal transfer matrix

2., 3. $$G(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ 0 & \frac{1}{s+1} \end{pmatrix}$$

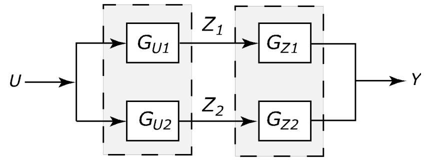
Couplings → nondiagonal G(s). Different A, B, C can give same G(s)

4. $$G(s) = \begin{pmatrix} \frac{1}{s+1} & -\frac{1}{(s+1)^2} \\ 0 & \frac{1}{s+1} \end{pmatrix}$$

1.-4. have the same poles (eigenvalues of A). Hard to see from G(s)

Example: series connection

Linear system viewed as interconnected multivariable systems

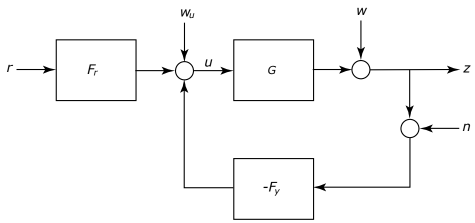


$$\begin{pmatrix} Z_1(s) \\ Z_2(s) \end{pmatrix} = \begin{pmatrix} G_{U1}(s) \\ G_{U2}(s) \end{pmatrix} U(s), \qquad Y(s) = \begin{pmatrix} G_{Z1}(s) & G_{Z2}(s) \end{pmatrix}$$

We see that $Y(s)=G_Z(s)Z(s)= G_Z(s)G_U(s)U(s)$.

Note that $G_Z G_U \leq G_U G_Z$ – **order matters**!

Block diagram calculations



Have to be careful when manipulating block diagrams.

Example. Let w_u,w,n=0 and derive transfer matrix from r to z

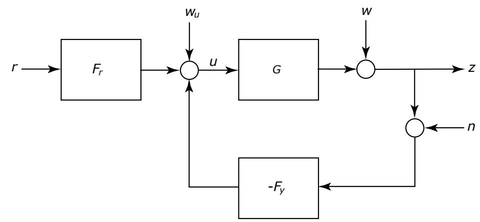
$$z = Gu = G(F_r r - F_y z) \Rightarrow z + GF_y z = GF_r r$$

so

$$z = (I + GF_y)^{-1} GF_r r \qquad (\neq (I + F_y G)^{-1} GF_r!)$$

Quiz: the closed-loop MIMO system

Determine the sensitivity and complementary sensitivity for the linear multivariable system



Recall: S is transfer matrix from w→z, T is transfer matrix from -n→z

Additional question: What is the relation between S and T?

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Frequency response and system gain

For a scalar linear system G(s) driven by u(t)=sin(ωt),

$$y(t) = |G(i\omega)| \sin(\omega t + \arg G(i\omega))$$

(after transients have died out). So

$$\frac{|Y(i\omega)|_2}{|U(i\omega)|_2} = |G(i\omega)|$$

The system gain (cf. Lecture 1) is defined as

$$\sup_u \frac{\|y\|_2}{\|u\|_2} = \sup_\omega |G(i\omega)| = \|G\|_\infty$$

Attained for sinusoidal input with frequency ω such that $|G(i\omega)|=||G||_1$

Q: What are the corresponding results for multivariable systems?

Operator norm of linear mapping

Consider the linear mapping $y = Ax$ (x, y, A complex-valued)

Since

$$|y|^2 = |Ax|^2 = (Ax)^* Ax = x^* A^* Ax$$

we have

$$|x|^2 \lambda_{\min}(A^* A) \leq |y|^2 \leq |x|^2 \lambda_{\max}(A^* A)$$

So

$$\underline{\sigma}(A) \leq \frac{|y|}{|x|} \leq \overline{\sigma}(A)$$

Where $\sigma(A) = \sqrt{\lambda(A^* A)}$ are called the **singular values** of A.

The singular value decomposition

A matrix A 2 Cm£ r (with r<=m, rank(A)=r), can be represented by its **singular value decomposition (SVD)**

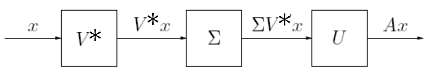
$$A = U\Sigma V^* = \begin{bmatrix} u_1 & \cdots u_r \end{bmatrix} \text{diag}(\sigma_i) \begin{bmatrix} v_1 & \cdots & v_r \end{bmatrix}^* = \sum_{i=1}^{r} \sigma_i u_i v_i^*$$

where

- The positive scalars σ_i are the **singular values** of A
- v_i are the **input singular vectors** of A, $V^*V=I$
- u_i are the **output singular vectors** of A, $U^*U=I$

SVD interpretation

$$A = U\Sigma V^* = \sum_{i=1}^{r} \sigma_i u_i v_i^*$$



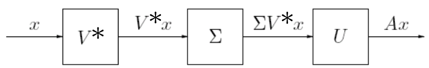
Interpretation: linear mapping y=Ax can be decomposed as

- compute coefficients of x along input directions v_i
- scale coefficients by σ_i
- reconstitute along output directions u_i

Since $\sigma_1 \leq \cdots \leq \sigma_r$ an **input in the v_r direction is amplified the most**. It **generates an output in the direction of u_r** (typically different from v_r).

Example

$$G(0) = \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 0.85 & 0.53 \\ -0.53 & 0.85 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 1.62 & 0 \\ 0 & 0.62 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.53 & 0.85 \\ -0.85 & 0.53 \end{bmatrix}^*}_{V}$$



$$\begin{bmatrix} 0.53 \\ -0.85 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1.62 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1.38 \\ -0.85 \end{bmatrix}$$

$x = v_1$

The multivariable frequency response

For a linear multivariable system Y(s)=G(s)U(s), we have

$$Y(i\omega) = G(i\omega)U(i\omega)$$

Since this is a linear mapping,

$$\underline{\sigma}(G(i\omega)) \leq \frac{|Y(i\omega)|}{|U(i\omega)|} \leq \overline{\sigma}(G(i\omega))$$

with equality if U(iω) parallell w. corresponding input singular vector.

For example,

$$\frac{|Y(i\omega)|}{|U(i\omega)|} = \overline{\sigma}(G(i\omega))$$

only if U(iω) parallell with input singular vector corresponding to $\overline{\sigma}$

The system gain

As for scalar systems, we can use Parseval's theorem to find

$$\|y\|_2 \leq \|G\|_\infty \|u\|_2$$

where

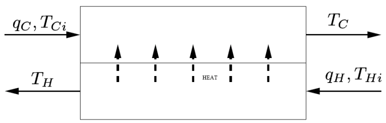
$$\|G\|_\infty = \sup_\omega |G(i\omega)| = \sup_\omega \overline{\sigma}(G(i\omega))$$

Note: Worst-case input is sinusoidal at the frequency that attains the supremum, but its components are appropriately scaled and phase shifted (as specified by the input singular vector of $\overline{\sigma}$)

Note: the infinity norm computes the maximum amplifications across frequency ($\sup_\omega$) and input directions ($\overline{\sigma}$)

Example: heat exchanger

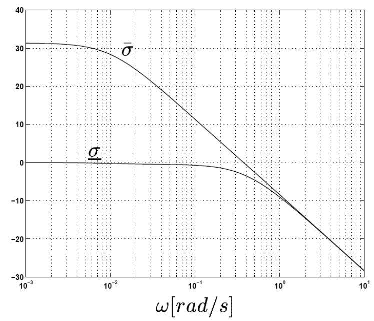
Objective: control outlet temperatures T_c, T_H by manipulating the flows q_C, q_H



Model:

$$\begin{pmatrix} T_C \\ T_H \end{pmatrix} = \frac{1}{(100s+1)(2.5s+1)} \begin{pmatrix} -19(5s+1) & 18 \\ -18 & 19(5s+1) \end{pmatrix} \begin{pmatrix} q_C \\ q_H \end{pmatrix}$$

Singular values of heat exchanger



System gain: $||G||_1$=31 dB (=37)

Heat exchanger steady-state

$$G(0) = \begin{bmatrix} -19 & 18 \\ -18 & 19 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 37 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

Singular value decomposition reveals

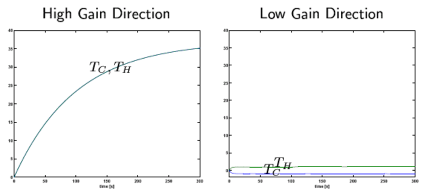
- Maximum effect input (input singular vector corresponding to $\bar{\sigma}$)

$$\begin{bmatrix} q_C \\ q_H \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} T_C \\ T_H \end{bmatrix} = \begin{bmatrix} 37 \\ 37 \end{bmatrix}$$

- Minimum effect input (corresponding to $\underline{\sigma}$)

$$\begin{bmatrix} q_C \\ q_H \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} T_C \\ T_H \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Heat exchanger step responses

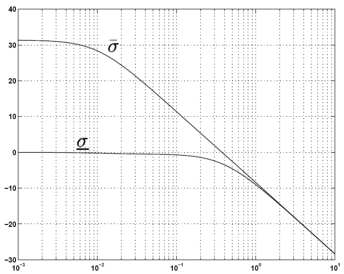


Input direction has dramatic effect! (agrees with physical intuition)

Note: large difference in time-scales!

Singular values and bandwidths

What is the bandwidth of the system?



No single value, but a range. Depends on input directions.

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- Transfer matrices
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- Poles and zeros

Poles

Definition. The **poles** of a linear systems are the eigenvalues of the system matrix in a minimal state-space realization.

Definition. The **pole polynomial** is the characteristic polynomial of the A matrix, $\lambda(s)$= det(sI-A).

Alternatively, the poles of a linear system are the zeros of the pole polynomial, i.e., the values p_i such that $\lambda(p_i) = 0$

Poles cont'd

Since the transfer matrix is given by

$$G(s) = C(sI - A)^{-1}B + D = \frac{1}{\det(sI - A)} r(s)$$

where r(s) is a polynomial in s (see book for precise expression), the pole polynomial must be "at least" the least common denominator of the the elements of the transfer matrix.

Example: The system

$$G(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} 2(s+2) & 3(s+1) \\ (s+2) & (s+2) \end{bmatrix}$$

must (at least) have poles in s=-1 and s=-2.

Poles of multivariable systems

Theorem. The pole polynomial of a system with transfer matrix G(s) is the common denominator of all minors of G(s)

Recall: a minor of a matrix M is the determinant of a (smaller) square matrix obtained by deleting some rows and columns of M

Example: The minors of

$$G(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

are $\frac{2}{s+1}$, $\frac{3}{s+2}$, $\frac{1}{s+1}$ and $\det G(s) = \frac{1-s}{(s+1)^2(s+2)}$

Thus, the system has poles in s=-1 (a double pole) and s=-2.

Zeros

Theorem. The **zero polynomial** of G(s) is the greatest common divisor of the maximal minors of G(s), normed so that they have the pole polynomial of G(s) as denominator. The **zeros** of G(s) are the roots of its zero polynomial.

Example: The maximal minor of

$$G(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

is $\det G(s) = \frac{1-s}{(s+1)^2(s+2)}$ (already normed!).

Thus, G(s) has a zero at s=1.

Quiz: multivariable poles and zeros

What are the poles and zeros of the multivariable system

$$G(s) = \frac{1}{(s+1)} \begin{pmatrix} 1 & s+1 \\ s-1 & 1 \end{pmatrix}$$

Notes on poles and zeros

For scalar system G(s) with poles p_i and zeros z_i,

$$G(z_i) = 0, \quad G(p_i) = \infty$$

For a multivariable system, directions matter!

For a system with pole p, there exist vectors u_p, v_p:

$$u_p^* G(p) = \infty \qquad G(p)v_p = \infty$$

Similarly, a zero at z_i implies the existence of vectors u_z, v_z:

$$u_z^* G(z) = 0 \qquad G(z)v_z = 0$$

As for scalar systems, a zero at s=z implies that there exists a signal on the form $u(t)=v_z e^{-zt}$ for t¸ 0, and u(t)=0 for t<0, and initial values $x(0)=x_z$ so that y(t)=0 for t¸ 0

Summary

An introduction to multivariable linear systems:

- Block diagram manipulations (order matters!)
- System gain (directions matter!)
- Poles and zeros

More next week!