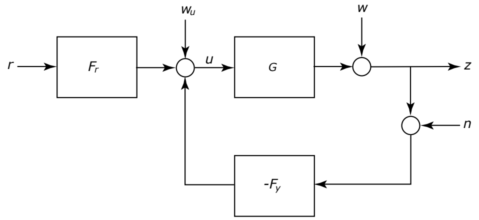
EL2520
Control Theory and Practice

Lecture 3: Robustness

Mikael Johansson
School of Electrical Engineering
KTH, Stockholm, Sweden

So far…



- Signal norms, system gains and the small gain theorem
- The closed-loop system and the design problem
  - Characterized by six transfer functions: need to look at all!
  - Internal stability: stability from all inputs to all outputs (sufficient to check that F_r, S, SG and SF_y are all stable)
  - Sensitivity function (suppression of load disturbances) and Complementary sensitivity (noise, robust stability)

Goals

After this lecture, you should

- Understand the concepts of robust stability and robust performance
- Be able to derive multiplicative uncertainty models
  - from parametric uncertainties (e.g. of process pole/zero locations)
  - from frequency responses of multiple plants
- Analyze robust stability using the small-gain theorem
  - "pull out" uncertainty and re-write system on standard form
  - assess robust stability in Bode and Nyquist diagrams

Robustness

Robustness=Insensitivity to model errors
(differences between modelled and actual system behavior)

To reason about uncertainty we need to model it!

- The *uncertainty set:* defines a family of possible models (quantifies how much we do not know about the system)

Would like to establish

- *Robust stability* (stability of all plants in uncertainty set)
- *Robust performance* (meet specs for all plants in uncertainty set)

Classes of uncertainty

Parametric uncertainty:

- Model structure known, but some parameters are uncertain

Dynamic uncertainty:

- Some (often high frequency) dynamics is missing, either by lack of understanding or in order to get a simpler model

Often, we have a combination of the two.

- Convenient to represent in “lumped” form

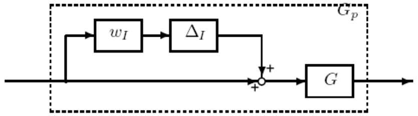
Multiplicative uncertainty

Multiplicative uncertainty

$$\Pi_I = \{G_p(s) = G(s)(1 + W_I(s)\Delta_I(s)) \mid \|\Delta_I\|_\infty \leq 1\}$$

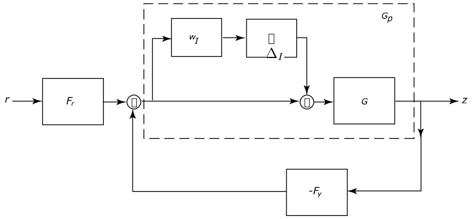
Here,

- Π_I is a *family* of possible behaviours of the physical plant
- Δ is *any* stable transfer function with gain less than one



Robust stability: closed-loop stability for all $G_p \in \Pi_I$

Robust stability w. multiplicative uncertainty



Small-gain theorem→ interconnection stable if
(a) nominal closed-loop system is internally stable, and
(b) $w_{I,}$ Δ are both stable, and
(c) $\|W_I T\|_\infty \leq 1$

Satisfied if $|T(i\omega)| \leq |W_I^{-1}(i\omega)| \quad \forall\omega$

Example: uncertain gain

Consider the set of possible plants

$$G_p(s) = k_p G_0(s), \quad k_{\text{min}} \leq k \leq k_{\text{max}}$$

Can re-write as

$$G_p(s) = \bar{k} G_0(s)(1 + r_k \Delta), \quad |\Delta| \leq 1$$

where

$$\bar{k} = \frac{k_{\text{min}} + k_{\text{max}}}{2}, \quad r_k = \frac{(k_{\text{max}} - k_{\text{min}})/2}{\bar{k}}$$

Note: here it is enough to let Δ be real (in standard form Δ is complex)

Example: uncertain zero location

Consider the set of possible plants

$$G_p(s) = (1 + s\tau)G_0(s), \quad \tau_{\text{min}} \leq \tau \leq \tau_{\text{max}}$$

Can be put into standard form via

$$\begin{aligned} \overline{\tau} &= (\tau_{\text{min}} + \tau_{\text{max}})/2 \\ r_\tau &= (\tau_{\text{max}} - \tau_{\text{min}})/(2\overline{\tau}) \\ G(s) &= (1 + \overline{\tau} s)G_0(s) \\ W_I(s) &= r_\tau \frac{\overline{\tau} s}{1 + \overline{\tau} s} \end{aligned}$$

Note: w_I is now frequency dependent, Δ still real

Alternative approach to obtain weight

Note that multiplicative uncertainty class

$$\Pi_I = \{G_p(s) = G(s)(1 + W_I(s)\Delta_I(s)) \mid \|\Delta_I\|_\infty \leq 1\}$$

can be re-written as

$$\Pi_I = \left\{G_p(s) \mid \|W_I(s)^{-1}G(s)^{-1}(G_p(s) - G(s))\|_\infty \leq 1\right\}$$

Thus, the uncertainty about the system captured by W_I if

$$|W_I(i\omega)| \geq \left|\frac{G_p(i\omega) - G(i\omega)}{G(i\omega)}\right| \qquad \forall\, G_p \in \Pi_I,\ \forall\, \omega$$

Note: RHS can be interpreted as relative error of nominal model G.

Example

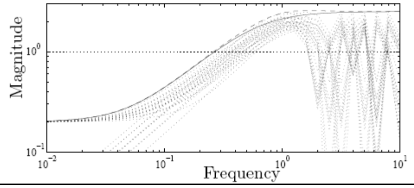
Consider the uncertain system

$$G_p(s) = \frac{k}{\tau s + 1} e^{-\theta s}, \quad k, \theta, \tau \in [2, 3]$$

with nominal plant

$$G(s) = \frac{\overline{k}}{\overline{\tau} s + 1}$$

Sample uncertainties (dotted) and corresponding w_I (dashed)



Example: robust stability

Consider the following nominal plant and controller

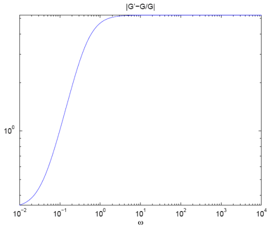
$$G(s) = \frac{3(1 - 2s)}{(5s + 1)(10s + 1)}, \quad K(s) = K_c \frac{12.7s + 1}{12.7s}$$

and assume that one “extreme” possible plant is

$$G'(s) = \frac{4(1 - 3s)}{(4s + 1)^2}$$

Example: robust stability

Relative error



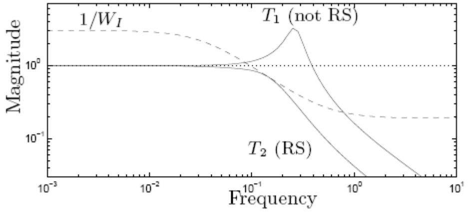
Is around 0.33 for low frequencies and 5.25 at high frequencies.

Suggests weight

$$W_I(s) = \frac{10s + 0.33}{(10/5.25)s + 1}$$

Example: robust stability

Uncertainty weight w_I and complementary sensitivities for two sets of controller parameters

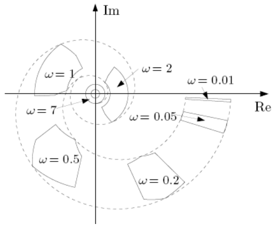


First setting (T1) is not robustly stable, second setting (T2) is.

Robust stability in the Nyquist curve

Uncertain system:

- G(iω) takes one of several possible values at each frequency → a family of Nyquist curves



- Robust stability if uncertainty regions do not encircle -1 point

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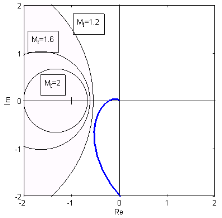
Complementary sensitivity in Nyquist

Constraint on complementary sensitivity

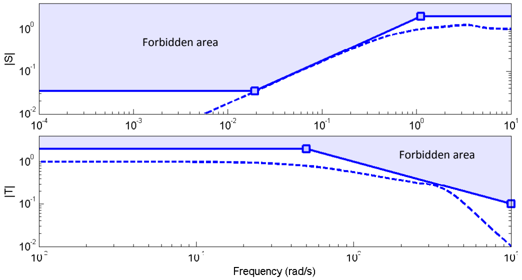
$$\|T(i\omega)\|_\infty \leq M_t$$

also yields circles that should be avoided by the Nyquist curve.



Circles centered at $(-M_t^2/(M_t^2-1),\ 0)$ with radius $M_t/(M_t^2-1)$

Frequency domain specifications



$$|S(i\omega)| \leq |W_S^{-1}(i\omega)|$$
$$|T(i\omega)| \leq |W_T^{-1}(i\omega)|$$

Can we choose weights w_S, w_T ("forbidden areas") freely?

- No, there are many constraints and limitations!

Extension: shaping the gang of six

Can shape all relevant transfer functions (in "the gang of six")

$$\|W_S(i\omega)S(i\omega)\|_\infty \leq 1$$
$$\|W_T(i\omega)T(i\omega)\|_\infty \leq 1$$
$$\vdots$$
$$\|W_{SF_r}(i\omega)S(i\omega)F_r(i\omega)\|_\infty 1$$

This is the topic of Computer Exercise 1b!

Robust performance

Nominal performance specified in terms of sensitivity function

$$|W_P S| \leq 1 \quad \forall\omega$$

Robust performance

$$|W_P S_p| \leq 1 \quad \text{for all } \omega \text{ and all } S_p$$

Since

$$W_P S_p = W_P \frac{1}{1+L_p} = \frac{W_P}{1+L+W_I \Delta L}$$

Worst-case Δ is such that 1+L and $w_I\Delta$ L point in opposite directions

$$|W_P S_p| \leq \frac{|W_P|}{|1+L|-|W_I L|} = \frac{|W_P S|}{1-|W_I T|} \quad \forall\omega$$

Robust performance cont'd

Robust performance

$$|W_P S_p| = \frac{|W_P S|}{1-|W_I T|} \leq 1$$

Can be expressed as

$$|W_P S| + |W_I T| \leq 1 \quad \forall\omega$$

Sometimes approximated by the *mixed* sensitivity constraint

$$\left\| \begin{pmatrix} W_P S \\ W_I T \end{pmatrix} \right\|_\infty \leq 1$$

Robust stability and performance

In summary

nominal performance	$\lvert W_P S\rvert \leq 1 \quad \forall\omega$
robust stability	$\lvert W_I T\rvert \leq 1 \quad \forall\omega$
robust performance	$\lvert W_P S\rvert + \lvert W_I T\rvert \leq 1 \quad \forall\omega$

Note that nominal performance and robust stability implies

$$|W_P S| + |W_I T| \leq 2 \quad \forall\omega$$

(i.e. robust stability cannot be “too bad”).

Only holds in SISO case.

Summary

Robustness
- Insensitivity to model errors

Can guarantee robustness if we model (or bound) uncertainty
- General tool: small gain theorem
- Sometimes need to ”pull out” uncertainty by hand
- Sometimes, can fall back onto standard forms (e.g. multiplicative input uncertainty)

Robustness typically introduces new constraints on T

Robust performance: acceptable S, despite uncertainties.