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Mats Bengtsson/Björn Ottersten

Homework # 1

1. Consider the following block matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $|A| \neq 0$, $|D| \neq 0$ ($|\cdot|$ denotes the determinant).

   a) Derive the block UDL factorization of

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

   i.e. find X, Y and Δ_1, Δ_2 such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix} .$$

   b) Derive the inverse of

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} .$$

   c) Show that

$$\left| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right| = \left| A - BD^{-1}C \right| |D| .$$

2. Let

$$\begin{bmatrix} x \\ y \end{bmatrix} \sim N\left(\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}, \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix} \right) .$$

Show that the conditional pdf, $f_{X|Y}(x|y)$, is Gaussian with mean

$$\bar{x} + \Sigma_{xy}\Sigma_{yy}^{-1}(y - \bar{y})$$

and covariance

$$\Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx} .$$

The results from 1. are useful.

3. Suppose X and Y are jointly distributed random variables. When Y is unknown, an intelligent estimate of the value of X is

$$\bar{x} = \mathrm{E}\{X\} \ .$$

This estimate has the property that

$$\mathrm{E}\{\|X - \bar{x}\|^2\} \leq \mathrm{E}\{\|X - z\|^2\}$$

for all z, and has average error $\mathrm{E}\{\|X - \bar{x}\|^2\}$. Now suppose that one is told that $Y = y$. Let $\hat{x} = \mathrm{E}\{X|Y = y\}$. Show that

$$\mathrm{E}\{\|X - \hat{x}\|^2|Y = y\} = \mathrm{E}\{\|X - \bar{x}\|^2|Y = y\} - \|\hat{x} - \bar{x}\|^2 \ .$$

Conclude the intuitively reasonable result that the mean estimation error $\mathrm{E}\{\|X - \hat{x}\|^2\}$ averaged over all values of X and Y will be bounded above by $\mathrm{E}\{\|X - \bar{x}\|^2\}$. When will the bound be attained, i.e., when will there not be a strict improvement in the knowledge of X?

4. Given a zero-mean process $y(\cdot)$ and its covariance function, find the linear least-squares estimate of

$$\int_0^T y(t)dt$$

in terms of its values at the end points $y(0)$ and $y(T)$. When the covariance function is

$$e^{-\alpha|t|}$$

show that the estimate is

$$\frac{1}{\alpha} \tanh \frac{\alpha T}{2} \left(y(0) + y(T)\right) \ .$$

What is the estimate for small T?

5. Assume X and N are independent Gaussian random variables with means $\bar{x}$ and $\bar{n}$ and with covariance matrices Σ_x and Σ_n. Then $Y = X + N$ is Gaussian and $f_{X|Y}(x|y)$ is a Gaussian density function. Show that the associated conditional mean and covariance are

$$\Sigma_n(\Sigma_x + \Sigma_n)^{-1}\bar{x} + \Sigma_x(\Sigma_x + \Sigma_n)^{-1}(y - \bar{n})$$

and

$$\Sigma_x - \Sigma_x(\Sigma_x + \Sigma_n)^{-1}\Sigma_x = \Sigma_x(\Sigma_x + \Sigma_n)^{-1}\Sigma_n = (\Sigma_x^{-1} + \Sigma_n^{-1})^{-1} \ .$$

Find the joint density $f_{X,Y}(x, y)$ first and assume that the inverses exist.