AUTOMATIC CONTROL
KTH

Nonlinear Control, EL2620 / 2E1262

Answers December 11 , 2012

1. (a) $x_1 = 0, x_2 = 0$

   (b) $A = [-6\ 3; -3\ \ -3]$ and $\lambda_{1,2} = -\frac{9}{2} \pm \sqrt{27/4}$, hence stable focus.

   (c) $\dot{V} = \frac{-12x_1^2}{(1+x_1^2)^4} + \frac{-6x_2^2}{(1+x_1^2)^2} \leq 0$ and equal to 0 only for $x = 0$, hence asymptotically stable. We can not conclude globally asymptotically stable since V is not radially unbounded.

2. (a) $\dot{V} = 4x_1^2 x_2 + 4x_2 x_1^4 + 6x_2 u$. To make V negative (semi)definite we choose

   $$u = \frac{1}{6}(-4x_1^2 - 4x_1^4 - x_2)$$

   to yield $\dot{V} = -x_2^2 \leq 0$. To check for global stability we consider LaSalle and determine invariant sets for which $\dot{V} = 0$, i.e., $x_2 = 0$ and $\dot{x}_2 = 0$. We see that $\dot{x}_2 = 2x_1^4 - 2x_1^2 - 2x_1^4 - 0.5x_2$ and hence $x_2 = 0, \dot{x}_2 = 0$ only if also $x_1 = 0$. Hence we have made the origin globally stable.

   (b) The system has only one equilibrium which furthermore is unstable. Hence if we can find an invariant subspace in the state-plane, then there must exist a stable limit cycle within that subspace. If we consider sets with level curves $V = x^2 + y^2 = c$, with $c > 0$ some positive constant, we get $\dot{V} = 2y^2 - 2y^2x^2 - 8y^4 = 2y^2(1 - x^2 - 4y^2) = 2y^2(1 - c - 3y^2)$. With $c > 1$ we have $\dot{V} < 0$ and hence all trajectories point inwards, hence there must be a stable limit cycle within the circle with radius 1 in the (x, y)-plane. Likewise, for $c < 1/4$ we have that $\dot{V} > 0$ and hence all trajectories pointing outwards. Thus, all trajectories startting outside the unit circle and inside the circle with radius 1/4 will be attracted to the region between these two circles. Assuming there is a unique limit cycle within the region, all trajectories will end up at the limit cycle and the limit cycle is then globally attracting.

3. (a) (i) With $y = x_2$ we have $\dot{y} = \dot{x}_2 = -x_1x_2 + x_2^3 + u$ and the choice

   $$u = x_1x_2 - x_2^3 + v$$

   yields $\dot{y} = v$ or $G(s) = 1/s$. The zero dynamics is given by $\dot{x}_1 = cos(x_1) - x_2$ which clearly is not stable since x_1 while increase (decrease) continuously if $x_2 < 0 (> 0)$.

   (ii) With $y = x_1$ we have $\dot{y} = cos(x_1) - x_2$, $\ddot{y} = -sin(x_1)(cos(x_1) - x_2) + x_1x_2 - x_2^3 - u$ and $u = -sin(x_1)(cos(x_1) - x_2) + x_1x_2 - x_2^3 - v$ yields $\ddot{y} = v$ or $G(s) = 1/s^2$. In this case there are no zero dynamics.

(b) On the sliding manifold we have $x_1 + ax_2 = 0$ or $x_2 = -\frac{1}{a}x_1$ and this yields $\dot{x}_1 = (1 - \frac{1}{a})x_1(t)$ and hence we should choose $0 < a < 1$ to get covergence to the origin on the sliding manifold. To globally stabilize S we consider the control Lyapunov function $V = 0.5\sigma^2$ and $\dot{V} = \sigma\dot{\sigma} = \sigma(x_1+x_2+ax_1^2+ax_2+au)$ and we choose $u = (-x_1-x_2-ax_1^2-ax_2-sign(\sigma))/a$ to yield $\dot{V} < 0$ for $\sigma \neq 0$. The equivalent control when on the manifold is $u = (-x_1 - x_2 - ax_1^2 - ax_2)/a$.

4. (a) $G(s)$ stable and hence we can consider the stationary frequency response of the linear part. The linear system has amplification $|G(j0.7)| = 1.3$ and phase shift $-\pi/2$. Hence the signal into f has amplitude 1.3 and in phase with the output of f. This implies that the nonliearity is $y = u$ when $|u| > 1$ and $y = 0$ when $|u| < 1$. Sketch not shown here.

(b) The gain $\gamma(f) = 1$ and the small gain theorem guarantees stability if $|k||G(j\omega)|\gamma(f) < 1 \forall\omega$. Since $|G(j\omega)| < \sqrt{2}$ we get stability for $|k| < 1/\sqrt{2}$.

(c) We have that $kG(s)$ has no poles in RHP and we can bound the nonlinearity f by

$$0 \leq k_1 \leq \frac{f(v)}{v} \leq k_2$$

with $k_1 = 0$ and $k_2 = 1$. The circle criterion guarantees stability of the closed loop if the Nyquist curve $kG(j\omega)$ does not encircle or intersect the circle defined by the points $-1/k_1$ and $-1/k_2$. Here $-1/k_1 = \infty$ and $-1/k_2 = -1$.

Hence we require that $\min_\omega \operatorname{Re} kG(\jmath\omega) > -1$. From the plot of the frequency response of $G(j\omega)$ we see that if $k > 0$ then $\min_\omega \operatorname{Re} kG(j\omega) \approx -0.7k$ and that if $k < 0$ then $\min_\omega \operatorname{Re} kG(j\omega) = \sqrt{2}k$.

The circle criterion hence guarantees stability for $-\frac{1}{\sqrt{2}} < k < 1.4$.

5. (a) The optimization problem, with states $(x_1, x_2) = (x, \dot{x})$ is

$$\min_u \int_0^T 1dt$$

subject to

$$\dot{x}_1 = x_2, \dot{x}_2 = u, x_1(0) = x_2(0) = 0, |u| < 5$$

and

$$x_1(T) = 0.5, x_2(T) = 0.2, \phi(x) = 0, \psi_1(x) = x_1 - 0.5, \psi_2(x) = x_2 - 0.2$$

(b) The fastest movement from $x = 0$ to $x = 0.5$ is obtained with maximum acceleration $\ddot{x} = 5$ or $x(t) = 5t^2/2$ and hence $5T^2/2 = 0.5$ yields $T = 0.45$ s. If we require rest at the end, i.e., $\dot{x}(T) = 0$, then we we need to have full acceleration for half the time and then full retardation under half the time. In this case we reach halfway at $5T_1^2/2 = 0.25$ or $T_1 = 0.316$ s, and then we have the same velocity profile reversed from T_1 to T such that $T = 2T_1 = 0.63$ s.

(c) We have $L = 1$ and with $n_0 = 1$

$$H = 1 + \lambda_1 x_2 + \lambda_2 u$$

$$\dot{\lambda}_1 = 0\ ; \quad \dot{\lambda}_2 = -\lambda_1$$

with $\lambda_1(T) = \mu_1,\ \lambda_2(T) = \mu_2$. This yields

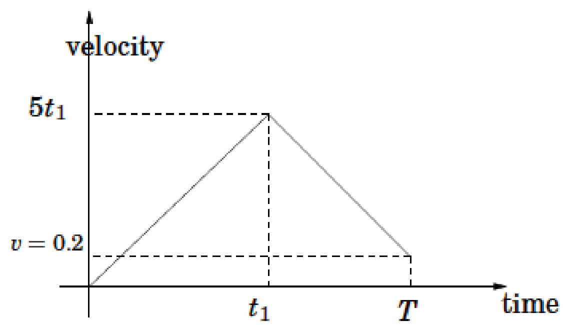
$$\lambda_1(t) = \mu_1 \ ; \quad \lambda_2(t) = -\mu_1 t + C$$

with $-\mu_1 T + C = \mu_2$ and hence

$$\lambda_1(t) = \mu_1 \ ; \quad \lambda_2(t) = \mu_1(T - t) + \mu_2$$

The optimal u is given by minimizing H wrt u and hence $u = 5$ for $\lambda_2 < 0$ and $u = -5$ for $\lambda_2 > 0$. Since $\lambda_2(t)$ is a continuously increasing function, there will be one switch at t_1 with $\lambda_2 = \mu_1(T - t_1) + \mu_2 = 0$.

To determine t_1 and T consider the sketch of the velocity below.



The area under the curve is the distance x and should satisfy $x(T) = 0.5$. The area is

$$5t_1^2/2 + (T - t_1)(5t_1 - 0.2)/2 + (T - t_1)0.2 = 0.5$$

The end speed should be 0.2

$$5t_1 - 5(T - t_1) = 0.2$$

which yields $t_1 = 0.3175\ s$ and $T = 0.595\ s$.