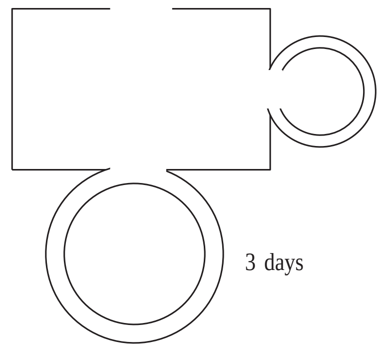
1. Repetition – probability theory and transforms

Ex. 1.1.

$\bar{S}$ – time to get out of the prison

$E(\bar{S}) = ?$



$$E(\bar{S}) = \sum_{i=1}^{3} E(\bar{S}|\text{door } i)P(\text{door } i)$$

$$P(\text{door } i) = \frac{1}{3}$$

$$E(\bar{S}|\text{door } 1) = 0$$

$$E(\bar{S}|\text{door } 2) = 1 + E(\bar{S})$$

$$E(\bar{S}|\text{door } 3) = 3 + E(\bar{S})$$

$$E(\bar{S}) = \frac{1}{3}(1+E(\bar{S}))+\frac{1}{3}(3+E(\bar{S})) = \frac{4}{3}+\frac{2}{3}E(\bar{S})$$

$$E(\bar{S}) = 4$$

Ex. 1.2.

$$\lambda_1 = 0.1,\ \lambda_2 = 0.02,\ a = 0.2\, b = 0.8$$

$$a\frac{1}{\lambda_1} + b\frac{1}{\lambda_2} = 42$$

$$a\frac{2}{\lambda_1^2} + b\frac{2}{\lambda_2^2} = 4040$$

$$a \cdot d_1 + b \cdot d_2 = 42$$

$$a \cdot d_1^2 + b \cdot d_2^2 = 4040$$

$$0.2 \cdot d_1 + 0.8 \cdot d_2 = 42$$

$$0.2 \cdot d_1^2 + 0.8 \cdot d_2^2 = 4040$$

$$0.2\left(\frac{42 - 0.8 \cdot d_2}{0.2}\right)^2 + 0.8 \cdot d_2^2 = 4040$$

$$d_2^2 - 84 \cdot d_2 + 1195 = 0 \qquad \Rightarrow \qquad d_{2/1} = 65.85, \qquad \underline{d_{2/2} = 18.15}$$

$$d_1 = \frac{42 - 0.8 \cdot d_2}{0.2} = \frac{42 - 0.8 \cdot 18.15}{0.2} = 137.41$$

Ex. 1.3.

a)

$$\sum_{k=0}^{\infty} P_k = \sum_{k=0}^{\infty} \frac{a^k}{k!}e^{-a} = e^{-a}\sum_{k=0}^{\infty}\frac{a^k}{k!} = e^{-a}e^{a} = 1$$

b)

$$P(z) = \sum_{k=0}^{\infty} z^k \frac{a^k}{k!}e^{-a} = e^{-a}\sum_{k=0}^{\infty}\frac{(za)^k}{k!} = e^{-a}e^{az} = e^{-a(1-z)}$$

c)

$$E\{X(X-1)\cdot \ldots \cdot (X-r+1)\} = \sum_{k=0}^{\infty} k(k-1)\cdot \ldots \cdot (k-r+1)P_k =$$

$$= \sum_{k=0}^{\infty} k(k-1)\cdot \ldots \cdot (k-r+1)\frac{a^k}{k!}e^{-a} = a^r e^{-a} \sum_{k=0}^{\infty} \frac{a^{(k-r)}}{(k-r)!} = a^r e^{-a} e^a = a^r \qquad (r = 1, 2, \ldots)$$

$$E(X) = a$$

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = E\{X(X-1)\} + E(X) - E(X)^2 = a^2 + a - a^2 = a$$

$$\frac{\mathrm{d}^r P(z)}{\mathrm{d}z^r} = \sum_{k=0}^{\infty} k(k-1)\cdot \ldots \cdot (k-r+1) P_k z^{(k-r)}$$

$$E\{X(X-1)\cdot \ldots \cdot (X-r+1)\} = \lim_{z\to 1} \frac{\mathrm{d}^r P(z)}{\mathrm{d}z^r} \qquad (r = 1, 2, \ldots)$$

$$\frac{\mathrm{d}^r P(z)}{\mathrm{d}z^r} = \frac{\mathrm{d}^r}{\mathrm{d}z^r}\left\{e^{-a(1-z)}\right\} = a^r e^{-a(1-z)} \qquad (r = 0, 1, \ldots)$$

$$\lim_{z\to 1} \frac{\mathrm{d}^r P(z)}{\mathrm{d}z^r} = a^r \qquad (r = 0, 1, \ldots)$$

$$E(X) = a$$

$$E\{X(X-1)\} = a^2; \quad \mathrm{Var}(X) = a; \quad E\{X(X-1)\cdot \ldots \cdot (X-r+1)\} = a^r \qquad (r = 1, 2, \ldots)$$

Ex. 1.4.

$$P(z) = E\left\{z^X\right\} = \sum_{k=0}^{\infty} z^k P(X=k) = E\left\{z^{X_1+X_2+\cdots+X_n}\right\} = E\left\{z^{X_1}\cdot z^{X_2}\cdot \ldots \cdot z^{X_n}\right\} =$$

$$E\left\{z^{X_1}\right\}\cdot E\left\{z^{X_2}\right\}\cdot \ldots \cdot E\left\{z^{X_n}\right\}$$

$$E\left\{z^{X_i}\right\} = \sum_{k=0}^{\infty} z^k P(X_i = k) = \sum_{k=0}^{\infty} z^k \frac{{a_i}^k}{k!} e^{-a_i} = e^{-a_i(1-z)} \qquad (r = 1, 2, \ldots)$$

$$P(z) = e^{-a_1(1-z)}\cdot e^{-a_2(1-z)}\cdot \ldots \cdot e^{-a_n(1-z)} = e^{-a(1-z)}$$

If $a = \sum_{i=1}^{n} a_i$, X is Poisson distributed with parameter a.

Ex. 1.5.

a)

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left\{1 - e^{-ax}\right\} = \begin{cases} ae^{-ax} & x > 0 \\ 0 & x \le 0 \end{cases}$$

b)

$$\overline{F}(x) = P(X > x) = 1 - P(X \le x) = 1 - F(x)$$

$$\overline{F}(x) = \begin{cases} 1 & x < 0 \\ ae^{-ax} & x \ge 0 \end{cases}$$

c)

$$F^*(s) = E(e^{-sX}) = \int_0^{\infty} e^{-sx} a e^{-ax}\mathrm{d}x = \frac{a}{s+a} \qquad (Re(s) > -a)$$

d)

$$E\{X^0\} = \int_0^{\infty} f(x)\mathrm{d}x = \int_0^{\infty} ae^{-ax}\mathrm{d}x = -e^{-ax}\Big|_0^{\infty} = 0 + 1 = 1$$

$$E\{X^k\} = \int_0^\infty x^k a e^{-ax} \mathrm{d}x = (x^k) - e^{-ax}\Big|_0^\infty + k\int_0^\infty x^{k-1} e^{-ax}\mathrm{d}x =$$

$$= 0 + 0 + \frac{k}{a} E X^{k-1} \qquad (k = 1, 2, \ldots)$$

$$E\{X^k\} = \frac{k}{a} \cdot \frac{k-1}{a} \cdot \ldots \cdot \frac{1}{a} = \frac{k!}{a^k} \qquad (k = 0, 1, \ldots)$$

$$E\{X\} = \frac{1}{a}; \quad \sigma^2 = \mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{2}{a^2} - \frac{1}{a^2} = \frac{1}{a^2}$$

$$\sigma = \sqrt{\mathrm{Var}(X)} = \frac{1}{a}; \quad C = \frac{\sigma}{m} = 1$$

$$F^*(s) = E(e^{-sX}) = \int_0^\infty e^{-sx} f(x)\mathrm{d}x \quad \Rightarrow \quad \frac{\mathrm{d}^k F^*(s)}{\mathrm{d}s^k} = (-1)^k \int_0^\infty x^k e^{-sx} f(x)\mathrm{d}x$$

$$E\{X^k\} = (-1)^k \lim_{s\to 0} \frac{\mathrm{d}^k F^*(s)}{\mathrm{d}s^k} \qquad (k = 0, 1, \ldots)$$

$$\frac{\mathrm{d}^k F^*(s)}{\mathrm{d}s^k} = \frac{\mathrm{d}^k}{\mathrm{d}s^k}\left\{\frac{a}{s+a}\right\} = \frac{(-1)^k a k!}{(s+a)^{k+1}} \qquad (k = 0, 1, \ldots)$$

$$E\{X^k\} = (-1)^k \cdot (-1)^k \cdot \frac{k!}{a^k} = \frac{k!}{a^k} \qquad (k = 0, 1, \ldots)$$

$$E\{X\} = \frac{1}{a}; \quad \sigma^2 = \mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{1}{a^2}$$

$$\sigma = \sqrt{\mathrm{Var}(X)} = \frac{1}{a}; \quad C = \frac{\sigma}{m} = 1$$

Ex. 1.6.

a)

$$\overline{F}_X(x) = P(X > x) = P\{X_1, X_2, \ldots, X_n > x\} = P\{X_1 > x, X_2 > x, \ldots, X_n > x\} =$$

$$= \{\text{because } X_i \text{ are independent}\} = P\{X_1 > x\} \cdot P\{X_2 > x\} \cdot \ldots \cdot P\{X_n > x\} =$$

$$= \prod_{i=1}^{n} \overline{F}_{X_i}(x) = \prod_{i=1}^{n} e^{-ax} = e^{-nax} \Rightarrow X \text{ is exponentially distributed with mean } \frac{1}{na}$$

$$F_X(x) = P(X \le x) = 1 - e^{-nax} \qquad (x \ge 0)$$

b)

$$F_X(x) = P(X \le x) = P\{\max(X_1, X_2, \ldots, X_n) \le x\} = P\{X_1 \le x, X_2 \le x, \ldots, X_n \le x\} =$$

$$= \{\text{because } X_i \text{ are independent}\} = P\{X_1 \le x\} \cdot P\{X_2 \le x\} \cdot \ldots \cdot P\{X_n \le x\} =$$

$$= \prod_{i=1}^{n} F_{X_i}(x) = (1 - e^{-ax})^n$$

$$\overline{F}_X(x) = P(X > x) = 1 - (1 - e^{-ax})^n \qquad (x \ge 0)$$

Ex. 1.7.

a)

$$F^*(s) = E(e^{-sX}) = \int_0^\infty e^{-sx} f(x)\mathrm{d}x = E\{e^{-s(X_1 + X_2 + \cdots + X_r)}\} =$$

$$= \{\text{because } X_i \text{ are independent}\} = E\{e^{-sX_1}\} \cdot E\{e^{-sX_2}\} \cdot \ldots \cdot E\{e^{-sX_r}\} =$$

$$= \frac{a}{s+a} \cdot \frac{a}{s+a} \cdot \ldots \cdot \frac{a}{s+a} = \left(\frac{s}{s+a}\right)^r \qquad (r = 1, 2, \ldots;\ Re(s) > -a)$$

We recognize that the above expression is the Laplace transform of the Erlang-r distribution:

$$f(x) = a \cdot \frac{(ax)^{r-1}}{(r-1)!} \cdot e^{-ax} \qquad (x > 0;\ r = 1, 2, \ldots)$$

$$\overline{F}(x) = \int_x^\infty f(u)\mathrm{d}u = \int_x^\infty a\frac{(an)^{r-1}}{(r-1)!}e^{-an}\mathrm{d}u =$$

$$= \Big| \left(\frac{(an)^{r-1}}{(r-1)!}\right)\left(-e^{-an}\right)\Big|_x^\infty + \int_x^\infty a\frac{(an)^{r-2}}{(r-2)!}e^{-an}\mathrm{d}u =$$

$$= \frac{(ax)^{r-1}}{(r-1)!}e^{-ax} + \Big| \left(\frac{(an)^{r-2}}{(r-2)!}\right)\left(-e^{-an}\right)\Big|_x^\infty + \int_x^\infty a\frac{(an)^{r-3}}{(r-3)!}e^{-an}\mathrm{d}u =$$

$$= \frac{(ax)^{r-1}}{(r-1)!}e^{-ax} + \frac{(ax)^{r-2}}{(r-2)!}e^{-ax} + \cdots + \frac{(ax)^1}{1!}e^{-ax} + \int_x^\infty ae^{-an}\mathrm{d}u =$$

$$= \sum_{j=0}^{r-1}\frac{(ax)^j}{j!}e^{-ax} \qquad (x \le 0)$$

$$F(x) = 1 - \overline{F}(x) = 1 - \sum_{j=0}^{r-1}\frac{(ax)^j}{j!}e^{-ax} = \sum_{j=r}^{\infty}\frac{(ax)^j}{j!}e^{-ax} \qquad (x \le 0)$$

$$m = E\{X\} = -\lim_{s\to 0}\frac{\mathrm{d}F^*(s)}{\mathrm{d}s} = -\lim_{s\to 0}\left\{ r\left(\frac{a}{s+a}\right)^{r-1} \cdot \frac{-a}{(s+a)^2}\right\} = \frac{r}{a}$$

$$E\{X^2\} = \lim_{s\to 0}\frac{\mathrm{d}^2F^*(s)}{\mathrm{d}s^2} = \lim_{s\to 0}\left\{\frac{\mathrm{d}}{\mathrm{d}s}\left[-\frac{r}{a}\left(\frac{a}{s+a}\right)^{r+1}\right]\right\} =$$

$$= \lim_{s\to 0}\left\{\frac{r(r+1)}{a^2}\left(\frac{a}{s+a}\right)^r\right\} = \frac{r(r+1)}{a^2}$$

$$\sigma^2 = \mathrm{Var}(X) = \frac{r(r+1)}{a^2} - \frac{r^2}{a^2} = \frac{r}{a^2} \quad \Rightarrow \quad \sigma = \frac{\sqrt{r}}{a}$$

$$C = \frac{\sigma}{m} = \frac{1}{\sqrt{r}} \qquad (r = 1, 2, \dots)$$

Alternative method:

$$m = E(X) = \int_0^\infty xf(x)\mathrm{d}x = \frac{r}{a}\int_0^\infty a\frac{(ax)^r}{r!}e^{-ax}\mathrm{d}x = \frac{r}{a}\cdot 1 = \frac{r}{a}$$

$$E(X^2) = \int_0^\infty x^2 f(x)\mathrm{d}x = \frac{(r+1)r}{a^2}\int_0^\infty a\frac{(ax)^{r+1}}{(r+1)!}e^{-ax}\mathrm{d}x = \frac{(r+1)r}{a^2}\cdot 1 = \frac{(r+1)r}{a^2}$$

$$\sigma^2 = \frac{r}{a^2};\ \sigma = \frac{\sqrt{r}}{a};\ C = \frac{1}{\sqrt{r}}$$

Ex. 1.8.

The memoryless property of the exponential distribution

$$P(\tilde{t} \le t) = 1 - e^{-\lambda t}, \qquad t \ge 0$$

$$P(\tilde{t} \le t + t_0 | \tilde{t} > t_0) = \frac{P[t_0 < \tilde{t} \le t + t_0]}{P[\tilde{t} > t_0]} = \frac{P[\tilde{t} \le t + t_0] - P[\tilde{t} \le t_0]}{P[\tilde{t} > t_0]} =$$

$$= \frac{1 - e^{-\lambda(t+t_0)} - (1 - e^{-\lambda t_0})}{1 - (1 - e^{-\lambda t_0})} = 1 - e^{-\lambda t}$$