# Principles of Wireless Sensor Networks

**https://www.kth.se/social/course/EL2745/**

Lecture 4
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**Carlo Fischione**
**Associate Professor of Sensor Networks**
e-mail: carlofi@kth.se
http://www.ee.kth.se/~carlofi/

*KTH Royal Institute of Technology*
*Stockholm, Sweden*

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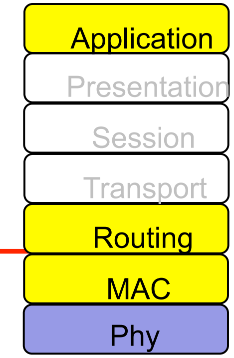
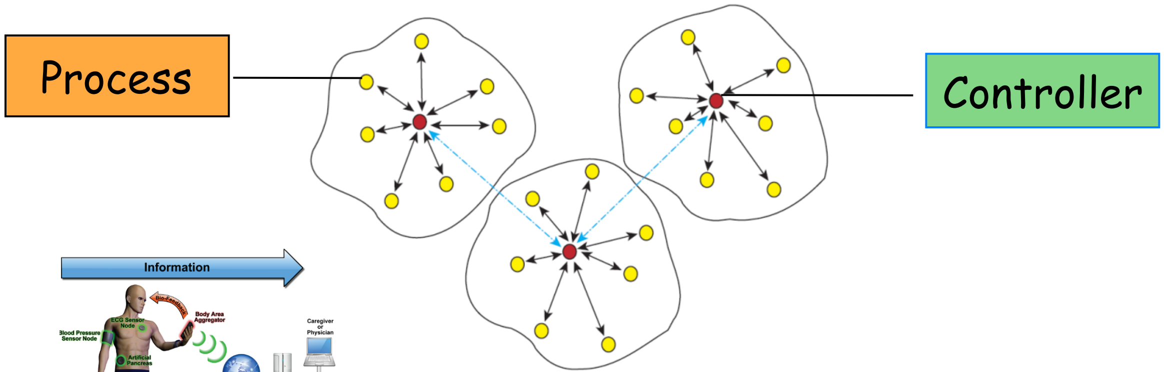
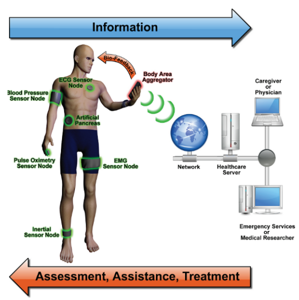
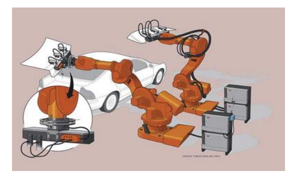
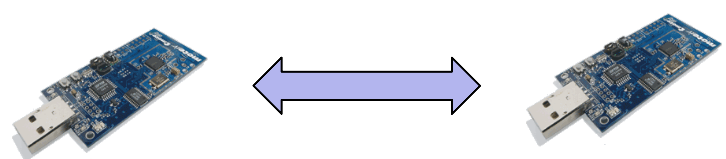
# Where we are





- How messages are successfully transmitted and received over the wireless channel?
- Aim: modeling the probability to successfully receive messages as function of the radio power, modulations, coding, and channel attenuations normally experienced in WSNs

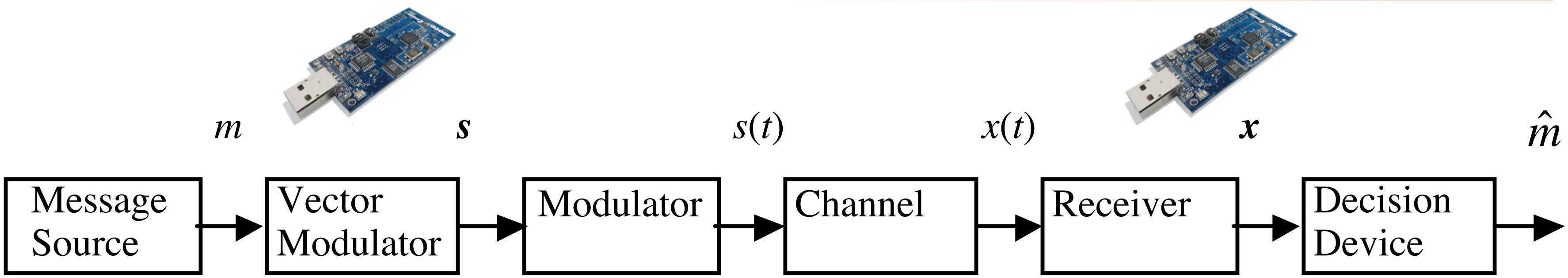
# Today's learning goals

- How bits of messages are transmitted over a channel?
- What is the probability to successfully receive messages over AWGN channels?
- What is the probability to successfully receive messages over fading channels?

# Today's lecture

- **Basic of modulation theory**
- Probability of error over AWGN channels
  - BPSK
  - Amplitude Modulation
- Probability of error over fading + AWGN channels

# Digital modulations




$$s(t) = \sum_{k=0}^{\infty} a_k(t) g(t - kT_s)$$

modulated signal transmitted by the sensor's antenna

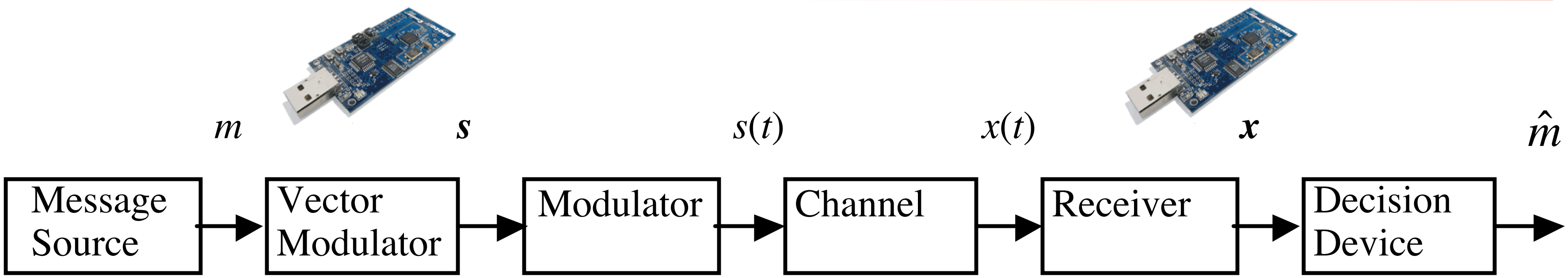
$$G(f) \triangleq \int_0^{T_s} a_0 g(t) e^{-2\pi f t} dt$$

spectrum of the signal over a symbol

$$\Phi_s(f) \triangleq \sigma_a^2 \frac{|G(f)|^2}{T_s}$$

power spectral density of the signal

# Example: Binary Phase Shift Keying (BPSK) modulation




$$s(t) = \sum_{k=0}^{\infty} a_k(t) g(t - kT_s)$$

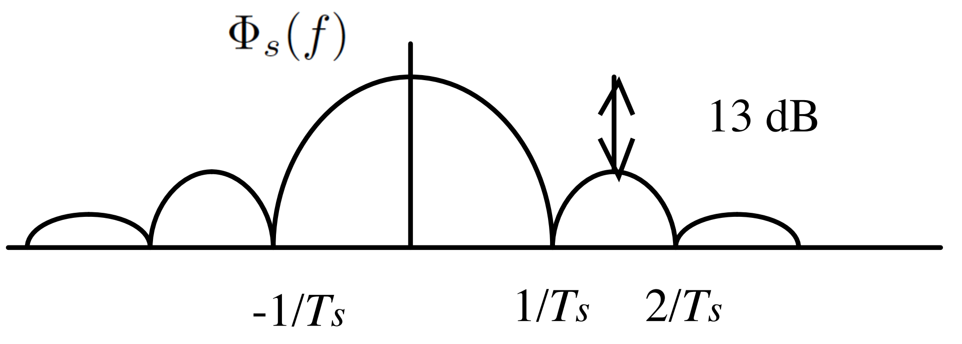
$$a_k(t) = \begin{cases} \cos(2\pi f_c t) & \text{if bit 0 at symbol time k,} \\ \cos(2\pi f_c t + \pi) & \text{if bit 1 at symbol time k.} \end{cases}$$

Modulation of the bits

$$g(t) = \sqrt{\frac{E}{T_s}}, 0 \le t \le T_s$$

$$P_t = \frac{E}{T_s}$$

# BPSK spectral density




$$G(f) = -\sqrt{\frac{E}{T_s}}\frac{e^{j\pi f T_s} - e^{-j\pi f T_s}}{j2\pi f} = -\sqrt{\frac{E}{T_s}}T_s\frac{\sin(\pi f T_s)}{\pi f T_s} = -\sqrt{\frac{E}{T_s}}T_s\text{sinc}(fT_s)$$

$$\Phi_s(f) \triangleq \frac{E}{T_s}\text{sinc}^2(fT_s)$$

# BPSK

Now, we would like to compute the probability that a bit is received successfully when it is transmitted by a modulation over a AWGN channel

# Today's lecture

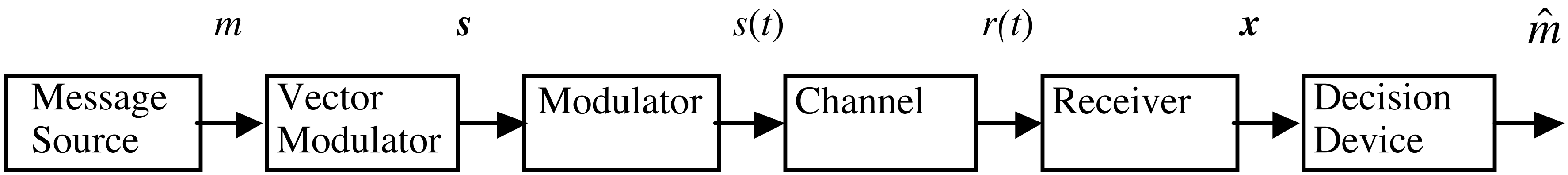
- Basic of modulation theory

- **Probability of error over AWGN channels**
    - BPSK
    - Amplitude Modulation

- Probability of error over fading + AWGN channels

- Basic of coding theory

# BPSK probability of error in AWGN wireless channels



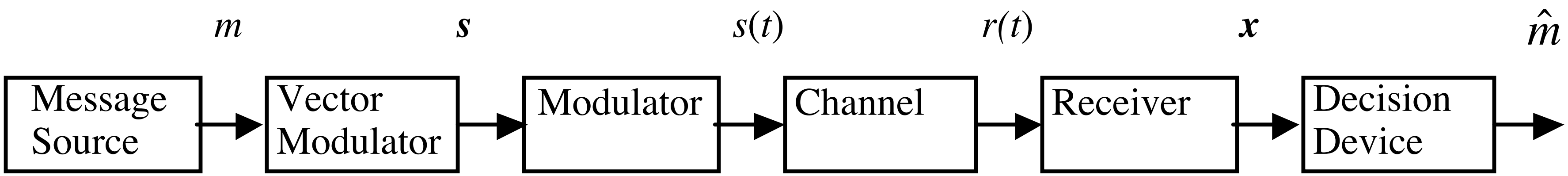

Assume that the transmitted signal is received together with an Additive White Gaussian Noise (AWGN)

$$r(t) = s(t) + n_0(t)$$

$$n_0(t) \in N\left(0, \sigma^2 = \frac{N_0}{2T_s}\right)$$

The demodulator in the receiver produces a signal

# BPSK detection in AWGN wireless channels




- The demodulator in the receiver block produces a signal

$$r'(t) = s'(t) + n_0'(t)$$

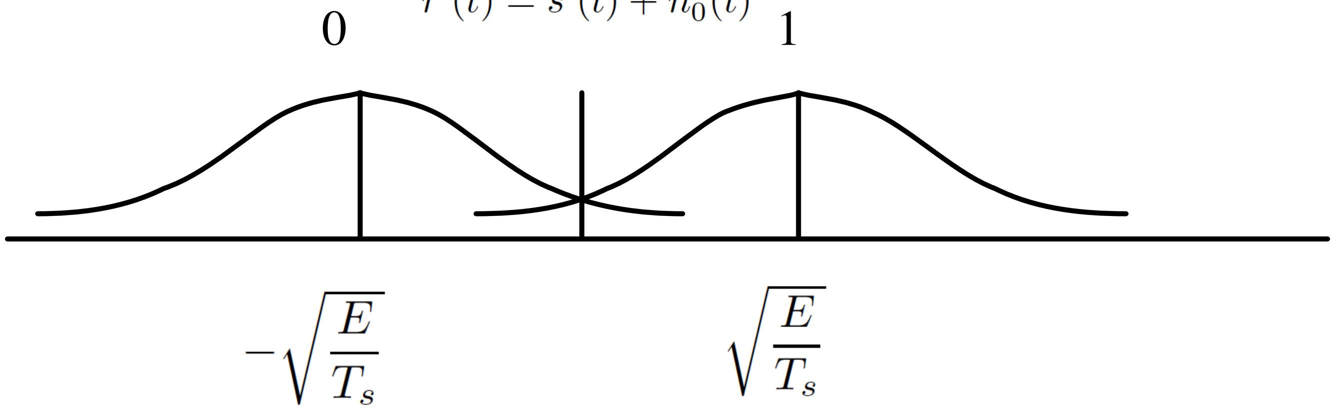
$$s'(t) = \begin{cases} \sqrt{\frac{E}{T_s}} & \text{if bit 0 was transmitted,} \\ -\sqrt{\frac{E}{T_s}} & \text{if bit 1 was transmitted.} \end{cases}$$

$$n_0'(t) \in N\left(0, \sigma^2 = \frac{N_0}{2T_s}\right)$$

- If $r'(t) \geq 0$ the detector decides for bit 1
- If $r'(t) < 0$ the detector decides for bit 0
- Given the AWGN, what is the error in this detection?

# BPSK probability of error

$$r'(t) = s'(t) + n_0'(t)$$

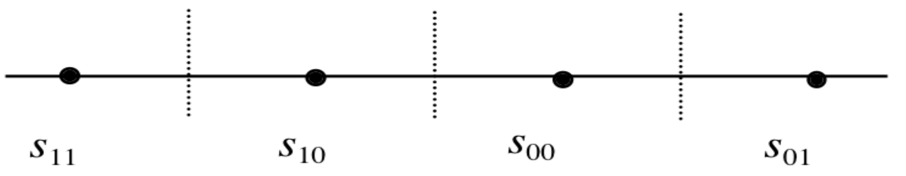



$$P_{e,0|1} = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(x-\sqrt{\frac{E}{T_s}}\right)^2}{2\sigma^2}} dx = Q\left(\sqrt{\frac{2E}{N_0}}\right)$$

$$P_{e,1|0} = \int_{0}^{-\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(x+\sqrt{\frac{E}{T_s}}\right)^2}{2\sigma^2}} dx = Q\left(\sqrt{\frac{2E}{N_0}}\right)$$

$$P_e = \frac{1}{2} P_{e,0|1} + \frac{1}{2} P_{e,1|0} = Q\left(\sqrt{\frac{2E}{N_0}}\right) \qquad \text{SNR} = \frac{E}{N_0}$$

# Amplitude modulation, AM




- BPSK is a simple example of binary Amplitude Modulation
- By adding more amplitude values, more general modulations are possible
- Modulations are characterized by constellation points
- Every point (symbol) is associated to a signal with specific amplitude
- Every signal is then associated to a code-word of bits

# Probability of error for AM

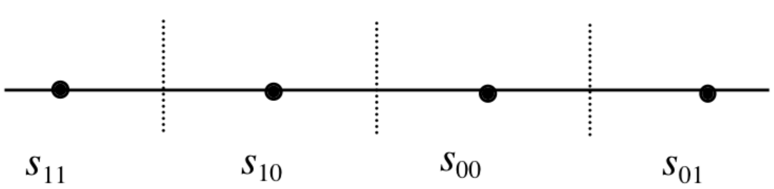
- The distance between the constellation points determines the probability that a symbol is detected with error

- It is possible to show that

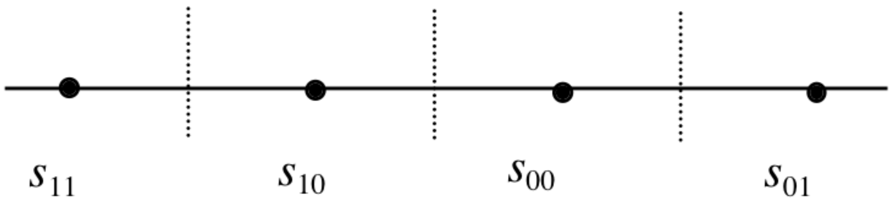
$$P_e \simeq N_{d_{\min}} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

$d_{\min}$ minimum distance among points

$N_{d_{\min}}$ average number of neighbors at minimum distance



# Exercise: 4-AM (or 4-PAM)




$$s_{00} = \sqrt{\frac{E}{5}} g(t) \qquad s_{01} = 3\sqrt{\frac{E}{5}} g(t)$$

$$s_{11} = -3\sqrt{\frac{E}{5}} g(t) \quad s_{10} = -\sqrt{\frac{E}{5}} g(t)$$

$$g(t) = \sqrt{\frac{1}{T_s}} \qquad 0 \leq t \leq T_s$$

What is the probability that a bit is erroneously received?

# Solution

$$d^2_{\min} = 4\frac{E}{5} \qquad N_{\min} = 1.5$$

- The probability of symbol error is

$$P_e \simeq 1.5Q\left(\sqrt{\frac{2E}{5N_0}}\right) = 1.5Q\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$

- Since per every symbol there are two bits, the probability of a bit in error is

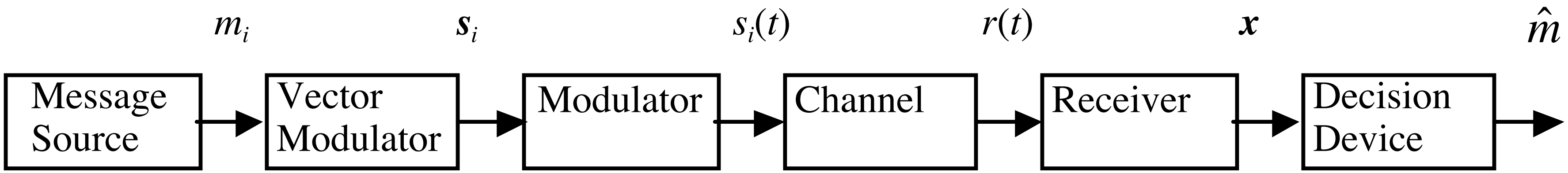
$$P_b \simeq 0.75Q\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$

$$E_b = \frac{E}{\log_2 M} = \frac{E}{2}$$ Energy per bit

# 4-Quadrature Amplitude Modulation

- Let's consider a more general amplitude modulation that is used in
  - TskyMotes
  - IEEE 802.15.4 standard (which we study next lecture)
- 4-Quadrature Amplitude Modulation

# 4-QAM



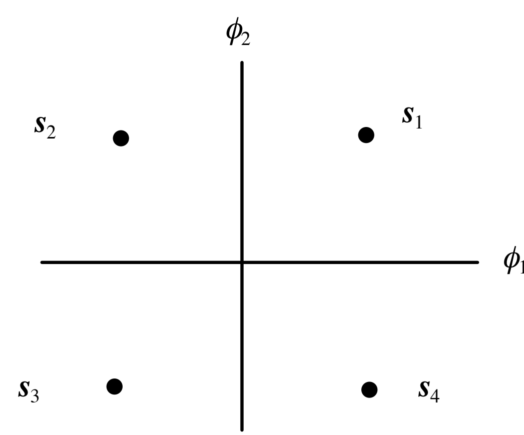

$$s_i(t) = \frac{2E}{T_s} \cos\left(2\pi f_c t + \frac{(2i-1)\pi}{4}\right) \quad 0 \le t \le T_s \qquad i = 1, \ldots, 4$$

$$s_i(t) = \frac{2E}{T_s} \cos\left(\frac{(2i-1)\pi}{4}\right) \cos(2\pi f_c t) - \frac{2E}{T_s} \sin\left(\frac{(2i-1)\pi}{4}\right) \sin(2\pi f_c t) \quad 0 \le t \le T_s$$

Signal space by two basis function

$$\phi_1(t) = \frac{2}{T_s} \cos(2\pi f_c t) \quad 0 \le t \le T_s$$

$$\phi_1(t) = \frac{2}{T_s} \sin(2\pi f_c t) \quad 0 \le t \le T_s$$



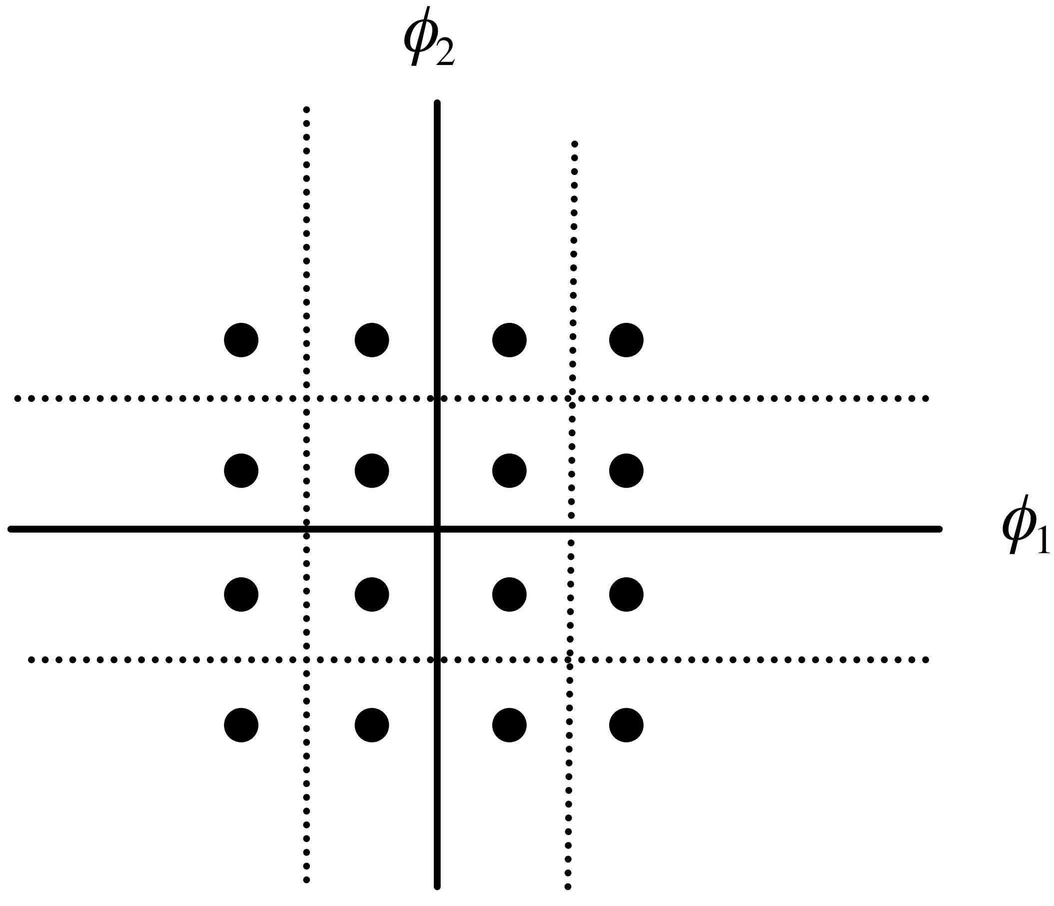
# Probability of error in 4-QAM

Minimum distance between two symbols $d_{\min} = \sqrt{2E}$

2 neighbors at minimum distance

$$P_e \simeq 2Q\left(\sqrt{\frac{E}{5N_0}}\right) = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

# A more complex modulation: 16-QAM




More points can be added, this increases the transmit bit rate, but increases also the probability of error

# Comparison of probabilities or error

$$P_{e,\mathrm{BPSK}} = Q\left(\sqrt{\frac{2E}{N_0}}\right)$$

$$\mathrm{SNR} = \frac{E}{N_0}$$

$$P_{e,4-\mathrm{PAM}} \simeq 1.5Q\left(\sqrt{\frac{2E}{5N_0}}\right)$$

$$P_{e,4-\mathrm{QAM}} \simeq 2Q\left(\sqrt{\frac{E}{N_0}}\right)$$

# Today's lecture

- Basic of modulation theory
- Probability of error over AWGN channels
    - BPSK
    - Amplitude Modulation
- **Probability of error over fading + AWGN channels**

# Communication over wireless channel

- So far, we have seen only channels where the transmitted signal is received corrupted by AWGN, but in real channel

$$r(t) = \sqrt{A}s(t) + n_0(t)$$

- The power of the transmitted signals is attenuated by the wireless channel

- How the probability of error is affected by the channel?

# Probability of error over fading channels

- Consider a AWGN + Rayleigh channel with path loss, Rayleigh fast fading, and fixed shadow fading

$$P_r = \underbrace{P_t G_t(\theta_t, \psi_t) G_r(\theta_r, \psi_r) \frac{\lambda^2}{(4\pi r)^2} \bar{\mathrm{PL}} \cdot y}_{C} \cdot z$$

$$\triangleq P_t C \cdot z$$

# Probability of error over fading channels

- The receiver sees the transmit power $P_t = \frac{E}{T_s}$ received with an attenuation $C \cdot z$ and thus

$$\text{SNR} = \frac{E}{N_0} C z$$

- To compute the probability, mathematically, it is AS if the transmit power WERE

$$P_t = \frac{E}{T_s} C z$$

and the channel WERE just AWGN and no fading

- Thus, the probability of error with fading has same expression of simple AWGN channels, but with the new SNR above

$$P_{e,\text{BPSK}} = Q\left(\sqrt{\frac{2E}{N_0}}\right) \longrightarrow P_{e,\text{BPSK}}(z) = Q\left(\sqrt{\frac{2ECz}{N_0}}\right)$$

AWGN AWGN+ Rayleigh fading

# Probability of error over fading channels

- The probability so derived over fading channel is instantaneous
  - Depends on the given realization of the fading channel $z$
- What is the average probability of error, where the average is taken over the distribution of the fading?

- Just take the expectation of $P_{e,\mathrm{BPSK}}(z)$ over the distribution of $z$

- Thus (remember that the square of a Rayleigh random variable is an exponential random variable)

$$p(z) = \frac{1}{\gamma^*} e^{-\frac{z}{\gamma^*}} \qquad \gamma^* \triangleq \mathbb{E}\,\mathrm{SNR} = \frac{E}{N_0} C \qquad \mathbb{E}\, z = 1$$

$$\bar{P}_{e,\mathrm{BPSK}} = \int_0^\infty P_{e,\mathrm{BPSK}}(z) p(z) dz = \frac{1}{2}\left[1 - \sqrt{\frac{\gamma^*}{1+\gamma^*}}\right] \simeq \frac{1}{4\gamma^*}$$

# Comparison of probability of error over fading channels

- Simple AWGN channel:

$$P_{e,\mathrm{BPSK}} = Q\left(\sqrt{\frac{2E}{N_0}}\right)$$

  - linear increase of SNR results in exponential decrease in the error probability

- AWGN + Rayleigh channel:

$$\bar{P}_{e,\mathrm{BPSK}} \simeq \frac{1}{4\gamma^*} \qquad \gamma^* \triangleq \mathbb{E}\,\mathrm{SNR} = \frac{E}{N_0}C$$

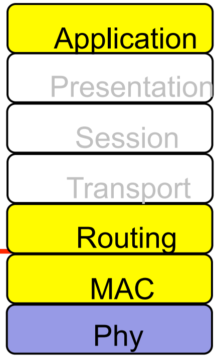
  - linear increase of the SNR gives only a linear decrease of error probability
  - 40dB higher SNR than the simple AWGN channel to have same probability $P_e = 10^{-6}$

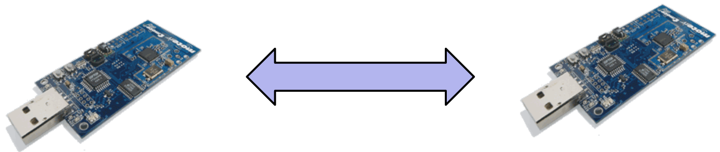
# Packet error probability

- Bits are grouped in unit denoted physical layer data frame, "physical layer message"
- What is the probability that the message is erroneously received?
- Suppose that the message is formed by L bits and BPSK is used, then the probability that the packet is in error is bounded by

$$P_m \leq 1 - (1 - P_{e,\mathrm{BPSK}})^L$$

# Conclusions



- We studied how bits are transmitted over the wireless channel by modulation and coding

- The probability of successful reception of messages was characterized for AWGN and for AWGN + fading channels

# Next lecture

- But how a node has the right to transmit a message?

- Medium access control
  - When a node gets the right to transmit?
  - What is the mechanism to get such a right?