EL2620 Nonlinear Control

KTH Electrical Engineering

Lecture 9

- Nonlinear control design based on high-gain control

Today’s Goal

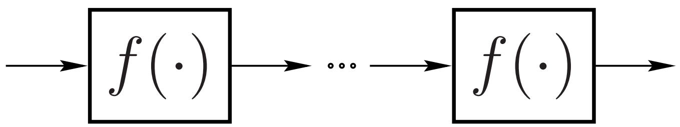
You should be able to analyze and design

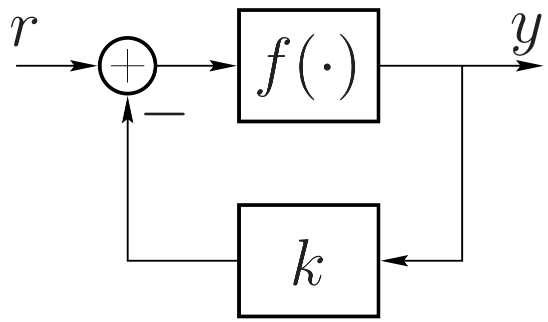
- High-gain control systems
- Sliding mode controllers

History of the Feedback Amplifier

New York–San Francisco communication link 1914.

High signal amplification with low distortion was needed.

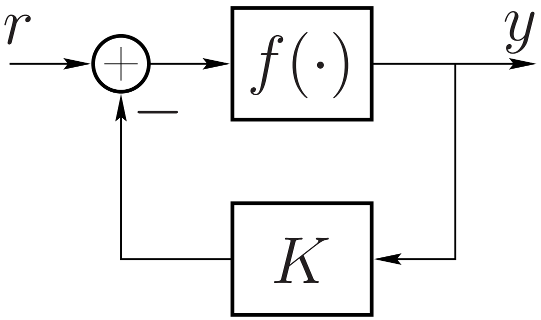


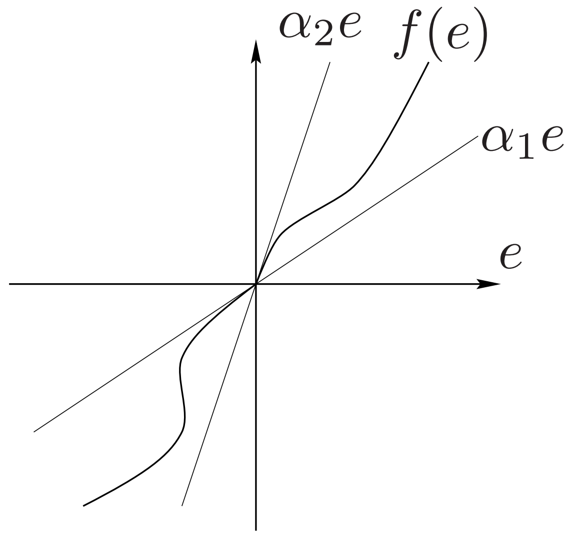


Feedback amplifiers were the solution!

Black, Bode, and Nyquist at Bell Labs 1920–1950.

Linearization Through High Gain Feedback





$$\alpha_1 \leq \frac{f(e)}{e} \leq \alpha_2 \qquad \Rightarrow \qquad \frac{\alpha_1}{1+\alpha_1 K} r \ \leq \ y \ \leq \ \frac{\alpha_2}{1+\alpha_2 K} r$$

choose $K \gg 1/\alpha_1$, yields

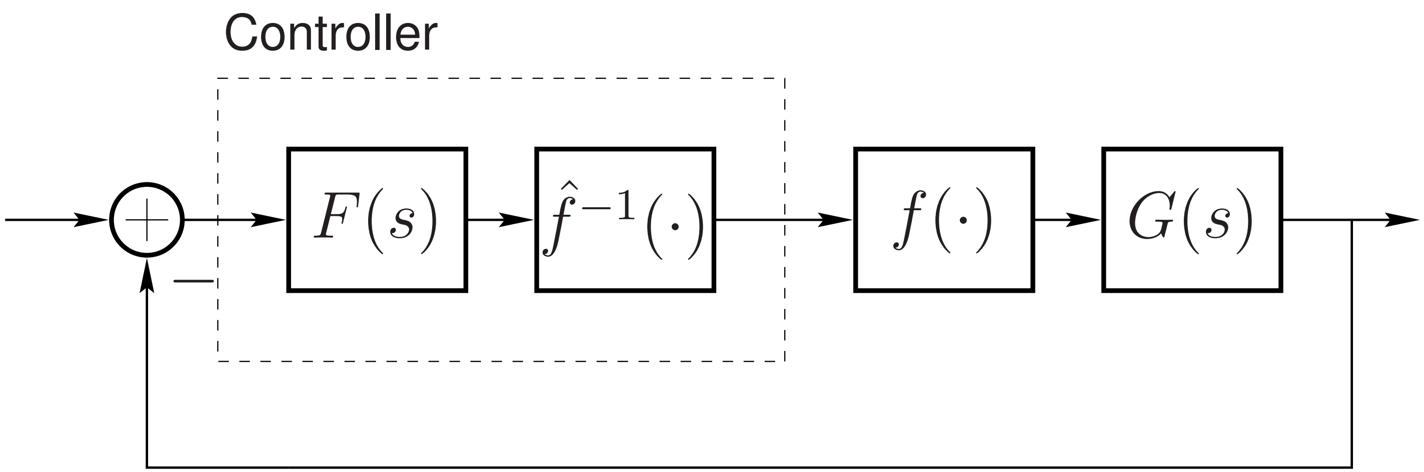
$$y \approx \frac{1}{K} r$$

A Word of Caution

Nyquist: high loop-gain may induce oscillations (due to dynamics)!

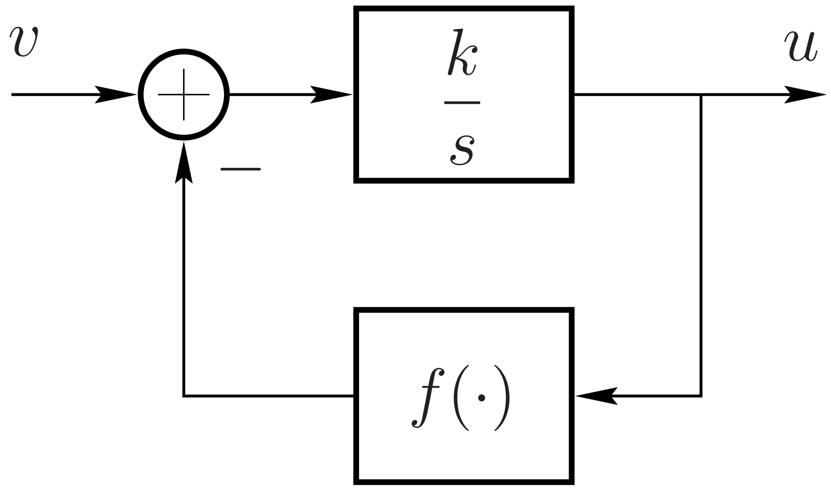
Inverting Nonlinearities

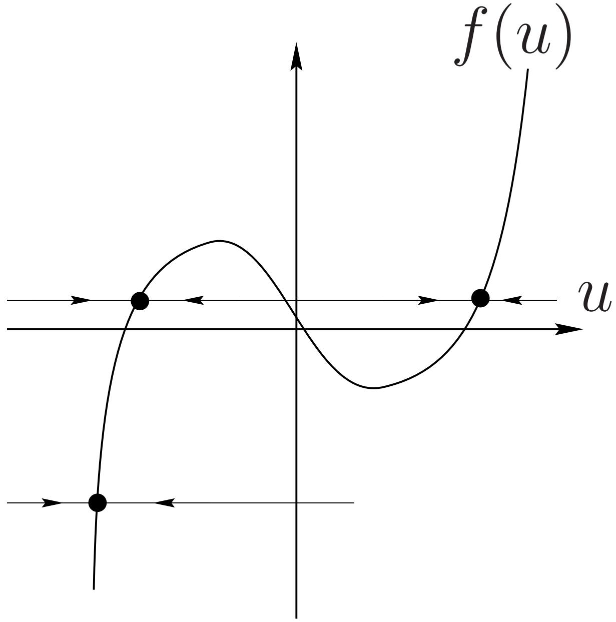
Compensation of static nonlinearity through inversion:



Should be combined with feedback as in the figure!

Remark: How to Obtain f^{-1} from f using Feedback



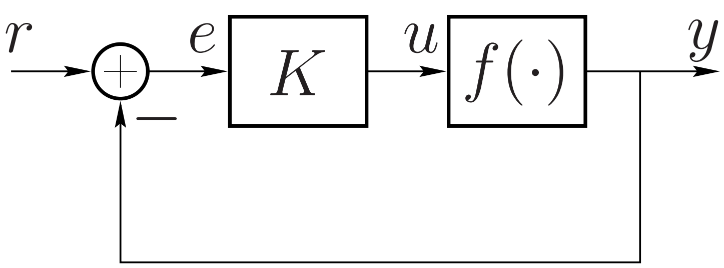


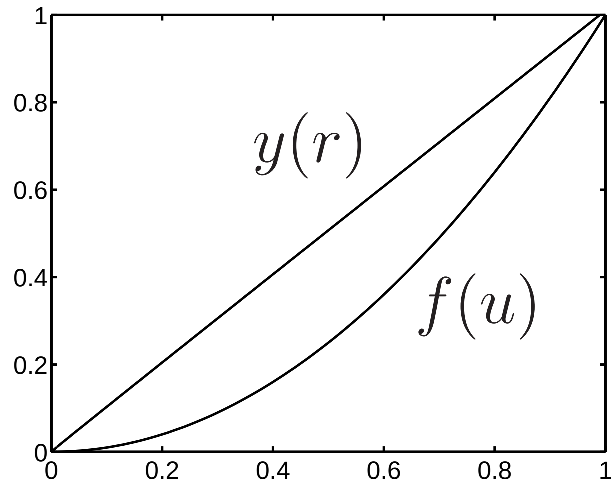
$$\dot{u} = k\big(v - f(u)\big)$$

If $k > 0$ large and $df/du > 0$, then $\dot{u} \to 0$ and

$$0 = k\big(v - f(u)\big) \quad \Leftrightarrow \quad f(u) = v \quad \Leftrightarrow \quad u = f^{-1}(v)$$

Example—Linearization of Static Nonlinearity

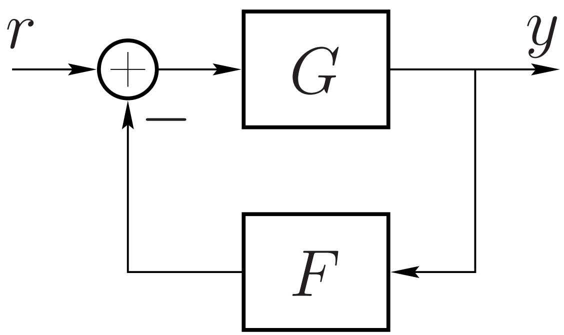




Linearization of $f(u) = u^2$ through feedback.

The case $K = 100$ is shown in the plot: $y(r) \approx r$.

The Sensitivity Function $S = (1 + GF)^{-1}$



The closed-loop system is

$$G_{\text{cl}} = \frac{G}{1 + GF}$$

Small perturbations dG in G gives

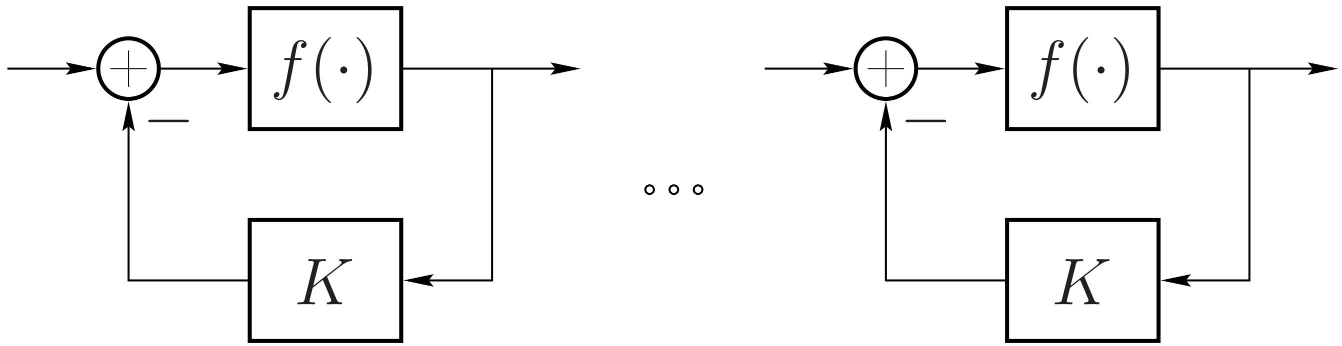
$$\frac{dG_{\text{cl}}}{dG} = \frac{1}{(1 + GF)^2} \quad \Rightarrow \quad \frac{dG_{\text{cl}}}{G_{\text{cl}}} = \frac{1}{1 + GF}\frac{dG}{G} = S\frac{dG}{G}$$

S is the closed-loop **sensitivity** to open-loop perturbations.

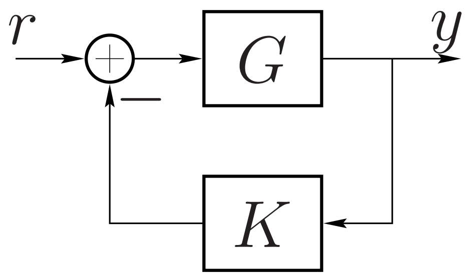
Distortion Reduction via Feedback

The feedback reduces distortion in each link.

Several links give distortion-free high gain.



Example—Distortion Reduction



Let $G = 1000$,
distortion $dG/G = 0.1$

Choose $K = 0.1 \quad \Rightarrow \quad S = (1 + GK)^{-1} \approx 0.01$. Then

$$\frac{dG_{\text{cl}}}{G_{\text{cl}}} = S\frac{dG}{G} \approx 0.001$$

100 feedback amplifiers in series give total amplification

$$G_{\text{tot}} = (G_{\text{cl}})^{100} \approx 10^{100}$$

and total distortion

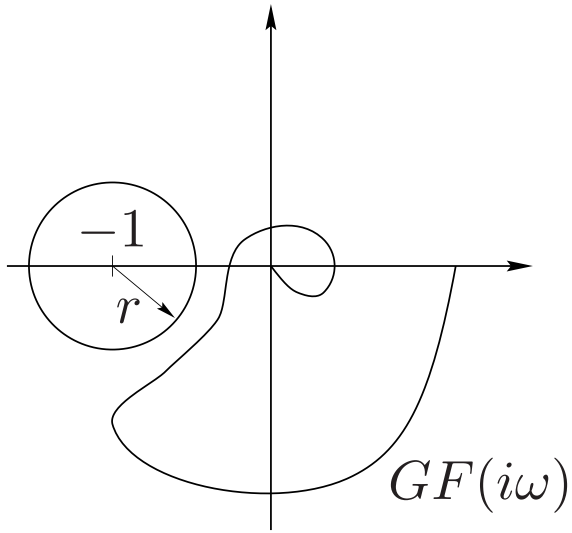
$$\frac{dG_{\text{tot}}}{G_{\text{tot}}} = (1 + 10^{-3})^{100} - 1 \approx 0.1$$

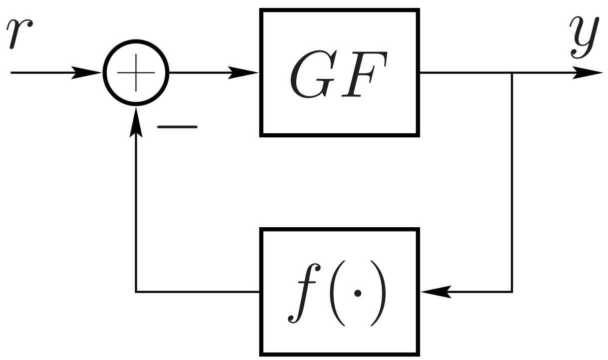
Transcontinental Communication Revolution

The feedback amplifier was patented by Black 1937.

Year	*Channels*	*Loss (dB)*	*No amp's*
1914	1	60	3–6
1923	1–4	150–400	6–20
1938	16	1000	40
1941	480	30000	600

Sensitivity and the Circle Criterion





Consider a circle $\mathcal{C} := \{z \in \mathbb{C} : |z+1| = r\}$, $r \in (0,1)$.

$GF(i\omega)$ stays outside $\mathcal{C}$ if

$$|1 + GF(i\omega)| > r \qquad \Leftrightarrow \qquad |S(i\omega)| \leq r^{-1}$$

Then, the Circle Criterion gives stability if $\dfrac{1}{1+r} \leq \dfrac{f(y)}{y} \leq \dfrac{1}{1-r}$

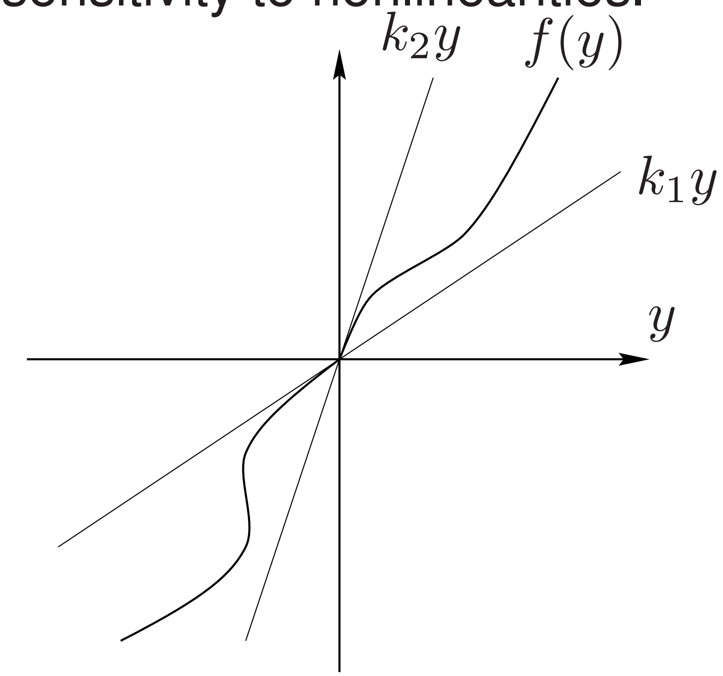
Small Sensitivity Allows Large Uncertainty

If $|S(i\omega)|$ is small, we can choose r large (close to one).

This corresponds to a large sector for $f(\cdot)$.

Hence, $|S(i\omega)|$ small implies low sensitivity to nonlinearities.

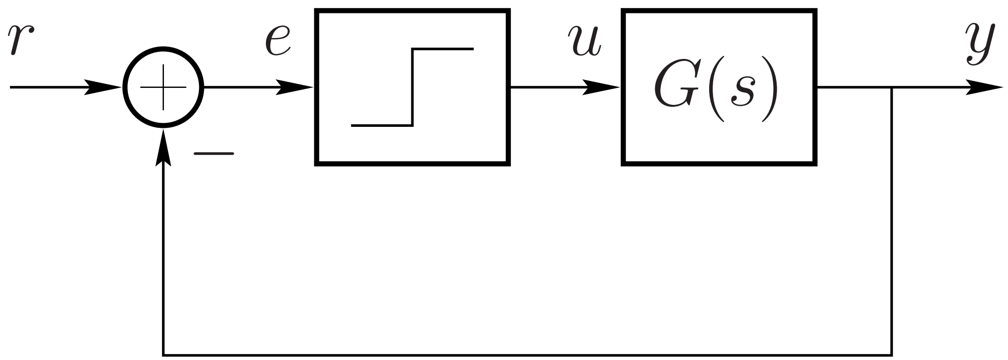
$$k_1 = \frac{1}{1+r}$$
$$k_2 = \frac{1}{1-r}$$



On–Off Control

On–off control is the simplest control strategy.

Common in temperature control, level control etc.



The relay corresponds to infinite high gain at the switching point

Example: Stabilizing control of inverted pendulum

Model with $x_1 = \theta - \alpha$, $\alpha = \pi/2$ and $x_2 = \dot{\theta}$

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin(x_1+\alpha) - \frac{k_0}{m}x_2 + \frac{1}{ml^2}u\end{aligned}$$

Thus,

$$f(x) = -\frac{g}{l}\sin(x_1+\alpha) - \frac{k_0}{m}x_2, \; g(x) = \frac{1}{ml^2}$$

We choose the sliding manifold

$$\sigma(x) = x_2 + a x_1 = 0$$

Control design

The control law

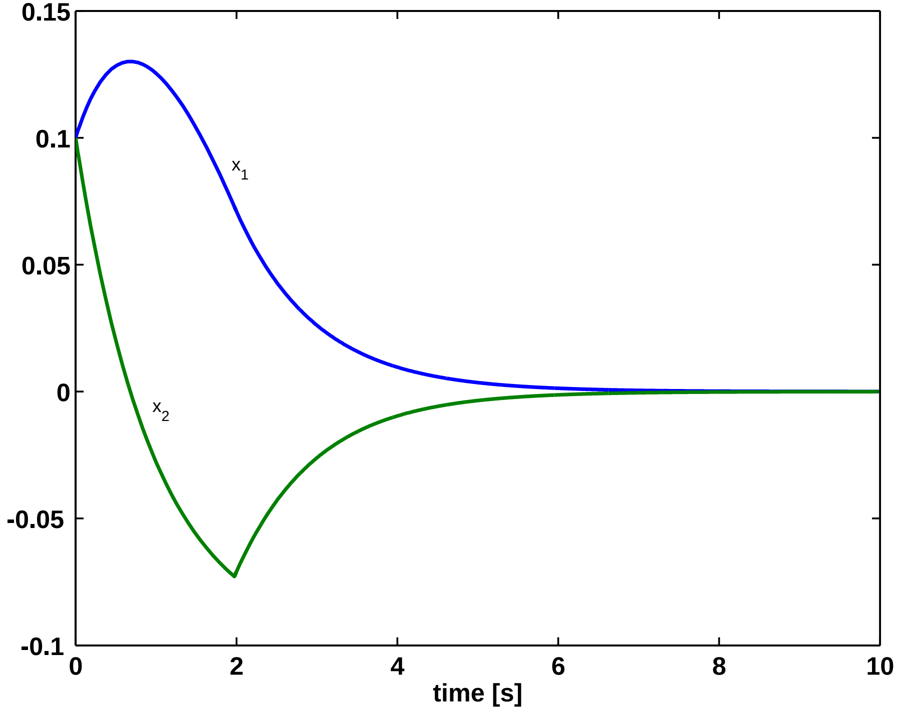
$$u(x) = \beta(x) - K\,sign(\sigma)$$

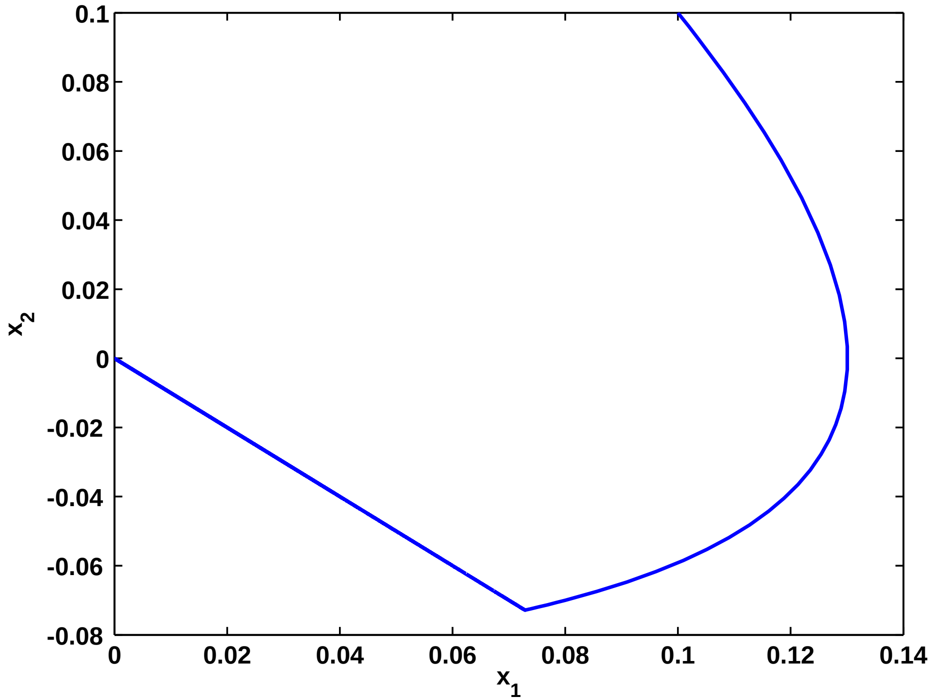
where

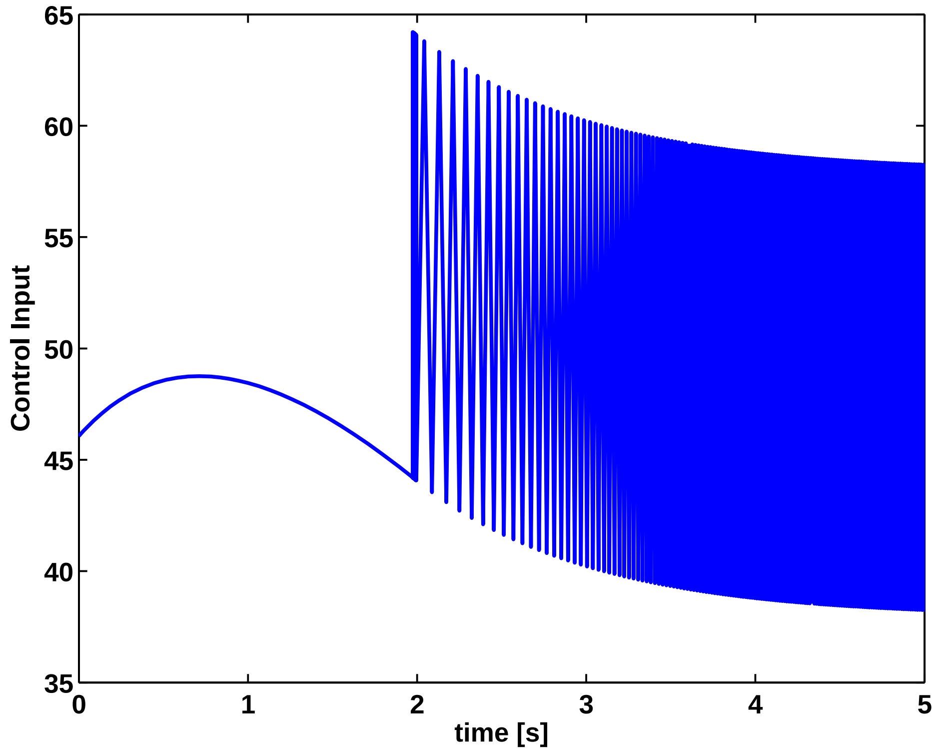
$$\beta(x) = -\frac{f(x) + ax_2}{g(x)} = gml\sin(x_1 + \alpha) + k_0 ml^2 x_2 - ml^2 a x_2$$

and we choose

$$K = k_1 + k_2 \sigma^2$$


0.15
0.1
0.05
0
-0.05
-0.1
0
2
4
6
8
10
time [s]
x1
x2


0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
x_1
x_2



65
60
55
50
45
40
35
Control Input
0
1
2
3
4
5
time [s]

Design of Sliding Mode Controller

Idea: Design a control law that forces the state to $\sigma(x) = 0$. Choose $\sigma(x)$ such that the sliding mode tends to the origin. Assume

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(x) + g_1(x)u \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = f(x) + g(x)u$$

Choose $\sigma(x) = p^T x$ with $p^T = \begin{pmatrix} p_1 & \dots & p_n \end{pmatrix}$ the coefficients of a stable polynomial Then the control law

$$u = -\frac{p^T f(x)}{p^T g(x)} - \frac{\mu}{p^T g(x)} \operatorname{sgn} \sigma(x),$$

where $\mu > 0$ is a design parameter, will make the sliding mode and the equilibrium globally asymptotically stable.

Closed-Loop Stability

Consider $V(x) = \sigma^2(x)/2$ with $\sigma(x) = p^T x$. Then,

$$\dot{V} = \sigma^T(x)\dot{\sigma}(x) = x^T p\big(p^T f(x) + p^T g(x)u\big)$$

With the chosen control law, we get

$$\dot{V} = -\mu\sigma(x)\,\mathrm{sgn}\,\sigma(x) < 0$$

so x tend to $\sigma(x) = 0$.

$$\begin{aligned} 0 = \sigma(x) &= p_1 x_1 + \cdots + p_{n-1}x_{n-1} + p_n x_n \\ &= p_1 x_n^{(n-1)} + \cdots + p_{n-1}x_n^{(1)} + p_n x_n^{(0)} \end{aligned}$$

where $x^{(k)}$ denote time derivative. Now p corresponds to a stable differential equation, and $x_n \to 0$ exponentially as $t \to \infty$. The state relations $x_{k-1} = \dot{x}_k$ now give $x \to 0$ exponentially as $t \to \infty$.

Example—Sliding Mode Controller

Design state-feedback controller for

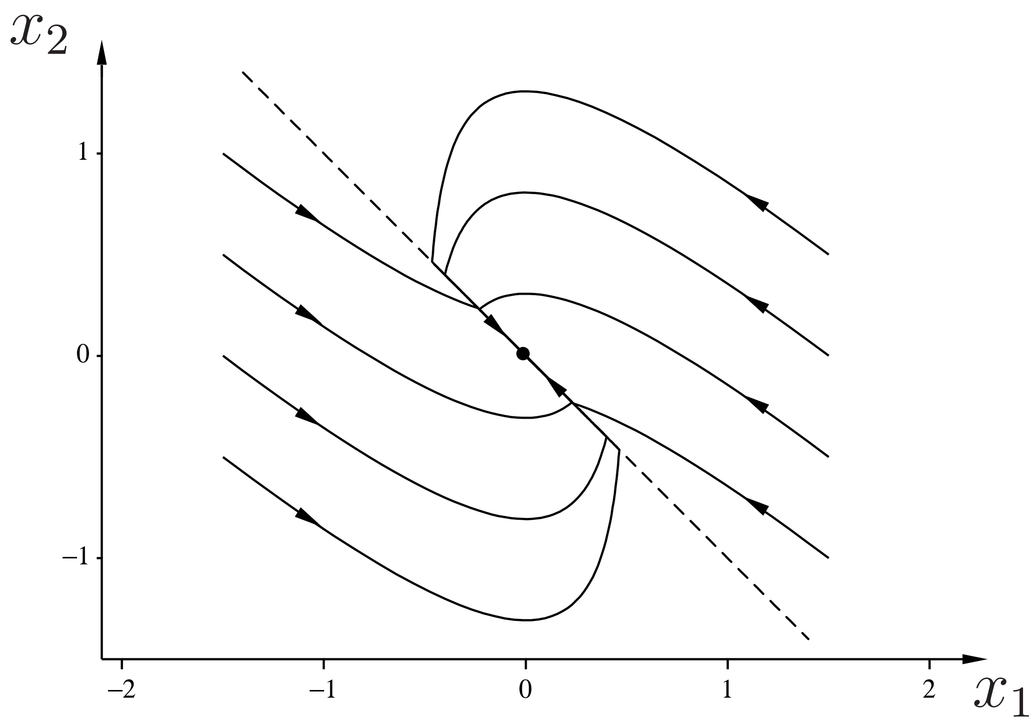
$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

Choose $p_1 s + p_2 = s + 1$ so that $\sigma(x) = x_1 + x_2$. The controller is given by

$$u = -\frac{p^T A x}{p^T B} - \frac{\mu}{p^T B} \operatorname{sgn} \sigma(x)$$
$$= 2x_1 - \mu \operatorname{sgn}(x_1 + x_2)$$

Phase Portrait

Simulation with $\mu = 0.5$. Note the sliding surface $\sigma(x) = x_1 + x_2$.



Time Plots

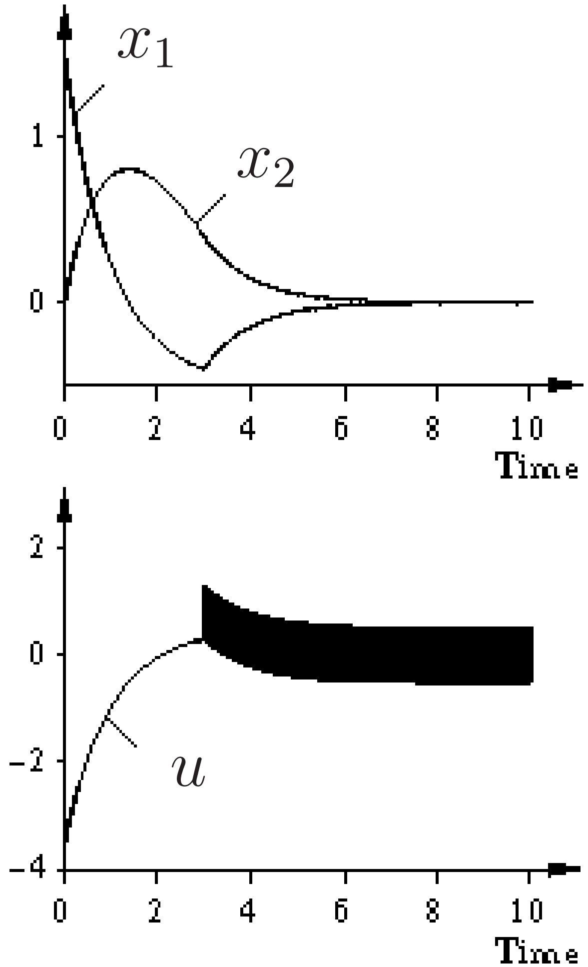
Initial condition
$x(0) = \begin{pmatrix} 1.5 & 0 \end{pmatrix}^T.$

Simulation agrees well with time to switch

$$t_{\mathrm{s}} = \frac{\sigma_0}{\mu} = 3$$

and sliding dynamics

$$\dot{y} = -y$$



The Sliding Mode Controller is Robust

Assume that only a model $\dot{x} = \widehat{f}(x) + \widehat{g}(x)u$ of the true system $\dot{x} = f(x) + g(x)u$ is known. Still, however,

$$\dot{V} = \sigma(x)\left[\frac{p^T(f\widehat{g}^T - \widehat{f}g^T)p}{p^T\widehat{g}} - \mu\frac{p^T g}{p^T\widehat{g}}\operatorname{sgn}\sigma(x)\right] < 0$$

if $\operatorname{sgn}(p^T g) = \operatorname{sgn}(p^T\widehat{g})$ and $\mu > 0$ is sufficiently large.

The closed-loop system is thus robust against model errors!

(High gain control with stable open loop zeros)

Comments on Sliding Mode Control

- Efficient handling of model uncertainties
- Often impossible to implement infinite fast switching
- Smooth version through low pass filter or boundary layer
- Applications in robotics and vehicle control
- Compare pulse-width modulated control signals

Today's Goal

You should be able to analyze and design

- High-gain control systems
- Sliding mode controllers

Next Lecture

- Lyapunov design methods
- Exact feedback linearization