1. Repetition – probability theory and transforms

1.1. A prisoner is kept in a cell with three doors. Through one of them he can get out of the prison. The other one leads to a tunnel: through this he is back to the cell after one day of walking. The third door leads to another tunnel, through which it takes the prisoner three days to get back to the cell. Assume that the prisoner selects a door randomly every time he gets back to his cell. Calculate the expected time it takes for the prisoner to get out of the prison.

1.2. X is a random variable chosen from X_1 with probability *a* and from X_2 with probability *b*. Calculate E[X] and σ_X for a=0.2, b=0.8, X_1 is an exponentially distributed r.v. with parameter λ_1=0.1 and X_2 is an exponentially distributed r.v. with parameter λ_2=0.02. Let the r.v. Y be chosen from D_1 with probability *a* and from D_2 with probability *b*, where D_1 and D_2 are deterministic r.v.s. Calculate the values D_1 and D_2 so that E[Y]=E[X] and $\sigma_Y=\sigma_X$.

1.3. X is a discrete stochastic variable, $p_k = P(X = k) = \frac{a^k}{k!}e^{-a}$, (k=0,1,2…), and *a* is a positive constant.

a) Prove that $\sum_{k=0}^{\infty} p_k = 1$.

b) Determine the z-transform (generating function) $P(z) = \sum_{k=0}^{\infty} z^k p_k$.

c) Calculate E(X), Var(X), and E{X(X-1)…(X-r+1)}, r=1,2…., with and without using z-transforms.

1.4. X_i's are Poisson distributed random variables, thus $p_k = P(X_i = k) = \frac{a_i^{\,k}}{k!}e^{-a_i}$, (k=0,1,2…) and a_i's are a positive constant (i=1,2,…n). X_1, X_2, … X_n are assumed to be independent. Give the probability distribution function of $X = \sum_{i=1}^{n} X_i$.

1.5. X is a positive continuous stochastic variable with probability distribution function (PDF)

$F(x) = P(X \le x) = \begin{cases} 0 & (x < 0) \\ 1 - e^{-ax} & (x \ge 0) \end{cases}$, where *a* is a positive constant.

a) Give the probability density function $f(x) = \frac{dF(x)}{dx}$.

b) Give $\overline{F}(x) = P(X > x)$.

c) Calculate $F^*(s) = E(e^{-sx}) = \int_0^{\infty} e^{-sx} f(x)dx.$

d) Calculate the expected values m=E(X), $E(X^k)$ (k=0,1,…),
the variance $\sigma^2 = Var(X) = E\{(X-m)^2\}$,
the standard deviation $\sigma = \sqrt{Var(X)}$ and
the coefficient of variation $c = \frac{\sigma}{m}$, with and without the transform F*(s).

1.6. X_i's are independent, exponentially distributed random variables with a mean value of 1/a (a>0) (i=1,2,…n). Calculate $P(X \le x)$ and $P(X \ge x)$, where

a) X=min(X_1, X_2,…X_n)

b) $X=\max(X_1, X_2,\ldots X_n)$.

1.7. X_i's are independent, exponentially distributed random variables with a mean value of 1/a (a>0) (i=1,2,…n).

a) Determine the distribution of $X = \sum_{i=1}^{r} X_i$.

b) Give f(x), the probability density function (pdf) of X;
F(X), the probability distribution function (PDF) of X;
the function $\overline{F}(x) = P(X > x)$ and
the Laplace transform of X.

c) Calculate the mean value m=E(X),
the variance $\sigma^2 = Var(X)$,
the standard deviation σ and
the coefficient of variation $c = \sigma/m$.

1.8. Prove the memoryless property of the exponential distribution