EL2520
Control Theory and Practice

Lecture 3: Robustness

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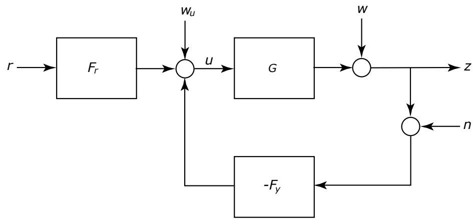
So far…



- Signal norms, system gains and the small gain theorem
- The closed-loop system and the design problem
  - Characterized by six transfer functions: need to look at all!
  - Internal stability: stability from all inputs to all outputs (sufficient to check that F_r, S, SG and SF_y are all stable)
  - Sensitivity function (suppression of load disturbances) and Complementary sensitivity (noise, robust stability)

Goals

After this lecture, you should

- Understand the concepts of robust stability and robust performance

- Be able to derive multiplicative uncertainty models
  - from parametric uncertainties (e.g. of process pole/zero locations)
  - from frequency responses of multiple plants

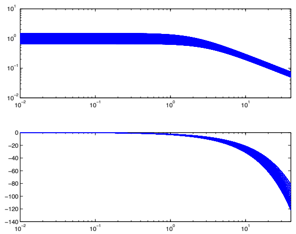
- Analyze robust stability using the small-gain theorem
  - "pull out" uncertainty and re-write system on standard form
  - assess robust stability in Bode and Nyquist diagrams

Motivating example

Assume that you want to control a system on the form

$$G_p(s) = \frac{k}{1+s\tau}e^{-s\theta}$$

but the values of $k,\ \tau,\ \theta$ are unknown. You only know that $k,\ \tau,\ \theta \in [2,3]$



How can we design a controller that is guaranteed to work for all G_p?

Robustness

Robustness=Insensitivity to model errors
(differences between modelled and actual system behavior)

To reason about uncertainty we need to model it!

- The *uncertainty set:* defines a family of possible models (quantifies how much we do not know about the system)

Would like to establish

- *Robust stability* (stability of all plants in uncertainty set)
- *Robust performance* (meet specs for all plants in uncertainty set)

Classes of uncertainty

Parametric uncertainty:

- Model structure known, but some parameters are uncertain

Dynamic uncertainty:

- Some (often high frequency) dynamics is missing, either by lack of understanding or in order to get a simpler model

Often, we have a combination of the two.

- Convenient to represent in “lumped” form

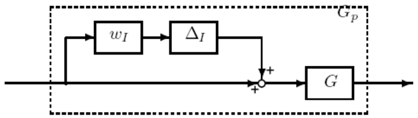
Multiplicative uncertainty

Multiplicative uncertainty

$$\Pi_I = \{G_p(s) = G(s)(1 + W_I(s)\Delta_I(s)) \mid \|\Delta_I\|_\infty \leq 1\}$$

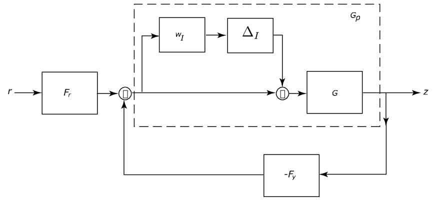
Here,

- Π_I is a *family* of possible behaviours of the physical plant
- Δ is *any* stable transfer function with gain less than one



Robust stability: closed-loop stability for all $G_p \in \Pi_I$

Robust stability w. multiplicative uncertainty



Small-gain theorem→ interconnection stable if
(a) nominal closed-loop system is internally stable and W_I stable, and
(b) $\|W_I T\|_\infty \leq 1$

To ensure robust stability:

- first write uncertain system on standard form (find G(s), W_I(s))
- make sure that (a) and (b) are satisfied

Example: uncertain gain

Consider the set of possible plants

$$G_p(s) = kG_0(s), \quad k_{\min} \le k \le k_{\max}$$

Any feasible k can be written as $k = \bar{k} + r_k\Delta$ for some $|\Delta| \le 1$ and

$$\bar{k} = \frac{k_{\min} + k_{\max}}{2}, \quad r_k = \frac{k_{\max} - k_{\min}}{2}$$

Hence, we can re-write the uncertainty in standard form

$$\Pi_I = \left\{ G_p(s) = \underbrace{\bar{k}G_0(s)}_{G(s)}\Big(1 + \underbrace{\frac{r_k}{\bar{k}}}_{W_I(s)}\Delta\Big) \mid |\Delta| \le 1 \right\}$$

Note: here it is enough to let Δ be real (in standard form Δ is complex)

Example: uncertain zero location

Consider the set of possible plants

$$G_p(s) = (1 + s\tau)G_0(s), \quad \tau_{\min} \le \tau \le \tau_{\max}$$

Can be put into standard form via

$$\begin{aligned}
\bar{\tau} &= (\tau_{\min} + \tau_{\max})/2 \\
r_\tau &= (\tau_{\max} - \tau_{\min})/2 \\
G(s) &= (1 + s\bar{\tau})G_0(s) \\
W_I(s) &= \frac{r_\tau s}{1 + \bar{\tau}s}
\end{aligned}$$

Note: W_I is now frequency dependent, Δ is still real

Alternative approach to obtain weight

Note that multiplicative uncertainty class

$$\Pi_I = \{G_p(s) = G(s)(1 + W_I(s)\Delta_I(s)) \mid \|\Delta_I\|_\infty \le 1\}$$

can be re-written as

$$\Pi_I = \{G_p(s) \mid \|W_I(s)^{-1}G(s)^{-1}(G_p(s) - G(s))\|_\infty \le 1\}$$

Thus, the uncertainty about the system captured by W_I if

$$|W_I(i\omega)| \ge \left| \frac{G_p(i\omega) - G(i\omega)}{G(i\omega)} \right| \qquad \forall G_p \in \Pi_I, \ \forall \omega$$

Note: RHS can be interpreted as relative error of nominal model G.

Example

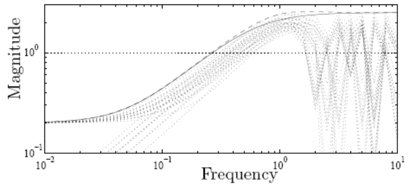
Consider the uncertain system

$$G_p(s) = \frac{k}{\tau s + 1}e^{-\theta s}, \quad k, \theta, \tau \in [2, 3]$$

with nominal plant

$$G(s) = \frac{\bar{k}}{\bar{\tau}s + 1}$$

Sample uncertainties (dotted) and corresponding w_I (dashed)



Example: robust stability

Consider the following nominal plant and controller

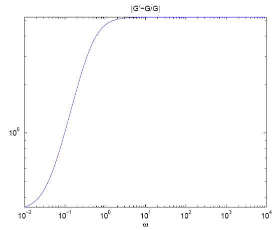
$$G(s) = \frac{3(1-2s)}{(5s+1)(10s+1)}, \quad K(s) = K_c\frac{12.7s+1}{12.7s}$$

and assume that one “extreme” possible plant is

$$G'(s) = \frac{4(1-3s)}{(4s+1)^2}$$

Example: robust stability

Relative error



Is around 0.33 for low frequencies and 5.25 at high frequencies.

Suggests weight

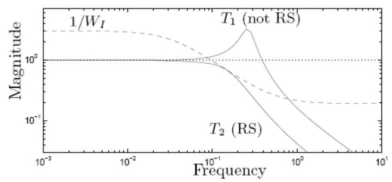
$$W_I(s) = \frac{10s+0.33}{(10/5.25)s+1}$$

Example: robust stability

Robust stability condition $\|W_I T\|_\infty \leq 1$ holds if $|T(i\omega)| \leq |W_I^{-1}(i\omega)| \quad \forall\omega$

Hence, we can validate robust stability in the bode diagram of T.

For two controller settings, we obtain two complementary sensitivities

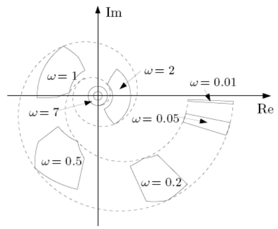


First setting (T1) is not robustly stable, second setting (T2) is.

Robust stability in the Nyquist curve

Uncertain system:

- G(iω) takes one of several possible values at each frequency → a family of Nyquist curves



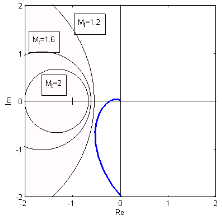
- Robust stability if uncertainty regions do not encircle -1 point

Complementary sensitivity in Nyquist

Constraint on complementary sensitivity

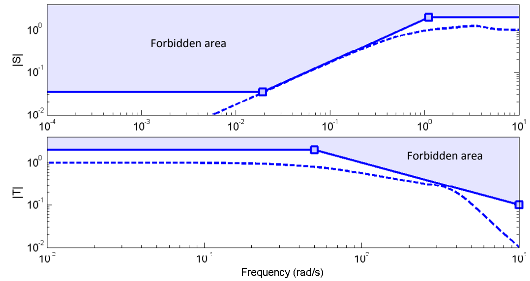
$$\|T(i\omega)\|_\infty \leq M_t$$

also yields circles that should be avoided by the Nyquist curve.



Circles centered at $(-M_t^2/(M_t^2-1),\ 0)$ with radius $M_t/(M_t^2-1)$

Frequency domain specifications



$$|S(i\omega)| \leq |W_S^{-1}(i\omega)|$$
$$|T(i\omega)| \leq |W_T^{-1}(i\omega)|$$

Can we choose weights w_S, w_T (“forbidden areas”) freely?

– No, there are many constraints and limitations!

Extension: shaping the gang of six

Can shape all relevant transfer functions (in “the gang of six”)

$$\|W_S(i\omega)S(i\omega)\|_\infty \leq 1$$
$$\|W_T(i\omega)T(i\omega)\|_\infty \leq 1$$
$$\vdots$$
$$\|W_{SF_r}(i\omega)S(i\omega)F_r(i\omega)\|_\infty 1$$

This is the topic of Computer Exercise 1b!

Robust performance

Nominal performance specified in terms of sensitivity function

$$|W_P S| \leq 1 \quad \forall\omega$$

Robust performance

$$|W_P S_p| \leq 1 \quad \text{for all } \omega \text{ and all } S_p$$

Since

$$W_P S_p = W_P \frac{1}{1+L_p} = \frac{W_P}{1+L+W_I \Delta L}$$

Worst-case Δ is such that 1+L and $w_I\Delta$ L point in opposite directions

$$|W_P S_p| \leq \frac{|W_P|}{|1+L|-|W_I L|} = \frac{|W_P S|}{1-|W_I T|} \quad \forall\omega$$

Robust performance cont'd

Robust performance

$$|W_P S_p| = \frac{|W_P S|}{1 - |W_I T|} \leq 1$$

Can be expressed as

$$|W_P S| + |W_I T| \leq 1 \quad \forall\omega$$

Sometimes approximated by the *mixed* sensitivity constraint

$$\left\| \begin{pmatrix} W_P S \\ W_I T \end{pmatrix} \right\|_\infty \leq 1$$

Robust stability and performance

In summary

nominal performance	$\lvert W_P S\rvert \leq 1 \quad \forall\omega$
robust stability	$\lvert W_I T\rvert \leq 1 \quad \forall\omega$
robust performance	$\lvert W_P S\rvert + \lvert W_I T\rvert \leq 1 \quad \forall\omega$

Note that nominal performance and robust stability implies

$$|W_P S| + |W_I T| \leq 2 \quad \forall\omega$$

(i.e. robust stability cannot be "too bad").

Only holds in SISO case.

Summary

Robustness
- Insensitivity to model errors

Can guarantee robustness if we model (or bound) uncertainty
- General tool: small gain theorem
- Sometimes need to "pull out" uncertainty by hand
- Sometimes, can fall back onto standard forms
  (e.g. multiplicative input uncertainty)

Robustness typically introduces new constraints on T

Robust performance: acceptable S, despite uncertainties.