Chapter 5 – M/M/1 Queuing Systems

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1 Exercise 5.1

In a computer network a link has a transmission rate of C bit/s. Messages arrive to this link in a Poisson fashion with rate λ messages per second. Assume that the messages have exponentially distributed length with a mean of $1/\mu$ bits and the messages are queued in a FCFS fashion if the link is busy.
a) Determine the minimum required C for given λ and μ such that the average system time (service time + waiting time) is less than a given time T_0.

Solution: System Description

- Single communication link: C bits per second
- Poisson arrivals: λ messages per second
- Exponential Service times: $E[T] = E[X]/C = 1/(\mu C)$, so the *exponential rate* is μC.
- First Come First Served policy
- Infinite Queue[1]

This is a typical M/M/1 System. We see the system diagram in Fig. 1. We first derive the state distribution (steady-state) of this system through the solution of the balance equations. We *define* $\rho = \lambda/(\mu C)$. For a no-loss system, ρ is the OFFERED and, at the same time, the ACTUAL load.

$$\lambda P_0 = (\mu C)P_1 \rightarrow P_1 = \rho P_0$$
$$\lambda P_1 = (\mu C)P_2 \rightarrow P_2 = \rho P_1 = \rho^2 P_0$$
$$\lambda P_2 = (\mu C)P_3 \rightarrow P_3 = \rho P_2 = \rho^3 P_0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\lambda P_k = (\mu C)P_{k+1} \rightarrow P_{k+1} = \rho P_k = \rho^k P_0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Then, we calculate the P_0 through the normalization equation:

$$\sum_{k=0}^{\infty} P_k = 1 \rightarrow \sum_{k=0}^{\infty} \rho^k P_0 = 1 \rightarrow P_0 \sum_{k=0}^{\infty} \rho^k = 1 \rightarrow P_0 \cdot \frac{1}{1-\rho} = 1 \rightarrow P_0 = 1 - \rho.$$

[1] If no buffer capacity is mentioned, we always assume that this is infinite.

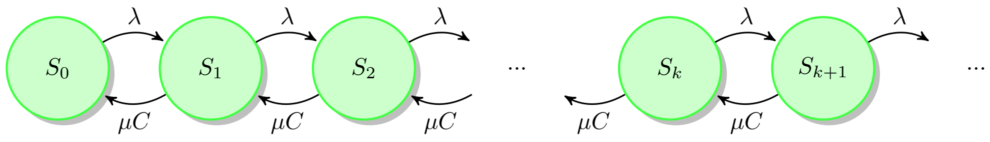

Figure 1: System diagram for the M/M/1 chain of exercise 5.1

Finally, the state distribution is given as

$$P_k = (1-\rho)\rho^k.$$

We, now, derive the average number of messages in the system, using the state distribution:

$$\overline{N} = \sum_{k=0}^{\infty} kP_k = \sum_{k=0}^{\infty} k(1-\rho)\rho^k = (1-\rho)\rho \sum_{k=0}^{\infty} k\rho^{k-1} =$$

$$= (1-\rho)\rho \sum_{k=0}^{\infty} \frac{d\rho^k}{d\rho} = (1-\rho)\rho \frac{d\left(\sum_{k=0}^{\infty} \rho^k\right)}{d\rho} = (1-\rho)\rho \frac{d(1/(1-\rho))}{d\rho} = \frac{\rho}{1-\rho}.$$

In order to solve the first question we can use the LITTLE's formula:

$$\overline{N} = \lambda_{\text{eff}} E[T_{total}] \to E[T_{\text{total}}] = \frac{\overline{N}}{\lambda} = \frac{\rho/(1-\rho)}{\lambda} = \frac{\lambda/(\mu C)/(1-\lambda/(\mu C))}{\lambda},$$

since $\lambda_{eff} = \lambda$, so, finally,

$$E[T_{total}] = \frac{1}{(\mu C) - \lambda}.$$

The minimum required C is determined by:

$$\frac{1}{\mu C - \lambda} \leq T_0 \to \mu C - \lambda \geq T_0^{-1} \to C \geq \frac{\lambda + T_0^{-1}}{\mu}.$$

2 Exercise 5.5

Consider a queuing system with a single server. The arrival events can be modeled with Poisson distribution, but two customers arrive at the system at each arrival event. Each customer requires an exponentially distributed service time.

1. Draw the state diagram
2. Determine p_k using local balance equations
3. Let $P(z) = \sum_{k=0}^{\infty} z^k p_k$. Calculate $P(z)$ for the system. Note, that $P(z)$ must be finite for $|z| < 1$, and we know $P(1) = 1$.

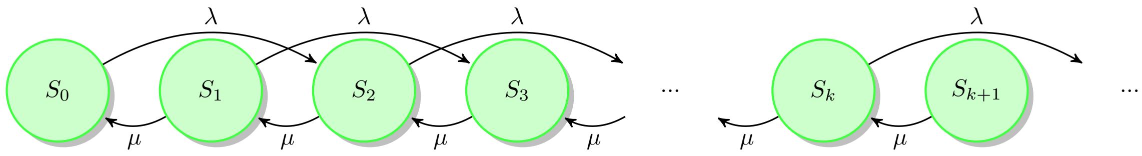

Figure 2: System diagram for the M/M/1 chain of exercise 5.5

4. Calculate the mean number of customers in the system with the help of $P(z)$ and compare it with the one of the M/M/1 system.

Solution: The system can be described by an M/M/1 model, since there is a single server, the service times are exponential service and the arrival process is Poisson. We must notice, however, that this Poisson Process models arrival events, but the events consist of two customer arrivals. (The departure events are still one-by-one, though.)

As always, for a Markovian System we must guarantee that all transitions are exponential. We define the usual state space: S_k : k customers in the system. Then, the state diagram is straightforward. Special care must be taken on determining the transitions and rates from state to state.

$$\textbf{Departure rate} = \mu$$
$$\textbf{Arrival Event rate} = \lambda$$

Clearly, the average customer arrival rate is 2λ and is NOT Poisson! What IS Poisson is the group arrival rate. We also DEFINE $\rho = \frac{\lambda}{\mu}$. This is neither the offered nor the actual load. We just use ρ to define this fraction.
The system diagram is given in Fig. 2.
Local Balance Equations:

$$\lambda P_0 = \mu P_1$$
$$\lambda P_{k-2} + \lambda P_{k-1} = \mu P_k, \quad k \geq 2$$

We can go ahead and solve them numerically. Alternatively, we can use the ZT methodology, since we only want to compute the average number of customers.

We consider the parametric local balance equation:

$$\mu P_k = \lambda P_{k-1} + \lambda P_{k-2} \rightarrow$$

$$\rightarrow \sum_{k=2}^{\infty} z^k \mu P_k = \sum_{k=2}^{\infty} z^k (\lambda P_{k-1} + \lambda P_{k-2})$$

$$\rightarrow \mu(P(z) - zP_1 - P_0) = \sum_{k=2}^{\infty} \lambda z^k P_{k-1} + \sum_{k=2}^{\infty} \lambda z^k P_{k-2}$$

$$\rightarrow \mu(P(z) - zP_1 - P_0) = \lambda z \sum_{k=2}^{\infty} \lambda z^{k-1} P_{k-1} + \lambda z^2 \sum_{k=2}^{\infty} \lambda z^{k-2} P_{k-2}$$

$$\rightarrow \mu(P(z) - zP_1 - P_0) = \lambda z(P(z) - P_0) + \lambda z^2 P(z)$$

We solve the equation with respect to $P(z)$

$$P(z) = \frac{\mu P_0 + \mu z P_1 - \lambda z P_0}{\mu - \lambda z - \lambda z^2} = \frac{P_0 + zP_1 - \rho z P_0}{1 - \rho z - \rho z^2}. \tag{1}$$

We need to apply two conditions that HOLD, in order to determine the unknown terms above. The first condition comes from the balance equation that we did not consider. We replace $P_1 = \rho P_0$ in (1), and obtain:

$$P(z) = \frac{P_0}{1 - \rho z - \rho z^2}. \tag{2}$$

The second condition comes from the NORMALIZATION in the probability or in the Z-domain:

$$\sum_{k=0}^{\infty} P_k = 1, \quad \text{or,} \quad P(z=1) = 1.$$

Replacing that in (2) we obtain $P_0 = 1 - 2\rho$, so finally

$$P(z) = \frac{1 - 2\rho}{1 - \rho z - \rho z^2} \tag{3}$$

Finally, we need to compute the mean number of customers. We have

$$\overline{N} = \left[\frac{dP(z)}{dz}\right]_{z=1}.$$

Proof:

$$\left[\frac{dP(z)}{dz}\right]_{z=1} = \left[\frac{d\sum_{k=0}^{\infty} z^k P_k}{dz}\right]_{z=1} = \left[\sum_{k=0}^{\infty} k z^{k-1} P_k\right]_{z=1} = \sum_{k=0}^{\infty} k P_k = \overline{N}.$$

So, this is what we will do. We differentiate the derived ZT in (3):

$$\frac{dP(z)}{dz} = \frac{(-1)(1-2\rho)(-\rho - 2\rho z)}{(1 - \rho z - \rho z^2)^2}$$

Replacing $z = 1$ we obtain

$$\overline{N} = \frac{3\rho}{1 - 2\rho} = \frac{3\lambda}{\mu - 2\lambda}.$$

The typical M/M/1 system with the same average customer arrival rate (2λ) and service rate (μ) has $\overline{N}_{M/M/1} = \frac{\rho}{1-\rho}$, where ρ is its offered load, and is equal to $\rho = 2\lambda/\mu$. So, finally,

$$\overline{N}_{M/M/1} = \frac{2\lambda}{\mu - 2\lambda}$$

so it is different, and, actually, less. Why?

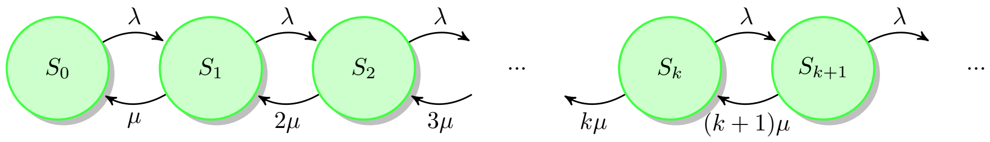

Figure 3: System diagram for the M/M/1 chain of exercise 5.6

3 Exercise 5.6

A queuing system has one server and infinite queuing capacity. The number of customers in the system can be modeled as a birth-death process with $\lambda_k = \lambda$ and $\mu_k = k\mu$, $k = 0, 1, 2, ...$ thus, the server increases the speed of the service with the number of customers in the queue. Calculate the average number of customers in the system as a function of $\rho = \lambda/\mu$.

Solution: The system is an M/M/1 queue, since it has infinite buffer, 1 server, and Markovian arrival and departure process. However, as we can see, it is not a typical M/M/1 case, as the service rates depend on the current system state. The system diagram is shown in Fig. 3. We need to solve the system of balance equations:

$$\begin{aligned}
&\lambda P_0 = \mu P_1 \rightarrow P_1 = \rho P_0 \\
&\lambda P_1 = 2\mu P_2 \rightarrow P_2 = \tfrac{1}{2}\rho P_1 = \tfrac{1}{2}\rho^2 P_0 \\
&\lambda P_2 = 3\mu P_3 \rightarrow P_3 = \tfrac{1}{3}\rho P_2 = \tfrac{1}{2\cdot 3}\rho^3 P_0 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\lambda P_{k-1} = k\mu P_k \rightarrow P_k = \tfrac{1}{k}\rho P_{k-1} = ... = \tfrac{1}{k!}\rho^k P_0 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\textstyle\sum_{k=0}^{\infty} P_k = 1 \quad \text{(normalization)}
\end{aligned}$$

From the last general equation and the normalization equation we obtain the state distribution:

$$\sum_{k=0}^{\infty} \frac{\rho^k}{k!} P_0 = 1 \rightarrow P_0 e^{\rho} = 1 \rightarrow P_0 = e^{-\rho}.$$

so finally, for each k

$$P_k = \frac{\rho^k}{k!} e^{-\rho}$$

so the state distribution is POISSON! Then, we can calculate the average number of customers from the state distribution

$$\overline{N} = \sum_{k=0}^{\infty} k P_k = \rho$$

or simply say that the average is ρ, from the Poisson distribution.
From LITTLE we can, also, calculate the average system time

$$E[T_{total}] = \frac{\overline{N}}{\lambda} = \frac{1}{\mu}.$$

This means that the arriving customers only stay in the system for an average time equal to the service time![2]

[2]This is equivalent to the case where there is no queue and each customer is served in parallel with the others, so actually this system is equivalent to an M/M/∞ system!