EQ2310 Digital Communications

Collection of Problems

KTH Electrical Engineering

Introduction

This text constitutes a collection of problems for use in the course *EQ2310 Digital Communications* given at the KTH School for Electrical Engineering.

The collection is based on old exam and homework problems (partly taken from the previous closely related courses 2E1431 Communication Theory and 2E1432 Digital Communications), and on the collection "Exempelsamling i Moduleringsmetoder," KTH–TTT 1994. Several problems have been translated from Swedish, and some problems have been modified compared with the originals.

The problems are numbered and allocated to three different chapters corresponding to different subject areas. All problems are provided with answers and hints (incomplete solutions). Most problems are provided with complete solutions.

Chapter 1

Information Sources and Source Coding

1–1 Alice and Bob play tennis regularly. In each game the first to win three sets is declared winner. Let V denote the winner in an arbitrary game. Alice is a better skilled player but Bob is in better physical shape, and can therefore typically play better when the duration of a game is long. Letting p_s denote the probability that Alice wins a set s we can assume $p_s = 0.6 - s/50$. The winner $X_s \in \{A, B\}$ of a set s is noted, and for each game a sequence $X = X_1 X_2 \dots$ is obtained (e.g., $X = ABAA$ or $X = BABAB$).

(a) When is more information obtained regarding the final winner in a game: getting to know the total number of sets played or the winner in the first set?

(b) Assume the number of sets played S is known. How many bits (on average) are then required to specify X?

1–2 Alice (A) and Bob (B) play tennis. The winner of a match is the one who first wins three sets. The winner of a match is denoted by $W \in \{A, B\}$ and the winner of set number k is denoted by $S_k \in \{A, B\}$. Note that the total number of sets in a match is a stochastic variable, denoted K. Alice has better technique than Bob, but Bob is stronger and thus plays better in long matches. The probability that Alice wins set number k is therefore $p_k = 0.6 - k/50$. The corresponding probability for Bob is of course equal to $1 - p_k$. The outcomes of the sets in a game form a sequence, X. Examples of possible sequences are $X = ABAA$ and $X = BABAB$.

In the following, assume that a large number of matches have been played and are used in the source coding process, i.e., the asymptotic results in information theory are applicable.

(a) Bob's parents have promised to serve Bob and Alice dinner when the match is over. They are therefore interested to know how many sets Bob and Alice play in the match. How many bits are needed on the average to transfer that information via a telephone channel from the tennis stadium to Bob's parents?

(b) Suppose that Bob's parents know the winner, W, of the match, but nothing else about the game. How many bits are needed on the average to transfer the number of played sets, K, in the match to Bob's parents?

(c) What gives most information about the number of played sets, K, in a match, the winner of the third set or the winner of the match?

To ease the computational burden of the problem, the following probability density functions are given:

$f_{KW}(k, w)$		w		
		A	B	Sum
k	3	0.17539	0.08501	0.26040
	4	0.21540	0.15618	0.37158
	5	0.18401	0.18401	0.36802
	Sum	0.57480	0.42520	

$f_{WS_1}(w, s_1)$		w		
		A	B	Sum
s_1	A	0.42420	0.15580	0.58000
	B	0.15060	0.26940	0.42000
	Sum	0.57480	0.42520	

$f_{KS_2}(k, s_2)$		s_2		
		A	B	Sum
k	3	0.17539	0.08501	0.26040
	4	0.19567	0.17591	0.37158
	5	0.18894	0.17908	0.36802
	Sum	0.56000	0.44000	

$f_{KS_3}(k, s_3)$		s_3		
		A	B	Sum
k	3	0.17539	0.08501	0.26040
	4	0.18560	0.18597	0.37158
	5	0.17900	0.18902	0.36802
	Sum	0.54000	0.46000	

1–3 Two sources, A and B, with entropies H_{A} and H_{B}, respectively, are connected to a switch (see Figure 1.1). The switch randomly selects source A with probability λ and source B with probability $1 - \lambda$. Express the entropy of S, the output of the switch, as a function of H_{A}, H_{B}, and λ!

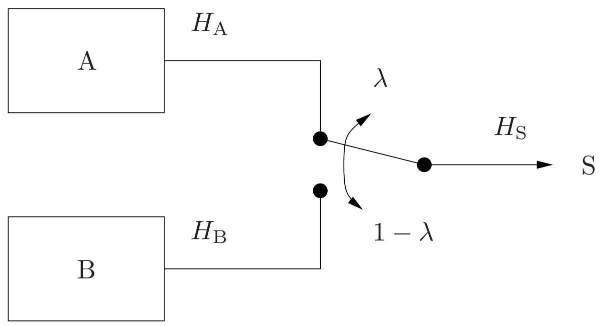


Figure 1.1: The source in Problem 1–3.

1–4 Consider a memoryless information source specified by the following table.

Symbol	Probability
s_0	0.04
s_1	0.25
s_2	0.10
s_3	0.21
s_4	0.20
s_5	0.15
s_6	0.05

(a) Is it possible to devise a lossless source code resulting in a representation at 2.50 bits/symbol?

(b) The source S is quantized resulting in a new source U specified below.

Old symbols	New symbol	Probability
$\{s_0, s_1\}$	u_0	0.29
$\{s_2, s_3, s_4\}$	u_1	0.51
$\{s_5, s_6\}$	u_2	0.20

How much information is lost in this process?

1–5 An iid binary source produces bits with probabilities $\Pr(0) = 0.2$ and $\Pr(1) = 0.8$. Consider coding groups of $m = 3$ bits using a binary Huffman code, and denote the resulting average codeword length L. Compute L and compare to the entropy of the source.

1–6 A source with alphabet $\{0, 1, 2, 3\}$ produces independent and equally likely symbols, with probabilities $P(0) = 1 - p$, $P(1) = P(2) = P(3) = p/3$. The source is to be compressed using lossless compression to a new representation with symbols from the set $\{0, 1, 2, 3\}$.

(a) Which is the lowest possible rate, in code symbols per source symbol, in lossless compression of the source?

(b) Assume that blocks of two source symbols are coded jointly using a Huffman code. Specify the code and the corresponding code rate in the case when $p = 1/10$.

(c) Consider an encoder that looks at a sequence produced by the source and divides the sequence into groups of source symbols according to the 16 different possibilities

$$\{1, 2, 3, 01, 02, 03, 001, 002, 003, 0001, 0002, 0003, 00000, 00001, 00002, 00003\}.$$

Then these are coded using a Huffman code. Specify this code and its rate.

(d) Consider the following sequence produced by the source.

$$10310000000300100000002000000000010030002030000000.$$

Compress the sequence using the Lempel-Ziv algorithm.

(e) Determine the compression ratio obtained for the sequence above when compressed using the two Huffman codes and the LZ algorithm.

1–7 English text can be modeled as being generated by a several different sources. A first, very simple model, is to consider a source generating sequences of independent symbols from a symbol alphabet of 26 letters. On average, how many bits per character are required for encoding English text according to this model? The probability of the different characters are found in Table 1.1. What would the result be if all characters were equiprobable?

Letter	Percentage
A	7.25
B	1.25
C	3.50
D	4.25
E	12.75
F	3.00
G	2.00
H	3.50
I	7.75
J	0.25
K	0.50
L	3.75
M	2.75
N	7.75
O	7.50
P	2.75
Q	0.50
R	8.50
S	6.00
T	9.25
U	3.00
V	1.50
W	1.50
X	0.50
Y	2.25
Z	0.25

Table 1.1: The probability of the letters A through Z for English text. Note that the total probability is larger than one due to roundoff errors (from Seberry/Pieprzyk, *Cryptography*).

1–8 Consider a discrete memoryless source with alphabet $\mathcal{S} = \{s_0, s_1, s_2\}$ and corresponding statistics $\{0.7, 0.15, 0.15\}$ for its output.

(a) Apply the Huffman algorithm to this source and show that the average codeword length of the Huffman code equals 1.3 bits/symbol.

(b) Let the source be extended to order two i.e., the outputs from the extended source are blocks consisting of 2 successive source symbols belonging to $\mathcal{S}$. Apply the Huffman algorithm to the resulting extended source and calculate the average codeword length of the new code.

(c) Compare and comment the average codeword length calculated in part (b) with the entropy of the original source.

1–9 The binary sequence

$$x_1^{24} = 000110011111110110111111$$

was generated by a binary discrete memoryless source with output X_n and probabilities $\Pr[X_n = 0] = 0.25$, $\Pr[X_n = 1] = 0.75$.

(a) Encode the sequence using a Huffman code for 2-bit symbols. Compute the average codeword length and the entropy of the source and compare these with each other and with the length of your encoded sequence. Conclusions?

(b) Repeat a) but use instead 3-bit symbols. Discuss the advantages/disadvantages of increasing the symbol length!

(c) Use the Lempel-Ziv algorithm to encode the sequence. Compare with the entropy of the source and the results of a) and b).

1–10 Consider a stationary and memoryless discrete source $\{X_n\}$. Each X_n can take on 8 different values, with probabilities

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64}$$

(a) Specify an instantaneous code that takes one source symbol X_n and maps this symbol into a binary codeword $c(X_n)$ and is optimal in the sense of minimizing the expected length of the code.

(b) To code sequences of source outputs one can employ the code from (a) on each consecutive output symbol. An alternative to this procedure would be to code length-N blocks of source symbols, X_1^N, into binary codewords $c(X_1^N)$. Is it possible to obtain a lower expected length (expressed in bits per source symbol) using this latter approach?

(c) Again consider coding isolated source symbols X_n into binary codewords. Specify an instantaneous code that is optimal in the sense of minimizing the length of the longest codeword.

(d) Finally, and still considering binary coding of isolated symbols, specify an instantaneous code optimal in the sense of minimizing the expected length subject to the constraint that no codeword is allowed to be longer than 4 bits.

1–11 Consider the random variables X, $Y \in \{1, 2, 3, 4\}$ with the joint probability mass function

$$\Pr(X = x, Y = y) = \begin{cases} 0 & x = y \\ K & \text{otherwise} \end{cases}$$

(a) Calculate numerical values for the entities $H(X), H(Y), H(X, Y)$ and $I(X; Y)$.

(b) Construct a Huffman code for outcomes of X, *given that* $Y = 1$ (and assuming that $Y = 1$ is known to both encoder and decoder).

(c) Is it possible to construct a uniquely decodable code with lower rate? Why?

1–12 So-called *run-length coding* is used e.g. in the JPEG image compression standard. This lossless source coding method is very simple, and works well when the source is skewed toward one of its outcomes. Consider for example a binary source $\{X_n\}$ producing i.i.d symbols 0 and 1, with probablity p for symbol 1; $p = \Pr(X_n = 1)$. Then run-length coding of a string simply counts the number of consequtive 1's that are produced before the next symbol 0 occurs. For example, the string

$$111011110$$

is coded as $3, 4$, that is "3 ones followed by a zero" and then "4 ones followed by a zero." The string 0 (one zero) is coded as 0 ("no ones"), so 011001110 gives $0, 2, 0, 3$, and so on. Since the indices used to count ones need to be stored with finite precision, assume they are stored using b bits and hence are limited to the integers $0, 1, \ldots, 2^b - 1$, letting the last index $2^b - 1$ mean "$2^b - 1$ ones have appeared and the next symbol may be another one; encoding needs to restart at next symbol." That is, with $b = 2$ the string 111110 is encoded as $3, 2$.

Let $p = 0.95$ and $b = 3$.

(a) What is the minimum possible average output rate (in code-bits per source-bit) for the i.i.d binary source with $p = 0.95$?

(b) How close to maximum possible compression is the resulting average rate of run-length coding for the given source?

1–13 A random variable uniformly distributed over the interval $[-1, +1]$ is quantized using a 4-bits uniform scalar quantizer. What is the resulting average quantization distortion?

1–14 A variable X with pdf according to Figure 1.2 is quantized using three bits per sample.

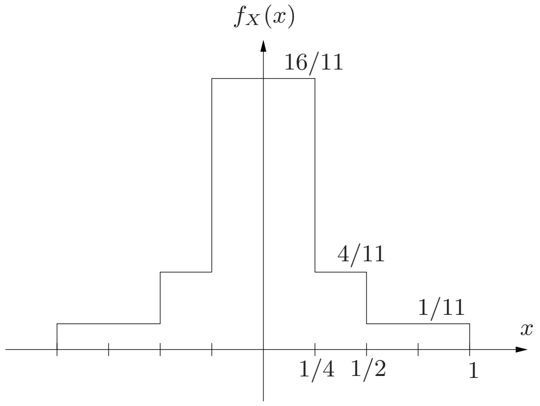


Figure 1.2: The probability density function of X.

Companding is employed in order to reduce quantization distortion. The compressor is illustrated in Figure 1.3. Compute the resulting distortion and compare the performance using companding with that obtained using linear quantization.

1–15 Samples from a speech signal are to be quantized using an 8-bit linear quantizer. The output range of the quantizer is $-V$ to V. The statistics of the speech samples can be modeled using the pdf

$$f(x) = \frac{1}{\sigma\sqrt{2}} \exp(-\sqrt{2}\,|x|/\sigma)$$

Since the signal samples can take on values outside the interval $[-V, V]$, with non-zero probability, the total distortion D introduced by the quantization can be written as

$$D = D_g + D_o$$

where D_g is the *granular* distortion and D_o is the *overload* distortion. The granular distortion stems from samples inside the interval $[-V, V]$, and the quantization errors in this interval, and the overload distortion is introduced when the input signal lies outside the interval $[-V, V]$. For a fixed quantization rate (in our case, 8 bits per sample) D_g is reduced and D_o is increased when V is decreased, and vice versa. The choice of V is therefore a tradeoff between granular and overload distortion.

Assume that V is chosen as $V = 4\sigma$. Which one of the distortions D_g and D_o will dominate?

1–16 A variable X with pdf

$$f_X(x) = \begin{cases} 2(1-|x|)^3, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

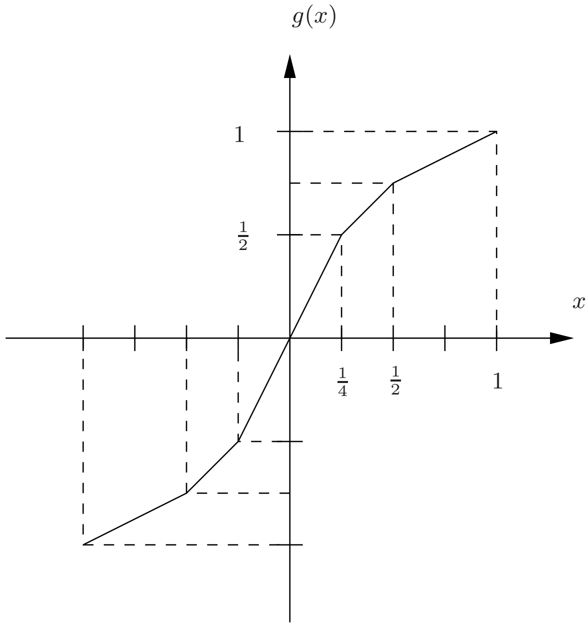


Figure 1.3: Compressor characteristics.

is to be quantized using a large number of quantization levels. Let D_{lin} be the distortion obtained using linear quantization and D_{opt} the distortion obtained with optimal companding. The optimal compressor function in this case is

$$g(x) = 1 - (1-x)^2, \quad x \geq 0$$

and $g(x) = -g(-x)$ for $x < 0$. Determine the ratio $D_{\text{opt}}/D_{\text{lin}}$.

1–17 A variable with a uniform distribution in the interval $[-V, V]$ is quantized using linear quantization with b bits per sample. Determine the required number of bits, b, to obtain a signal-to-quantization-noise ratio (SQNR) greater than 40 dB.

1–18 A PCM system works with sampling frequency 8 kHz and uses linear quantization. The indices of the quantization regions are transmitted using binary antipodal baseband transmission based on the pulse

$$p(t) = \begin{cases} A, & 0 \leq t \leq 1/R \\ 0, & \text{otherwise} \end{cases}$$

where the bitrate is $R = 64$ kbit/s. The transmission is subject to AWGN with spectral density $N_0/2 = 10^{-5}$ V^2/Hz. Maximum likelihood detection is employed at the receiver.

The system is used for transmitting a sinusoidal input signal of frequency 800 Hz and amplitude corresponding to fully loaded quantization. Let P_e be the probability that one or several of the transmitted bits corresponding to one source sample is detected erroneously. The resulting overall distortion can then be written

$$D = (1 - P_e)\, D_q + P_e\, D_e$$

where D_q is the quantization distortion in absence of transmission errors and D_e is the distortion obtained when transmission errors have occurred. A simple model to specify D_e is that when a transmission error has occurred, the reconstructed source sample is uniformly distributed over the range of the quantization and independent of the original source sample.

Determine the required pulse amplitude A to make the contribution from transmission errors to D less than the contribution from quantization errors.

1–19 A PCM system with linear quantization and transmission based on binary PAM is employed in transmitting a 3 kHz bandwidth speech signal, amplitude-limited to 1 V, over a system with 20 repeaters. The PAM signal is detected and reproduced in each repeater. The transmission between repeaters is subjected to AWGN of spectral density 10^{-10} W/Hz. The maximum allowable transmission power results in a received signal power of 10^{-4} W. Assume that the probability that a speech sample is reconstructed erroneously due to transmission errors must be kept below 10^{-4}. Which is then the lowest possible obtainable quantization distortion (distortion due to quantization errors alone, in the absence of transmission errors).

1–20 A conventional system (System 1) and an adaptive delta modulation system (System 2) are to be compared. The conventional system has step-size d, and the adaptive uses step-size $2d$ when the two latest binary symbols are equal to the present one and d otherwise. Both systems use the same updating frequency f_d.

(a) The binary sequence

$$0101011011011110100001000011100101$$

is received. Decode this sequence using both of the described methods.

(b) Compare the two systems when coding a sinusoid of frequency f_m with regard to the maximum input power that can be used without slope-overload distortion and the minimum input power that results in a signal that differs from the signal obtained with no input.

1–21 Consider a source producing independent symbols uniformly distributed on the interval $(-a, +a)$. The source output is quantized to four levels. Let X be a source sample and let $\hat{X}$ be the corresponding quantized version of X. The mapping of the quantizer is then defined according to

$$\hat{X} = \begin{cases} -\frac{3}{4}a & X \le -\gamma a \\ -\frac{1}{4}a & -\gamma a < X \le 0 \\ \frac{1}{4}a & 0 < X \le \gamma a \\ \frac{3}{4}a & \gamma a < X \end{cases}, \qquad 0 \le \gamma \le 1\,.$$

The four possible output levels of the quantizer can be coded "directly" using two bits per symbol. However, in grouping blocks of output symbols together a lower rate, in terms of average number of bits per symbols, can be achieved.

For which γ will it be *impossible* to code $\hat{X}$ at an average rate below 1.5 bits per symbol without loss?

1–22 Consider quantizing a zero-mean Gaussian variable X with variance $E[X^2] = \sigma^2$ using four-level linear quantization. Let $\hat{X}$ be the quantized version of X. Then the quantizer is defined according to

$$\hat{X} = \begin{cases} -3\sigma/2 & X \le -\sigma \\ -\sigma/2 & -\sigma < X \le 0 \\ \sigma/2 & 0 < X \le \sigma \\ 3\sigma/2 & X > \sigma \end{cases}$$

(a) Determine the average distortion $E[(X - \hat{X})^2]$.

(b) Determine the entropy $H(\hat{X})$.

(c) Assume that the thresholds defining the quantizer are changed to $\pm a\sigma$ (where we had $a = 1$ earlier). Which $a > 0$ maximizes the resulting entropy $H(\hat{X})$?

(d) Assume, finally, that the four possible values of $\hat{X}$ are coded according to $-3\sigma/2 \leftrightarrow 00$, $-\sigma/2 \leftrightarrow 01$, $\sigma/2 \leftrightarrow 10$ and $3\sigma/2 \leftrightarrow 11$, and then transmitted over a binary symmetric channel with crossover probability $q = 0.01$. Letting $\hat{Y}$ denote the corresponding output levels produced at the receiver, determine the mutual information $I(\hat{X}, \hat{Y})$.

1–23 A random variable X with pdf

$$f_X(x) = \frac{1}{2}\, e^{-|x|}$$

is quantized using a 3-level quantizer with output $\hat{X}$ specified according to

$$\hat{X} = \begin{cases} -b, & X < -a \\ 0, & -a \le X \le a \\ b, & X > a \end{cases}$$

where $0 \le a \le b$.

(a) Derive an expression for the resulting mean squared-error distortion $D = E[(X - \hat{X})^2]$ as a function of the parameters a and b.

(b) Derive the optimal values for a and b, minimizing the distortion D, and specify the resulting minimum value for D.

1–24 Consider a zero-mean Gaussian variable X with variance $E[X^2] = 1$.

(a) The variable X is quantized using a 4-level uniform quantizer with step-size Δ, according to

$$\hat{X} = \begin{cases} -3\Delta/2, & X < -\Delta \\ -\Delta/2, & -\Delta \le X < 0 \\ \Delta/2, & 0 \le X < \Delta \\ 3\Delta/2, & X \ge \Delta \end{cases}$$

Let Δ^* be the value of Δ that minimizes the distortion $D(\Delta) = E[(X - \hat{X})^2]$. Determine Δ^* and the resulting distortion $D(\Delta^*)$.

Hint: You can verify your results in Table 6.2 of the second edition of the textbook (Table 4.2 in the first edition).

(b) Let Δ^* be the optimal step-size from (a), and consider the quantizer

$$\hat{X} = \begin{cases} -\hat{x}_1, & X < -\Delta^* \\ -\hat{x}_2, & -\Delta^* \le X < 0 \\ \hat{x}_3, & 0 \le X < \Delta^* \\ \hat{x}_4, & X \ge \Delta^* \end{cases}$$

That is, a quantizer with arbitrary reproduction points, $\hat{x}_1, \dots, \hat{x}_4$, but with uniform encoding defined by the step-size Δ^* from (a). How much can the distortion be decreased, compared with (a), by choosing the reproduction points optimally?

(c) The quantizer in (a) is uniform in both encoding and decoding. The one in (b) has a uniform encoder. By allowing both the encoding and decoding to be non-uniform, further improvements on performance are possible. Optimal non-uniform quantizers for a zero-mean Gaussian variable X with $E[X^2] = 1$ are specified in Table 6.3 (in the second edition; Table 4.3 in the first edition). Consider the quantizer corresponding to $N = 4$ levels in the table. Verify that the specified quantizer fulfills the necessary conditions for optimality and compare the distortion it gives (as specified in the table) with the results from (a) and (b).

Some of the calculations involved in solving this problem will be quite messy. To get numerical results and to solve non-linear equations numerically, you may use mathematical software, like Matlab. Note that in parts of the problem it is possible to get closed-form results in terms of the Q-, erf-, or erfc-functions, where

$$Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt, \quad \mathrm{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad \mathrm{erfc}(x) \triangleq 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

The Q-function will be used frequently throughout the course. Unfortunately it is not implemented in Matlab. However, the erf- and erfc-functions are (as 'erf(x)' and 'erfc(x)'). Noting that

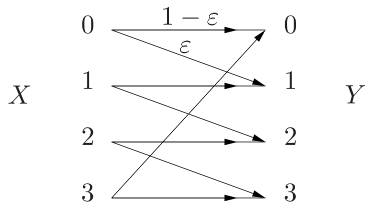
$$Q(x) = \frac{1}{2}\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right), \quad \mathrm{erfc}(x) = 2\, Q(\sqrt{2}\, x)$$

it is hence quite straightforward to get numerical values also for $Q(x)$ using Matlab.

Finally, as an additional hint regarding the calculations involved in this problem it is useful to note that

$$\int_x^\infty te^{-t^2/2}dt = e^{-x^2/2}, \qquad \int_x^\infty t^2 e^{-t^2/2}dt = xe^{-x^2/2} + \sqrt{2\pi}\, Q(x)$$

1–25 Consider the discrete memoryless channel depicted below.



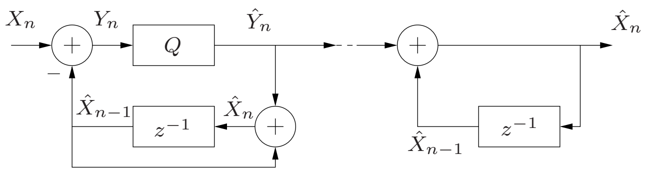
As illustrated, any realization of the input variable X can either be correctly received, with probability $1-\varepsilon$, or incorrectly received, with probability ε, as one of the other Y-values. (Assume $0 \le \varepsilon \le 1$.)

(a) Consider a random variable Z that is uniformly distributed over the interval $[-1, 1]$. Assume that Z is quantized using a uniform/linear quantizer with step-size $\Delta = 0.5$, and that the index generated by the quantizer encoder is transmitted as a realization of X and received as Y over the discrete channel (that is, $-1 \le Z < -0.5 \implies X = 0$ is transmitted and received as $Y = 0$ or $Y = 1$, and so on.) Assume that the quantizer decoder uses the received value of Y to produce an output $\hat{Z}$ (that is, $Y = 0 \implies \hat{Z} = -0.75$, $Y = 1 \implies \hat{Z} = -0.25$, and so on). Compute the distortion

$$D = E[(Z - \hat{Z})^2] \tag{1.1}$$

(b) Still assuming uniform *encoding*, but that you are allowed to change the reproduction points (the $\hat{Z}$'s). Which values for the different reproduction points would you choose to minimize D in the special case $\varepsilon = 0.5$

1–26 Consider the DPCM system depicted below.



Assume that the system is "started" at time $n = 1$, with initial value $\hat{X}_0 = 0$ for the output signal. Furthermore, assume that the X_n's, for $n = 1, 2, 3, \ldots$, are independent and identically distributed Gaussian random variables with $E[X_n] = 0$ and $E[X_n^2] = 1$, and that Q is a 2-level uniform quantizer with stepsize $\Delta = 2$ (i.e., $Q[X]$ = sign of X, and the DPCM system is in fact a delta-modulation system). Let

$$D_n = E[(X_n - \hat{X}_n)^2]$$

be the distortion of the DPCM system at time-instant n.

(a) Assume that

$$X_1 = 0.9,\ \ X_2 = 0.3,\ \ X_3 = 1.2,\ \ X_4 = -0.2,\ \ X_5 = -0.8$$

Which are the corresponding values for $\hat{X}_n$, for $n = 1, 2, 3, 4, 5$?

(b) Compute D_n for $n = 1$

(c) Compute D_n for $n = 2$

(d) Compare the results in (b) and (c). Is D_2 lower or higher than D_1? Is the result what you had expected? Explain!

1–27 A random variable X with pdf

$$f_X(x) = \frac{1}{2}e^{-|x|}$$

is quantized using a 4-level quantizer to produce the output variable $Y \in \{y_1, y_2, y_3, y_4\}$, with symmetric encoding specified as

$$Y = \begin{cases} y_1, & X < -a \\ y_2, & -a \le X < 0 \\ y_3, & 0 \le X < a \\ y_4, & X \ge a \end{cases}$$

where $a > 0$. The resulting distortion is

$$D = E[(X - Y)^2]$$

Let $H(Y)$ be the entropy of Y. Compute the minimum possible D under the simultaneous constraint that $H(Y)$ is maximized.

1–28 Let W_1 and W_2 be two independent random variables with uniform probability density functions (pdf's),

$$f_{W_1} = \begin{cases} \frac{1}{2}, & |W_1| \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$f_{W_2} = \begin{cases} \frac{1}{4}, & |W_2| \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

Consider the random variable $X = W_1 + W_2$

(a) Compute the differential entropy $h(X)$ of X

(b) Consider using a scalar quantizer of rate $R = 2$ bits per source symbol to code samples of the random variable X. The quantizer has the form

$$Q(X) = \begin{cases} a_1, & X \in (-\infty, -1] \\ a_2, & X \in (-1, 0] \\ a_3, & X \in (0, 1] \\ a_4, & X \in (1, \infty) \end{cases}$$

Find a_1, a_2, a_3 and a_4 that minimize the distortion

$$D = E[(X - Q(X))^2]$$

and evaluate the corresponding minimum value for D.

1–29 Consider quantizing a Gaussian zero-mean random variable X with variance 1. Before quantization, the variable X is clipped by a compressor $\mathcal{C}$. The compressed value $Z = \mathcal{C}(X)$ is given by:

$$\mathcal{C}(x) = \begin{cases} x, & |x| \le A \\ A, & \text{otherwise} \end{cases}$$

where A is a constant $0 < A < \infty$.

The variable Z is then quantized using a 2-bit symmetric quantizer, producing the output

$$Y = \begin{cases} -a, & -\infty < Z \le -c \\ -b, & -c < Z \le 0 \\ +b, & 0 < Z \le c \\ +a, & c < Z < \infty \end{cases}$$

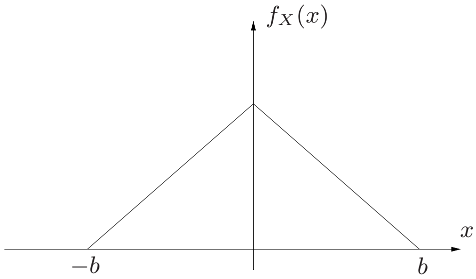
where $a > b$ and c are positive real numbers.

Consider optimizing the quantizer to minimize the mean square error between the quantized variable Y and the variable Z.

(a) Determine the pdf of Z.

(b) Derive a system of equations that describe the optimal solution for a, b and c in terms of the variable A. Simplify the equations as far as possible. (But you need not to solve them.)

1–30 A source $\{X_n\}$ produces independent and equally distributed samples X_n. The marginal distribution $f_X(x)$ of the source is illustrated below.



At time-instant n the sample $X = X_n$ is quantized using a 4-level quantizer with output $\hat{X}$, specified as follows

$$\hat{X} = \begin{cases} y_1, & X \leq -a \\ y_2, & -a < X \leq 0 \\ y_3, & 0 < X \leq a \\ y_4, & X > a \end{cases}$$

where $0 \leq a \leq b$. Now consider observing a long source sequence, quantizing each sample and then storing the corresponding sequence of quantizer outputs. Since there are 4 quantization levels, the variable $\hat{X}$ can be straightforwardly encoded using 2 bits per symbol. However, aiming for a more efficient representation we know that grouping long blocks of subsequent quantization outputs together and coding these blocks using, for example, a Huffman code generally results in a lower storage requirement (below 2 bits per symbol, on the average).

Determine the variables a and $y_1, \ldots, y_4$ as functions of the constant b so that the quantization distortion $D = E[(X - \hat{X})^2]$ is minimized under the constraint that it must be possible to losslessly store a sequence of quantizer output values $\hat{X}$ at rates above, but at no rates below, the value of 1.5 bits per symbol on the average.

1–31 (a) Consider a discrete memoryless source that can produce 8 different symbols with probabilities:

$$0.04, 0.07, 0.09, 0.1, 0.1, 0.15, 0.2, 0.25$$

For this source, design a binary Huffman code and compare its expected codeword-length with the entropy rate of the source.

(b) A random variable X with pdf

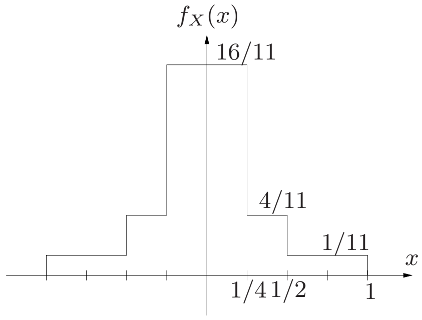
$$f_X(x) = \frac{1}{2}\, e^{-|x|}$$

is quantized using a 4-level quantizer with output $\hat{X}$. The operation of the quantizer can be described as

$$\hat{X} = \begin{cases} -3\Delta/2, & X < -\Delta \\ -\Delta/2, & -\Delta \leq X < 0 \\ \Delta/2, & 0 \leq X < \Delta \\ 3\Delta/2, & X \geq \Delta \end{cases}$$

Let Δ^* be the value of Δ that minimizes the expected distortion $D = E[(X - \hat{X})^2]$. Show that $1.50 < \Delta^* < 1.55$.

1–32 A variable X with pdf according to the figure below is quantized into eight reconstruction levels $\hat{x}_i,\ i = 0, \ldots, 7$, using uniform encoding with step-size $\Delta = 1/4$.



Denoting the resulting output $\hat{X} \in \{\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_7\}$, the operation of the quantizer can be described as

$$(i-4)\Delta \le X < (i-3)\Delta \Rightarrow \hat{X} = \hat{x}_i, \quad i = 0, 1, \ldots, 7$$

(a) Determine the entropy $H(\hat{X})$ of the discrete output variable $\hat{X}$.

(b) Design a Huffman code for the variable $\hat{X}$ and compute the resulting expected codeword length L. Answer, in addition, the following question: Why can a student who gets $L < H(\hat{X})$ be certain to have made an error in part (a), when designing the Huffman code or in computing L?

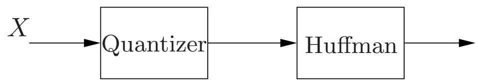
(c) Compute the resulting quantization distortion $D = E[(X - \hat{X})^2]$ when the output-levels $\hat{x}_i$ are set to be uniformly spaced according to $\hat{x}_i = (i - 7/2)\Delta, \quad i = 0, \ldots, 7$.

(d) Compute the resulting distortion $D = E[(X - \hat{X})^2]$ when the output-levels $\hat{x}_i$ are optimized to minimize D (for fixed uniform encoding with $\Delta = 1/4$ as before).

1–33 Consider the random variable X having a symmetric and piece-wise constant pdf, according to

$$f(x) = \frac{128}{255} 2^{-m}, \quad m-1 < |x| \le m$$

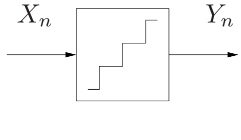
for $m = 1, 2, \ldots, 8$, and $f(x) = 0$ for $|x| > 8$. The variable is quantized using a uniform quantizer (i.e., uniform, or linear, encoding and decoding) with step-size $\Delta = 16/K$ and with $K = 2^k$ levels. The output-index of the quantizer is then coded using a binary Huffman code, according to the figure below.

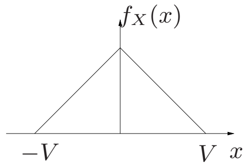


The output of the Huffman code is transmitted without errors, decoded at the receiver and then used to reconstruct an estimate $\hat{X}$ of X. Let L be the expected length of the Huffman code in bits per symbol and let $\bar{L}$ be L rounded off to the nearest integer (e.g $L = 7.3 \Rightarrow \bar{L} = 7$).

Compare the average distortion $E[(X - \hat{X})^2]$ of the system described above, with $k = 4$ bits and a resulting $\bar{L}$, with that of of a system that spends $\bar{L}$ bits on quantization and does not use Huffman coding (and instead transmits the $\bar{L}$ output bits of the quantizer directly). Notice that these two schemes give approximately the same transmission rate. That is, the task of the problem is to examine the performance gain in the quantization due to the fact that the resolution of the quantizer can be increased when using Huffman coding without increase in the resulting transmission rate.

1–34 The i.i.d continuous random variables X_n with pdf $f_X(x)$, as illustrated below, are quantized to generate the random variables Y_n $(n = -\infty \ldots \infty)$.





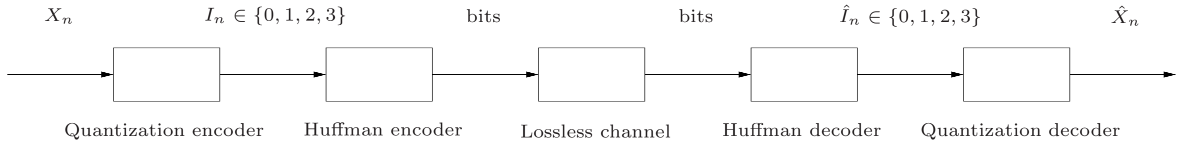
The quantizer is linear with quantization regions $[-\infty, -V/2)$, $[-V/2, 0)$, $[\,0, V/2)$ and $[V/2, \infty]$ and respective quantization points $y_0 = -3V/4$, $y_1 = -V/4$, $y_2 = V/4$ and $y_3 = 3V/4$.

(a) The approximation of the granular quantization noise $D_g \approx \Delta^2/12$, where Δ is the size of the quantization regions, generally assumes many levels, a small Δ and a nice input distribution. Compute the approximation error when using this formula compared with the exact expression for the situation described.

(b) Construct a Huffman code for Y_n.

(c) Give an example of a typical sequence of Y_n of length 8.

1–35 The i.i.d. continuous random variables X_n are quantized, Huffman encoded, transmitted over a channel with negligible errors, Huffman decoded and reconstructed.



The probability density function of X_n is

$$f_X(x) = \frac{1}{2} e^{-|x|}$$

The quantization regions in the quantizer are $\mathcal{A}_0 = [-\infty, -1]$, $\mathcal{A}_1 = (-1, 0]$, $\mathcal{A}_2 = (0, 1]$, $\mathcal{A}_3 = (1, \infty]$ and the corresponding reconstruction points are $\hat{x}_0 = -\frac{3}{2}$, $\hat{x}_1 = -\frac{1}{2}$, $\hat{x}_2 = \frac{1}{2}$, $\hat{x}_3 = \frac{3}{2}$.

(a) Compute the signal power to quantization noise power ratio $SQNR = \frac{E\left[X_n^2\right]}{E\left[\left(X_n - \hat{X}_n\right)^2\right]}$.

(b) Compute $H(I_n)$, $H(\hat{X}_n)$, $H(\hat{X}_n|I_n)$, $H(I_n|\hat{X}_n)$, $H(\hat{X}_n|X_n)$ and $h(X_n)$.

(c) Design a Huffman code for I_n. What is the rate of this code?

(d) Design a Huffman code for (I_{n-1}, I_n), i.e. blocks of two quantizer outputs are encoded ($N = 2$). What is the rate of this code?

(e) What would happen to the rate of the code if the block-length of the code would increase ($N \to \infty$)?

1–36 Consider Gaussian random variables X_n being generated according to[1]

$$X_n = aX_{n-1} + W_n$$

where the W_n's are independent zero-mean Gaussian variables with variance $E[W_n^2] = 1$, and where $0 \leq a < 1$.

(a) Compute the differential entropies $h(X_n)$ and $h(X_n, X_{n-1})$.

(b) Assume that each X_n is fed through a 3-level quantizer with output Y_n, such that

$$Y_n = \begin{cases} -b, & X_n < -1 \\ 0, & -1 \leq X_n < +1 \\ +b, & X_n \geq +1 \end{cases}$$

where $b > 0$. Which value of b minimizes the distortion $D = E[(X_n - Y_n)^2]$?

[1] Assume that the equation $X_n = aX_{n-1} + W_n$ was "started" at $n = -\infty$, so that $\{X_n\}$ is a stationary random process.

(c) Set $a = 0$. Construct a binary Huffman code for the 2-dimensional random variable (Y_n, Y_{n-1}), and compare the resulting rate of the code with the minimum possible rate.

Chapter 2

Modulation and Detection

2–1 Express the four signals in Figure 2.1 as two-dimensional vectors and plot them in a vector diagram against orthonormal base-vectors.

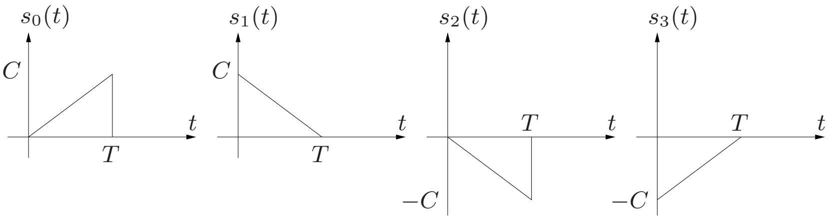


Figure 2.1: The signals in Problem 2–1.

2–2 Two antipodal and equally probable signal vectors, $\mathbf{s}_1 = -\mathbf{s}_2 = (2, 1)$, are employed in transmission over a vector channel with additive noise $\mathbf{w} = (w_1, w_2)$. Determine the optimal (minimum symbol error probability) decision regions, if the pdf of the noise is

(a) $f(w_1, w_1) = (2\pi\sigma^2)^{-1} \exp\big(- (2\sigma^2)^{-1}(w_1^2 + w_2^2)\big)$

(b) $f(w_1, w_2) = 4^{-1} \exp(-|w_1| - |w_2|)$

2–3 Let S be a binary random variable $S \in \{\pm 1\}$ with $p_0 = \Pr(S = -1) = 3/4$ and $p_1 = \Pr(S = +1) = 1/4$, and consider the variable r formed as

$$r = S + w$$

where w is independent of S and with pdf

(a) $f(w) = (2\pi)^{-1/2} \exp(-w^2/2)$

(b) $f(w) = 2^{-1} \exp(-|w|)$

Determine and sketch the conditional pdfs $f(r|s_0)$ and $f(r|s_1)$ and determine the optimal (minimum symbol error probability) decision regions for a decision about S based on r.

2–4 Consider the received scalar signal

$$r = a\,S + w$$

where a is a (possibly random) amplitude, $S \in \{\pm 5\}$ is an equiprobable binary information symbol and w is zero-mean Gaussian noise with $E[w^2] = 1$. The variables a, S and w are mutually independent.

Describe the optimal (minimum error probability) detector for the symbol S based on the value of r when

(a) $a = 1$ (constant)

(b) a is Gaussian with $E[a] = 1$ and $\mathrm{Var}[a] = 0.2$

Which case, (a) or (b), gives the lowest error probability?

2–5 Consider a communication system employing two equiprobable signal alternatives, $s_0 = 0$ and $s_1 = \sqrt{E}$, for non-repetitive signaling. The symbol s_i is transmitted over an AWGN channel, which adds zero-mean white Gaussian noise n to the signal, resulting in the received signal $r = s_i + n$. Due to various effects in the channel, the variance of the noise depends on the transmitted signal according to

$$E\{n^2\} = \begin{cases} \sigma_0^2 & s_0 \text{ transmitted} \\ \sigma_1^2 = 2\sigma_0^2 & s_1 \text{ transmitted} \end{cases}$$

A similar behavior is found in optical communication systems. For which values of r does an optimal receiver (i.e., a receiver minimizing the probability of error) decide s_0 and s_1, respectively?

Hint: an optimal receiver does not necessarily use the simple decision rule $r \underset{s_1}{\overset{s_0}{\lessgtr}} \gamma$, where γ is a scalar.

2–6 A binary source symbol A can take on the values a_0 and a_1 with probabilities $p = \Pr(A = a_0)$ and $1-p = \Pr(A = a_1)$. The symbol A is transmitted over a discrete channel that can be modeled as in Figure 2.2, with received symbol $B \in \{b_0, b_1\}$. That is, given $A = a_0$ the received symbol can take on the values b_0 and b_1 with probabilities 3/4 and 1/4, respectively. Similarly, given $A = a_1$ the received symbol takes on the values b_0 and b_1 with probabilities 1/8 and 7/8.

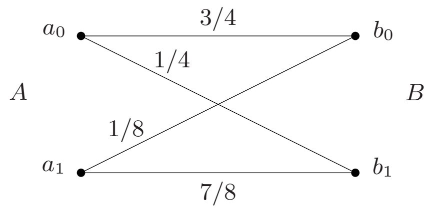


Figure 2.2: The channel in Problem 2–6.

Given the value of the received symbol, a detector produces an estimate $\hat{A} \in \{a_0, a_1\}$ of the transmitted symbol. Consider the following decision rules for the detector

1. $\hat{A} = a_0$ always
2. $\hat{A} = a_0$ when $B = b_0$ and $\hat{A} = a_1$ when $B = b_1$
3. $\hat{A} = a_1$ when $B = b_0$ and $\hat{A} = a_0$ when $B = b_1$
4. $\hat{A} = a_1$ always

(a) Let p_i, $i = 1, 2, 3, 4$, be the probability of error when decision rule i is used. Determine how p_i depends on the *a priori* probability p in the four cases.

(b) Which is the optimal (minimum error probability) decision strategy?

(c) Which rule i is the maximum likelihood strategy?

(d) Which rule i is the "minimax" rule (i.e., gives minimum error probability given that p takes on the "worst possible" value)?

2–7 For lunch during a hike in the mountains, the two friends A and B have brought warm soup, chocolate bars and cold stewed fruit. They are planning to pass one summit each day and want to take a break and eat something on each peak. Only three types of peaks appear in the area, high (1100 m, 20% of the peaks), medium (1000 m, 60% of the peaks) and low (900 m, 20% of the peaks). On high peaks it is cold and they prefer warm soup, on moderately high peaks chocolate, and on low peaks fruit. Before climbing a peak, they pack today's lunch easily accessible in the backpack, while leaving the rest of the food in the tent at the base of the mountain. They therefore try to find out the height of the peak in order to bring the right kind of food (soup, chocolate or fruit). B, who is responsible for the lunch, thinks maps are for wimps and tries

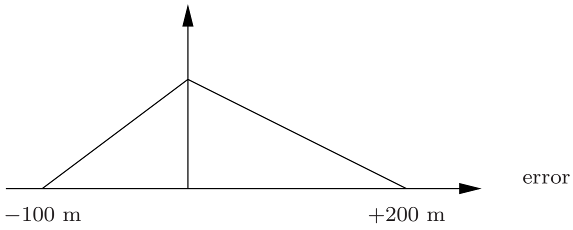


Figure 2.3: Pdf of estimation error.

to estimate the height visually. Each time, his estimated height deviates from the true height according to the pdf shown in Figure 2.3.

A, who trusts that B packs the right type of lunch each time, gets very angry if he does not get the right type of lunch.

(a) Given his height estimate, which type of food should B pack in order to keep A as happy as possible during the hike?

(b) Give the probability that B does not pack the right type of food for the peak they are about to climb!

2–8 Consider the communication system depicted in Figure 2.4,

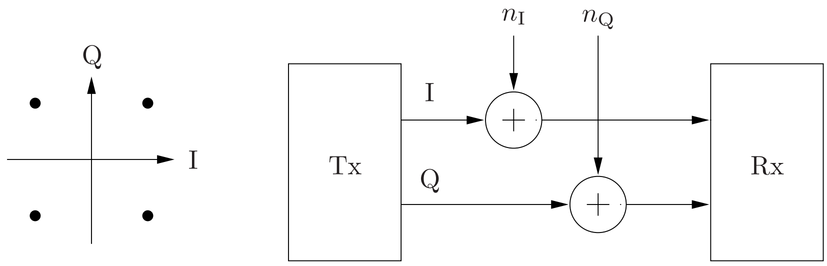


Figure 2.4: Communication system.

where I and Q denote the in-phase and quadrature phase channels, respectively. The white Gaussian noise added to I and Q by the channel is denoted by n_I and n_Q, respectively, both having the variance σ^2. The transmitter, Tx, uses QPSK modulation as illustrated in the figure, and the receiver, Rx, makes decisions by observing the decision region in which signal is received. Two equiprobable independent bits are transmitted for each symbol.

(a) Assume that the two noise components n_I and n_Q are *uncorrelated*. Find the mapping of the bits to the symbols in the transmitter that gives the lowest *bit* error probability. Illustrate the decision boundaries used by an optimum receiver and compute the *average* bit error probability as a function of the bit energy and σ^2.

(b) Assume that the two noise components are *fully correlated*. Find the answer to the three previous questions in this case (i.e., find the mapping, illustrate the decision boundaries and find the average bit error probability).

2–9 Consider a channel, taking an input $x \in \{\pm 1\}$, with equally likely alternatives, and adding two independent noise components, n_1 and n_2, resulting in the two outputs $y_1 = x + n_1$ and $y_2 = x + n_2$. The density function for the noise is given by

$$p_{n_1}(n) = p_{n_2}(n) = \frac{1}{2} e^{-|n|} \ .$$

Derive and plot the decision regions for an optimal (minimum symbol error probability) receiver! Be careful so you do not miss any decision boundary.

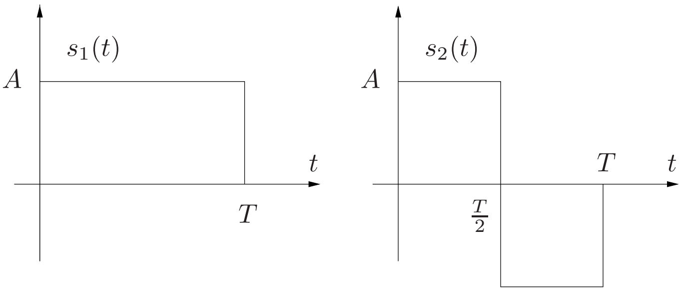


Figure 2.5: Signal alternatives.

2–10 Consider the two signal alternatives shown in Figure 2.5.

Assume that these signals are to be used in the transmission of binary data, from a source producing independent symbols, $X \in \{0, 1\}$, such that "0" is transmitted as $s_1(t)$ and "1" as $s_2(t)$. Assume, furthermore, that $\Pr(X = 1) = p$, where p is known, that the transmitted signals are subject to AWGN of power spectral density $N_0/2$, and that the received signal is $r(t)$.

(a) Assume that $10 \log(2A^2T/N_0) = 8.0$ [dB], and that $p = 0.15$. Describe how to implement the optimal detector that minimizes the average probability of error. What error probability does this detector give?

(b) Assume that instead of optimal detection, the rule: "Say 1 if $\int_0^T r(t)s_2(t)dt > \int_0^T r(t)s_1(t)dt$ otherwise say 0" is used. Which average error probability is obtained using this rule (for $10 \log(2A^2T/N_0) = 8.0$, and $p = 0.15$)?

2–11 One common way of error control in packet data systems are so-called ARQ systems, where an error detection code is used. If the receiver detects an error, it simply requests retransmission of the erroneous data packet by sending a NAK (negative acknowledgment) to the transmitter. Otherwise, an ACK (acknowledgment) is sent back, indicating the successful transfer of the packet.

Of course the ACK/NAK signal sent on the feedback link is subject to noise and other impairments, so there is a risk of misinterpreting the feedback signal in the transmitter. Mistaking an ACK as a NAK only causes an unnecessary retransmission of an already correctly received packet, which is not catastrophic. However, mistaking a NAK for an ACK is catastrophic, as the faulty packet never will be retransmitted and hence is lost.

Consider the ARQ scheme in Figure 2.6. The feedback link uses on-off-keying for sending back one bit of ACK/NAK information, where presence of a signal denotes an ACK and absence a NAK. The ACK feedback bit has the energy E_b and the noise spectral density on the feedback link is $N_0/2$. In this particular implementation, the risk of loosing a packet must be no higher than 10^{-4}, while the probability of unnecessary retransmissions can be at maximum 10^{-1}.

(a) To what value should the threshold in the ACK/NAK detector be set?

(b) What is the lowest possible value of E_b/N_0 on the feedback link fulfilling all the requirements above?

2–12 Consider the two dimensional vector channel

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \mathbf{s}_i + \mathbf{n}\,,$$

where $\mathbf{r}$ represents the received (demodulated) vector, $\mathbf{s}_i$ represents the transmitted symbol and $\mathbf{n}$ is correlated Gaussian noise such that $\mathrm{E}[n_1] = \mathrm{E}[n_2] = 0$, $\mathrm{E}[n_1^2] = \mathrm{E}[n_2^2] = 0.1$ and $\mathrm{E}[n_1 n_2] = 0.05$. Suppose that the rotated BPSK signal constellation shown in Figure 2.7 is used with equally probable symbol alternatives.

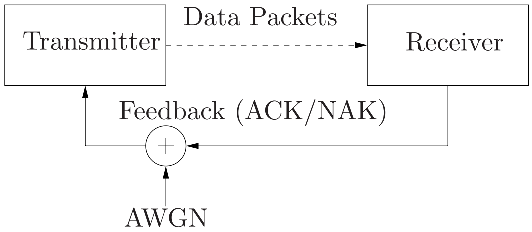


Figure 2.6: An ARQ system.

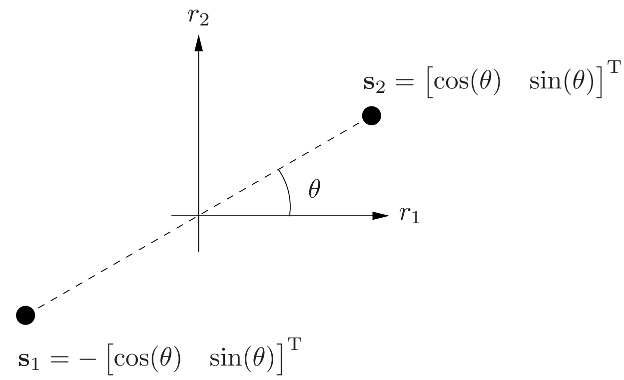


Figure 2.7: Signal constellation.

(a) Derive the probability of a symbol error, $P_{\mathrm{e}1}$, as a function of θ when a suboptimal maximum likelihood (ML) detector designed for *uncorrelated* noise is used. Also, find the two θ:s which minimize $P_{\mathrm{e}1}$ and compute the corresponding $P_{\mathrm{e}1}$. Give an intuitive explanation for why the θ:s you obtained are optimal.

Hint: Project the noise components onto a certain line (do not forget to motivate why this makes sense)!

(b) Find the probability of a symbol error, $P_{\mathrm{e}2}$, as a function of θ when an optimum detector is used (which takes the correlation of the noise into account).
Hint: The noise in the received vector can be made uncorrelated by multiplying it by the correct matrix.

(c) The performance of the suboptimal detector and the performance of the optimal detector is the same for only four values of θ - the values obtained in (a) and two other angles. Find these other angles and explain in an intuitive manner why the performance of the detectors are equal!

2–13 Consider a digital communication system in which the received signal, after matched filtering and sampling, can be written on the form

$$\boldsymbol{r} = \begin{bmatrix} r_{\mathrm{I}} \\ r_{\mathrm{Q}} \end{bmatrix} = \begin{bmatrix} s_{\mathrm{I}}(b_0, b_1) \\ s_{\mathrm{Q}}(b_0, b_1) \end{bmatrix} + \begin{bmatrix} w_{\mathrm{I}} \\ w_{\mathrm{Q}} \end{bmatrix} = \boldsymbol{s}(b_0, b_1) + \boldsymbol{w}\,,$$

where the in-phase and quadrature components are denoted by 'I' and 'Q', respectively and where a pair of independent and uniformly distributed bits b_0, b_1 are mapped into a symbol $\boldsymbol{s}(b_0, b_1)$, taken from the signal constellation $\{\boldsymbol{s}(0,0), \boldsymbol{s}(0,1), \boldsymbol{s}(1,0), \boldsymbol{s}(1,1)\}$. The system suffers from additive white Gaussian noise so the noise term $\boldsymbol{w}$ is assumed to be zero-mean Gaussian distributed with statistically independent components w_{I} and w_{Q}, each of variance σ^2. The above model arises for example in a system using QPSK modulation to transmit a sequence of bits. In this problem, we will consider two different ways of detecting the transmitted message – optimal bit-detection and optimal symbol detection.

(a) In the first detection method, two optimal bit-detectors are used in parallel to recover the transmitted message (b_0, b_1). Each bit-detector is designed so that the probability of a *bit-error* in the kth bit, i.e., $\Pr[\hat{b}_k \neq b_k]$, where $k \in \{0, 1\}$, is minimized. Derive such an optimal bit-detector and express it in terms of the quantities $\boldsymbol{r}$, $\boldsymbol{s}(b_0, b_1)$ and σ^2.

(b) Another detection approach is of course to first detect the symbol $\boldsymbol{s}$ by means of an optimal symbol detector, which minimizes the symbol error probability $\Pr[\tilde{\boldsymbol{s}} \neq \boldsymbol{s}]$, and then to map the detected symbol $\tilde{\boldsymbol{s}}$ into its corresponding bits $(\tilde{b}_0, \tilde{b}_1)$. Generally, this detection method is not equivalent to optimal bit-detection, i.e., $(\tilde{b}_0, \tilde{b}_1)$ is not necessarily equal to $(\hat{b}_0, \hat{b}_1)$. However, the two methods become more and more similar as the signal to noise ratio increases. Demonstrate this fact by showing that in the limit $\sigma^2 \to 0$, optimal bit-detection and optimal symbol detection are equivalent in the sense that $(\hat{b}_0, \hat{b}_1) = (\tilde{b}_0, \tilde{b}_1)$ with probability one.

Hint: In your proof, you may assume that expressions on the form $\sum_{k=1}^{K} \exp(-z_k \lambda)$, where $z_k > 0$ and $\lambda \to \infty$, can be replaced with $\max_k\{\exp(-z_k\lambda)\}_{k=1}^{K}$. This is motivated by the fact that for a sufficiently large λ, the largest term in the sum dominates and we have $\sum_k \exp(-z_k\lambda) \approx \max_k\{\exp(-z_k\lambda)\}_{k=1}^{K}$ (just as when the worst pairwise error probability dominates the union bound).

2–14 A particular binary disc storage channel can be approximated as a discrete-time additive Gaussian noise channel with input $x_n \in \{0, 1\}$ and output

$$y_n = x_n + w_n$$

where y_n is the decision variable in the disc-reading device, at time n. The noise variance depends on the value of x_n in the sense that w_n has the Gaussian conditional pdf

$$f_w(w|x) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{w^2}{2\sigma_0^2}} & x = 0 \\ \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{w^2}{2\sigma_1^2}} & x = 1 \end{cases}$$

with $\sigma_1^2 > \sigma_0^2$. For any n it holds that $\Pr(x_n = 0) = \Pr(x_n = 1) = 1/2$.

(a) Find a detection rule to decide $\hat{x}_n \in \{0, 1\}$ based on the value of y_n, and formulated in terms of σ_0^2 and σ_1^2. The detection should be optimal in the sense of minimum $P_b = \Pr(\hat{x}_n \neq x_n)$.

(b) Find the corresponding bit error probability P_b.

(c) What happens to the detector and P_b derived in (a) and (b) if $\sigma_0^2 \to 0$ and σ_1^2 remains constant?

2–15 Consider the signals depicted in Figure 2.8. Determine a causal matched filter with minimum possible delay to the signal

(a) $s_0(t)$

(b) $s_1(t)$

and sketch the resulting output signal when the input to the filter is the signal to which it is matched, and finally

(c) sketch the output signal from the filter in (b) when the input signal is $s_0(t)$.

2–16 A binary system uses the equally probable antipodal signal alternatives

$$s_1(t) = \begin{cases} \sqrt{E/T}, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

and $s_0(t) = -s_1(t)$ over an AWGN channel, with noise spectral density $N_0/2$, resulting in the received (time-continuous) signal $r(t)$. Let $y(t)$ be the output of a filter matched to $s_1(t)$ at the

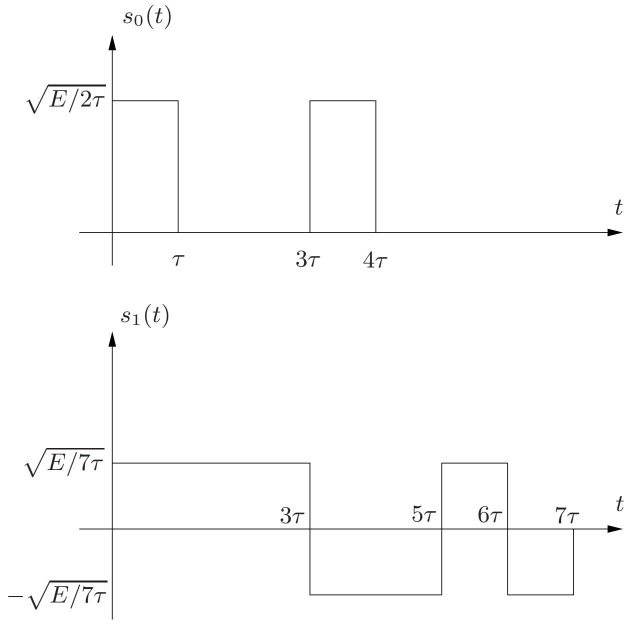


Figure 2.8: The signals $s_0(t)$ and $s_1(t)$.

receiver, and let y be the value of $y(t)$ sampled at the time-instant T_s, that is $y = y(T_s)$. Symbol detection is based on the decision rule

$$y \underset{s_0}{\overset{s_1}{\gtrless}} 0$$

(That is, decide that $s_1(t)$ was sent if $y > 0$ and decide $s_0(t)$ if $y < 0$.)

(a) Sketch the two possible forms for the signal $y(t)$ in absence of noise (i.e., for $r(t) = s_0(t)$ and $r(t) = s_1(t)$).

(b) The optimal sampling instance is $T_s = T$. In practice it is impossible to sample completely without error in the sampling instance. This can be modeled as $T_s = T + \Delta$ where Δ is a (small) error. Determine the probability of symbol error as a function of the sampling error Δ (assuming $|\Delta| < T$).

2–17 A PAM system uses the three equally likely signal alternatives shown in Figure 2.9.

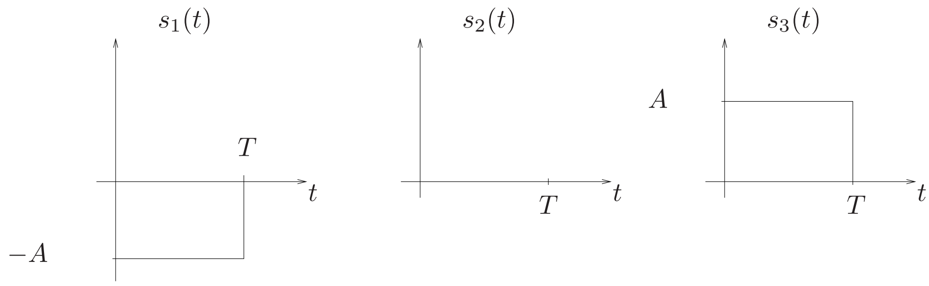


Figure 2.9: Signal alternatives.

$$s_3(t) = -s_1(t) = \begin{cases} A & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \qquad s_2(t) = 0$$

One-shot signalling over an AWGN channel with noise spectral density $N_0/2$ is studied. The receiver shown in Figure 2.10 is used. An ML receiver uses a filter matched to the pulse $m(t) = g_T(t)$, defined in Figure 2.11.

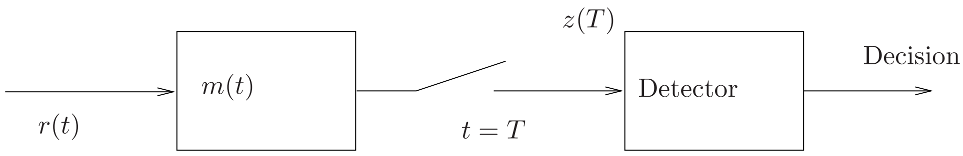


Figure 2.10: Receiver.

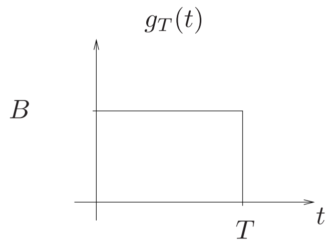


Figure 2.11: Pulse.

Decisions are made according to

$$\begin{cases} s_1(t): & z(T) \le -\gamma \\ s_2(t): & -\gamma \le z(T) \le \gamma \\ s_3(t): & z(T) \ge \gamma \end{cases},$$

where the threshold γ is chosen so that optimal decisions are obtained with $m(t) = g_T(t)$.

Consider now the situation obtained when instead $m(t) = K g_T(t)$ where K is an unknown constant, $0.5 \le K \le 1.5$. Determine an exact expression for the probability of symbol error as a function of K.

2–18 A fire alarm system works as follows: each KT second, where K is a large positive integer, a pulse $+s(t)$ or $-s(t)$, of duration T seconds, is transmitted, carrying the information

$+s(t)$: "no fire"

$-s(t)$: "fire"

The signal $s(t)$ has energy E and is subjected to AWGN of spectral density $N_0/2$ in the transmission. The received signal is detected using a matched filter followed by a threshold detector, and fire is declared to be present or non-present.

The system is designed such that the "miss probability" (the probability that the system declares "no fire" in the case of fire) is 10^{-7} when $2E/N_0 = 12$ dB. Determine the false alarm probability!

2–19 Consider the baseband PAM system in Figure 2.12, using a correlation-type demodulator. $n(t)$ is AWGN with spectral density $N_0/2$ and $\Psi(t)$ is the unit energy basis function. The receiver uses ML detection.

The transmitter maps each source symbol I onto one of the equiprobable waveforms $s_m(t)$, $m =$

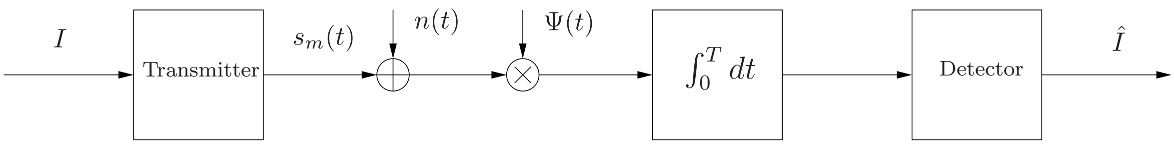
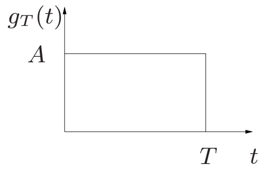


Figure 2.12: PAM system

$1 \ldots 4$.

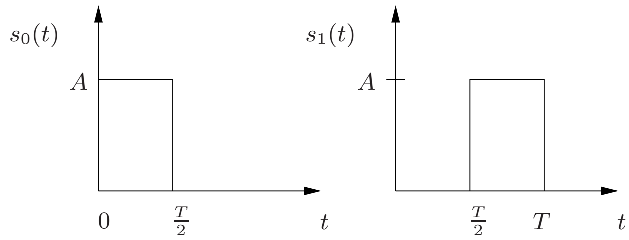
$$
\begin{aligned}
s_1(t) &= \frac{3}{2} g_T(t) \\
s_2(t) &= \frac{1}{2} g_T(t) \\
s_3(t) &= -s_2(t) \\
s_4(t) &= -s_1(t)
\end{aligned}
$$

where $g_T(t)$ is a rectangular pulse of amplitude A.



However, due to manufacturing problems, the amplitude of the used basis function is corrupted by a factor $b > 1$. I.e. the used basis function is $\bar{\Psi}(t) = b\Psi(t)$. Note that the detector is unchanged. Compute the symbol-error probability ($\Pr(\hat{I} \neq I)$).

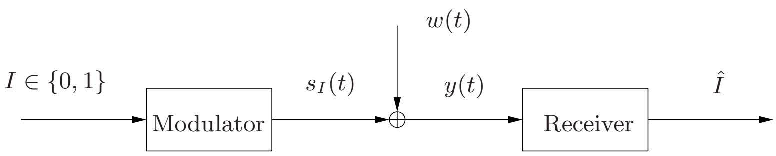
2–20 The information variable $I \in \{0, 1\}$ is mapped onto the signals $s_0(t)$ and $s_1(t)$.



The probabilities of the outcomes of I are given as

$$
\begin{aligned}
\Pr(I = 0) = \Pr(s_0(t) \text{ is transmitted}) &= p \\
\Pr(I = 1) = \Pr(s_1(t) \text{ is transmitted}) &= 1 - p
\end{aligned}
$$

The signal is transmitted over a channel with additive white Gaussian noise, $W(t)$, with power spectral density $N_0/2$.



(a) Find the receiver (demodulator and detector) that minimizes $\Pr(\hat{I} \neq I)$.

(b) Find the range of different p's for which $y(t) = s_1(t) \Rightarrow \hat{I} = 0$.

2–21 Consider binary antipodal signaling with equally probable waveforms

$$
s_1(t) = 0, \quad 0 \leq t \leq T
$$

$$
s_2(t) = \sqrt{\frac{E}{T}}, \quad 0 \leq t \leq T
$$

in AWGN with spectral density $N_0/2$.

The optimal receiver can be implemented using a matched filter with impulse reponse

$$h_{\text{opt}}(t) = \sqrt{\frac{1}{T}}, \quad 0 \le t \le T$$

sampled at $t = T$. However in this problem we consider using the (suboptimal) filter

$$h(t) = e^{-t/T}, \quad t \ge 0$$

($h(t) = 0$ for $t < 0$) instead of the macthed filter. More precicely, letting y_T denote the value of the output of this filter sampled at $t = T$, when fed by the received signal in AWGN, the decision is

$$\begin{aligned} y_T < b &\implies \text{choose } s_1 \\ y_T \ge b &\implies \text{choose } s_2 \end{aligned}$$

where $b > 0$ is a decision threshold.

(a) Determine the resulting error probability P_e, as a function of b, E, T and N_0.

(b) Which value for the threshold b minimizes P_e?

2–22 In the IS-95 standard, a CDMA based second generation cellular system, so-called Walsh modulation is used on the uplink (from the mobile phone to the base station). Groups of six bits are used to select one of 64 Walsh sequences. The base station determines which of the 64 Walsh sequences that were transmitted and can thus decode the six information bits. The set of L Walsh sequences of length L are characterized by being mutual orthogonal sequences consisting of $+1$ and -1. Assume that the sequences used in the transmitter are normalized such that the energy of one length L sequence equals unity.

Determine the bit error probability and the block (group of six bits) error probability for the simplified IS-95 system illustrated in Figure 2.13 when it operates over an AWGN channel with $E_b/N_0 = 4$ dB!

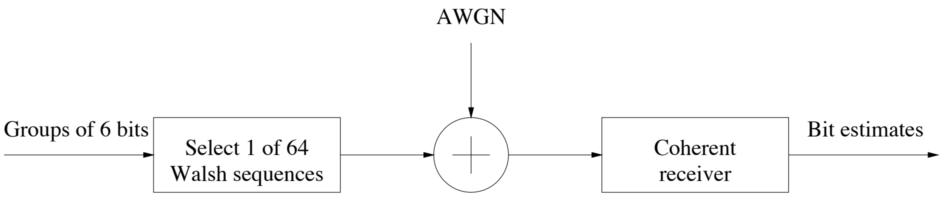
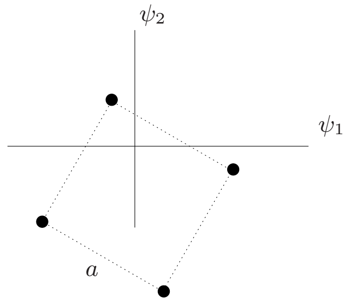


Figure 2.13: Communication system.

Example: the set of $L = 4$ Walsh sequences (NB! $L = 64$ in the problem above).

$$\begin{array}{cccc} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 \end{array}$$

2–23 Four signal points are located on the corners of a square with side length a, as shown in the figure below.



These points are used over an AWGN channel with noise spectral density $N_0/2$ to signal equally probable alternatives and with an optimum (minimum symbol error probability) receiver.

Derive an exact expression for the symbol error probability (in terms of the Q-function, and the variables a and N_0).

2–24 Two different signal constellations, depicted in Figure 2.14, are considered for a communication system. Compute the average *bit* error probability expressed as a function of d and $N_0/2$ for the two cases, assuming equiprobable and independent bits, an AWGN channel with power spectral density of $N_0/2$, and a detector that is optimal in the sense of minimum *symbol* error probability.

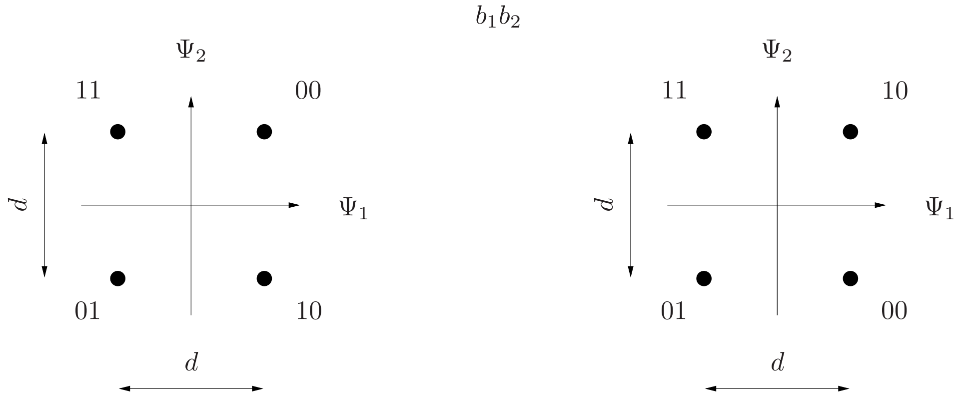


Figure 2.14: Signal constellations.

2–25 Consider a digital communication system with the signal constellation shown in Figure 2.15. The

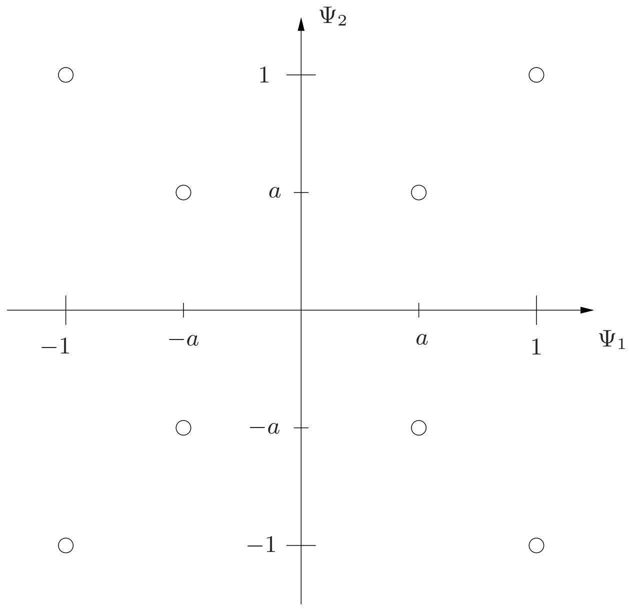


Figure 2.15: The signal constellation in Problem 2–25.

signal is transmitted over an AWGN channel with noise variance $\sigma_n^2 = 0.025$ and an ML receiver is used to detect the symbols. The symbols are equally probable and independent. Determine a value of $0 < a < 1$ such that the error rate is less than 10^{-2}.

Hint: Guess a good value for a. Describe the eight decision regions. Use these regions to bound the error probability.

2–26 Consider a communication system operating over an AWGN channel with noise power spectral density $N_0/2$. Groups of two bits are used to select one out of four possible transmitted signal alternatives, given by

$$s_i(t) = \begin{cases} \sqrt{\frac{2E_s}{T}}\cos(2\pi f t + i\pi/2 + \pi/4) & 0 \le t < T, i \in \{0, \dots, 3\} \\ 0 & \text{otherwise,} \end{cases}$$

where f is a multiple of $1/T$. The receiver, illustrated in Figure 2.16, uses a correlator-based front-end with the two basis functions

$$\Psi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f t) \qquad 0 \le t < T$$
$$\Psi_2(t) = \sqrt{\frac{2}{T}}\cos(2\pi f t + \pi/4) \qquad 0 \le t < T \ ,$$

followed by an optimal detection device.

From the MAP criterion, derive the decision rule used in the receiver expressed in y_1 and y_2. Determine the corresponding symbol error probability expressed in E_b and N_0 using the Q-function.

Hint: Carefully examine the basis functions used in the receiver. Does the choice of basis functions matter?

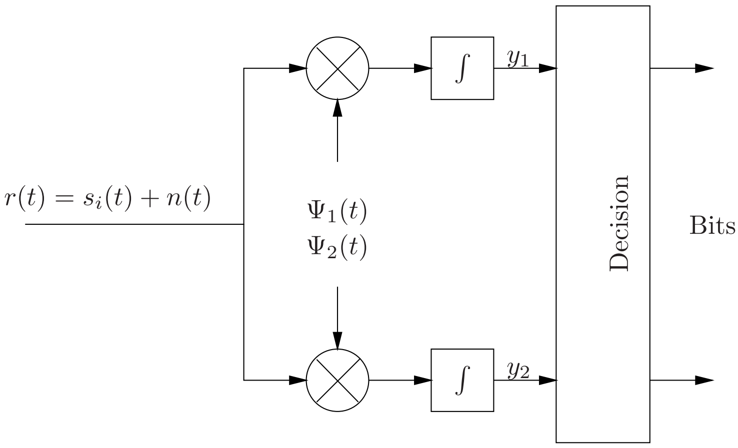
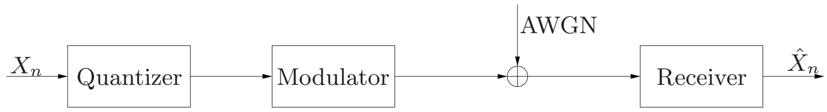
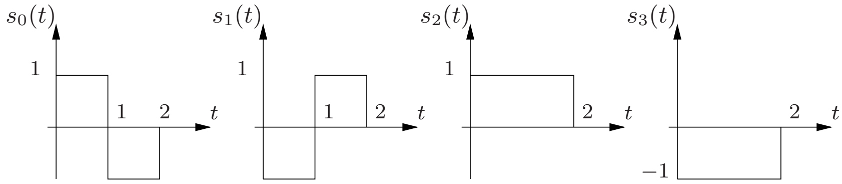


Figure 2.16: The receiver in Problem 2–26.

2–27 In the figure below a uniformly distributed random variable X_n, with $E[X_n] = 0$ and $E[X_n^2] = 1$, is quantized using a uniform quantizer with $2^8 = 256$ levels. The dynamic range of the quantizer is fully utilized and not overloaded.



The 8 output bits $b_1, \dots, b_8$ from the quantizer are mapped onto 4 uses of the signals in the figure below.



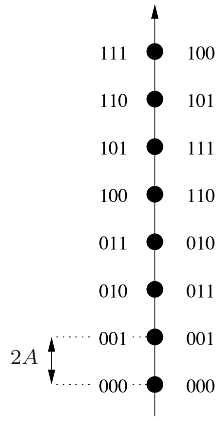


Figure 2.17: The signals in Problem 2–28.

The resulting signals are transmitted over a channel with AWGN of spectral density $N_0/2$. The receiver is optimal in the sense that it minimizes the symbol error probability. $E_s/N_0 = 16$ at the receiver, where $E_s = \|s_i(t)\|^2$ is the transmitted signal energy.

The expected total distortion $D = E[(X_n - \hat{X}_n)^2]$ due to quantization noise and transmission errors can be divided as

$$D = D_c \ \Pr(\text{ all 8 bits correct }) + D_e \left(1 - \Pr(\text{ all 8 bits correct })\right)$$

where D_c is the distortion conditioned that no transmission errors occurred in the transmission of the 8 bits, and D_e is the distortion conditioned that at least one bit was in error.

Take on the (crude) assumption that $\hat{X}_n$ is uniformly distributed over the whole dynamic range of the quantizer and independent of X_n when a transmission error occurs.

Compute D.

2–28 Consider transmitting 3 equally likely and independent bits (b_1, b_2, b_3) using 8-PAM/ASK, and the two different bit-labelings illustrated in Figure 2.17. The left labeling is the natural labeling (NL) and the right is the Gray labeling (GL). Assume that the PAM/ASK signal is transmitted over an AWGN channel, with noise spectral density $N_0/2$, and using an optimal (minimum symbol error probability) receiver, producing decisions $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$ on the bits. Define

$$P_{b,i} = \Pr(\hat{b}_i \neq b_i), \quad i = 1, 2, 3$$

and the average bit-error probability

$$P_b = \frac{1}{3} \sum_{i=1}^{3} P_{b,i}$$

(a) Compute P_b for the NL, in terms of the parameters A and N_0.

(b) Compute P_b for the GL, in terms of the parameters A and N_0.

(c) Which mapping gives the lowest value for P_b as $A^2/N_0 \to \infty$ (high SNR's)?

2–29 Consider a digital communication system where two i.i.d. source bits $(b_1 b_0)$ are mapped onto one of four signals according to table below. The probability mass function of the bits is $\Pr(b_i = 0) = 2/3$ and $\Pr(b_i = 1) = 1/3$.

Source bits $b_1 b_0$	Transmitted signal
00	$s_0(t)$
01	$s_1(t)$
11	$s_2(t)$
10	$s_3(t)$

The signals are given by

$$s_0(t) = \begin{cases} 3A & 0 \le t < T \\ 0 & \text{otherwise} \end{cases} \qquad s_1(t) = \begin{cases} A & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

$$s_2(t) = \begin{cases} -A & 0 \le t < T \\ 0 & \text{otherwise} \end{cases} \qquad s_3(t) = \begin{cases} -3A & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

In the communication channel, additive white Gaussian noise with p.s.d. $N_0/2$ is added to the transmitted signal. The received signal is passed through a demodulator and detector to find which signal and which bits were transmitted.

(a) Find the demodulator and detector that minimize the probability that the wrong signal is detected at the receiver.

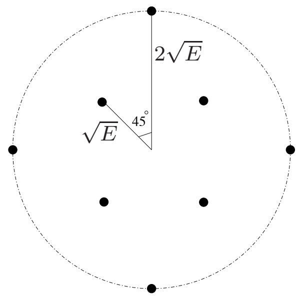
(b) Determine the probability that the wrong signal is detected at the receiver.

(c) Determine the bit error probability

$$P_b = \frac{1}{2}\Big(\Pr(\hat{b}_1 \neq b_1) + \Pr(\hat{b}_0 \neq b_0)\Big)$$

where $\hat{b}_1$ and $\hat{b}_0$ are the detected bits. Assume that $N_0 \ll A^2T$.

2–30 Use the union bound to upper bound the symbol error probability for the constellation shown below.



Symbols are equally likely and the transmission is subjected to AWGN with spectral density $N_0/2$. An optimal receiver is used.

2–31 Consider transmitting the equally probable decimal numbers $i \in \{0, \dots, 9\}$ based on the signal alternatives $s_i(t)$, with

$$s_i(t) = A\left(\sin(2\pi n_i t/T) + \sin(2\pi m_i t/T)\right) \quad 0 \le t \le T$$

where the integers n_i, m_i are chosen according to the following table:

i	n_i	m_i
0	1	2
1	3	1
2	1	4
3	5	1
4	2	3
5	4	2
6	2	5
7	3	4
8	5	3
9	4	5

The signals are transmitted over an AWGN channel with noise spectral density $N_0/2$. Use the Union Bound technique to establish an upper bound to the error probability obtained with maximum likelihood detection.

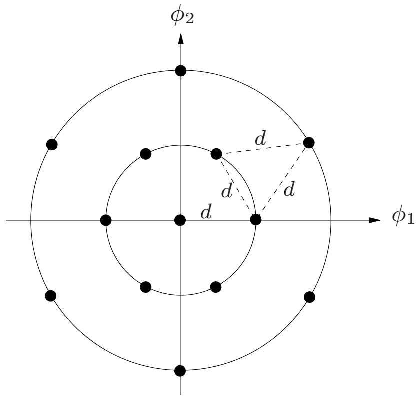


Figure 2.18: Signal alternatives.

2–32 Consider transmitting 13 equiprobable signal alternatives, as displayed in Figure 2.18, over an AWGN channel with noise spectral density $N_0/2$ and with maximum likelihood detection at the receiver.

For high values of the "signal-to-noise ratio" d^2/N_0 the symbol error probability P_e for this scheme can be tightly estimated as

$$P_e \approx K\,Q\left(\frac{d}{\sqrt{2N_0}}\right)$$

Determine the constant K in this expression.

2–33 Consider the signal set $\{s_0(t), \dots, s_{L-1}(t)\}$, with

$$s_i(t) = \begin{cases} \sqrt{2E/T}\,\cos(\pi K t/T + 2\pi i/L) & 0 < t < T \\ 0 & \text{otherwise} \end{cases}$$

where K is a (large) integer. The signals are employed in transmitting equally likely information symbols over an AWGN channel with noise spectral density $N_0/2$.

Show, based on a geometrical treatment of the signals, that the resulting symbol error probability P_e can be bounded as

$$Q(\beta) < P_e < 2\,Q(\beta),$$

where $\beta = \sqrt{2E/N_0}\,\sin(\pi/L)$.

2–34 A communication system with eight equiprobable signal alternatives uses the signal constellation depicted in Figure 2.19. The energies are $E_1 = 1$ and $E_2 = 3$, and a decision rule resulting in minimal symbol error probability is employed at the receiver. A memoryless AWGN channel is assumed. Derive upper and lower bounds on the symbol error probability as a function of the average signal to noise ratio, $\text{SNR} = 2E_{\text{mean}}/N_0$. Keeping the same average symbol energy, can the symbol error probability be improved by changing E_1 and/or E_2? How and to which values of E_1 and E_2?

2–35 A communication system utilizes the three different signal alternatives

$$s_1(t) = \begin{cases} \sqrt{\frac{2E}{T}}\cos\left(4\pi\frac{t}{T}\right) & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

$$s_2(t) = \begin{cases} \sqrt{\frac{2E}{T}}\cos\left(4\pi\frac{t}{T} - \frac{\pi}{2}\right) & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

$$s_3(t) = \begin{cases} \sqrt{\frac{E}{T}}\sin\left(4\pi\frac{t}{T} + \frac{\pi}{4}\right) & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

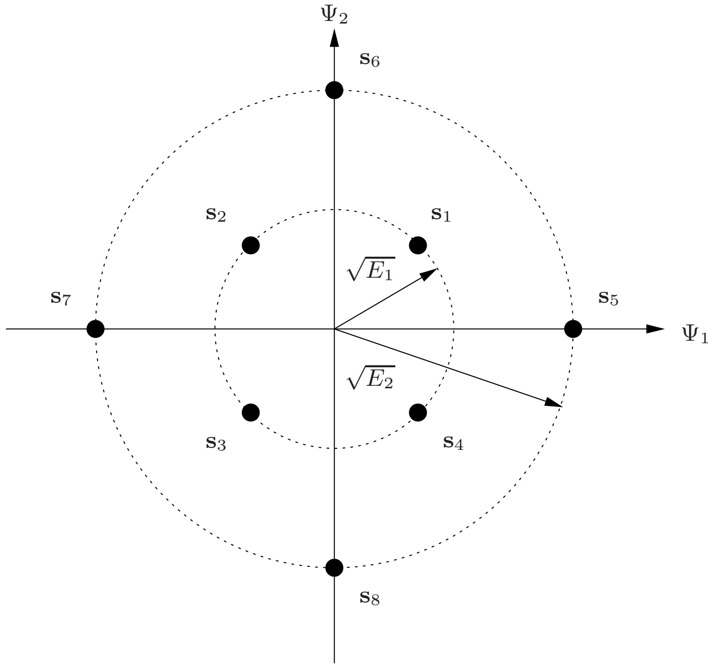


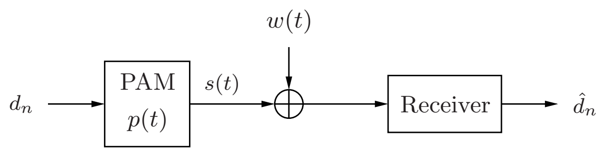
Figure 2.19: The signal constellation in Problem 2–34.

The signals are used over an AWGN channel, with noise spectral density $N_0/2$. An optimal (minimum symbol error probability) receiver is used.

Derive an expression for the resulting symbol error probability and compare with the Union Bound.

2–36 Consider the communication system illustrated below. Two independent and equally likely symbols $d_0 \in \{\pm 1\}$ and $d_1 \in \{\pm 1\}$ are transmitted in immediate succession. At all other times, a symbol value of zero is transmitted. The symbol period is $T = 1$ and the pulse is $p(t) = 1$ for $0 \leq t \leq 2$ and zero otherwise. Additive white Gaussian noise $w(t)$ with power spectral density $N_0/2$ disturbs the transmitted signal. The receiver detects the sequence $\{d_0, d_1\}$ such that the probability of a sequence error is as low as possible, i.e. the receiver is implemented such that $\Pr[\hat{d}_0 \neq d_0 \cup \hat{d}_1 \neq d_1]$ is minimized.

Find an upper bound on $\Pr[\hat{d}_0 \neq d_0 \cup \hat{d}_1 \neq d_1]$ that is as tight as possible at high signal-to-noise ratios!



2–37 Consider the four different signals in Figure 2.20.

These are used, together with an optimal receiver, in signaling four equally probable symbol alternatives over an AWGN channel with noise spectral density $N_0/2$.

(a) Use the Gram–Schmidt procedure to find orthonormal basis waveforms that can be used to represent the four signal alternatives and plot the resulting signal space.

(b) With $1/N_0 = 0\,\text{dB}$ show that the symbol error probability P_e lies in the interval

$$0.029 < P_e < 0.031$$

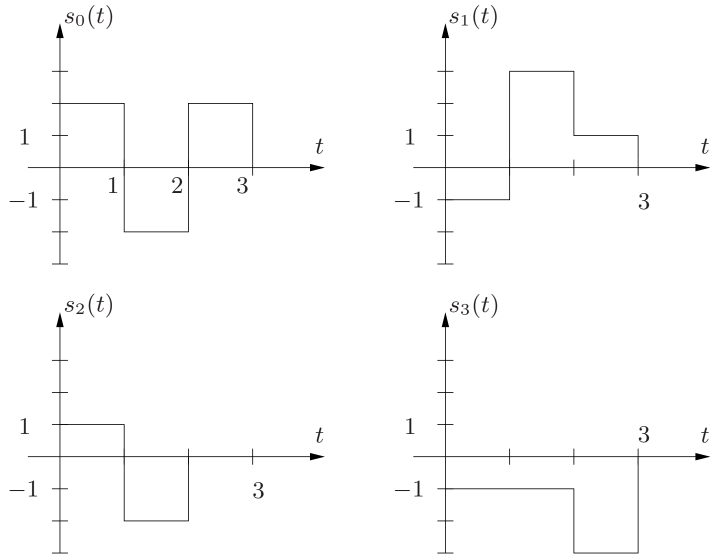
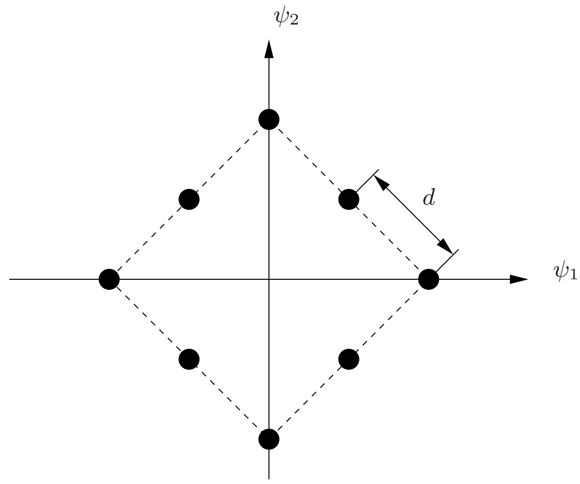


Figure 2.20: Signals

2–38 In a wireless link the main obstacle is generally not additive white noise (thermal noise in the receiver) but a phenomenon referred to as signal *fading*. That is, time-varying fluctuations in the received signal energy. Fading can be classified as *large-scale* or *small-scale*. Large-scale fading is due to large objects (e.g. buildings) that are "shadowing" the transmitted radio signal. If the receiver is moving, this will result in relatively slow amplitude variation as the receiver passes by shadowing objects. Small-scale fading, on the other hand, is due to destructive interference between signal components that reach the receiver having traveled different paths from the transmitter. This can result in relatively fast time-variation as the receiver moves. In this problem we study a simplified scenario taking into account small-scale fading only, and AWGN (as usual).

Consider signalling based on the constellation depicted below, denoting the different equiprobable signal alternatives $\mathbf{s}_0, \ldots, \mathbf{s}_7$.



The signals are used over a channel with small-scale fading and AWGN. The received signal can be modeled as

$$\mathbf{r} = a\,\mathbf{s} + \mathbf{w}$$

where $\mathbf{w} = (w_1, w_2)^T$ is 2-dimensional AWGN, with $E[w_1^2] = E[w_2^2] = N_0/2$ and $E[w_1 w_2] = 0$. The scalar variable a models an amplitude attenuation due to fading. A common assumption about the statistics of the amplitude attenuation is that a is a *Rayleigh* variable, with a pdf according to

$$f(a) = \begin{cases} \frac{a}{\sigma^2} \exp\left(-\frac{a^2}{2\sigma^2}\right), & a \geq 0 \\ 0, & a < 0 \end{cases}$$

where σ^2 is a fixed parameter. It holds that $E[a^2] = 2\sigma^2$. At the receiver, a detector is used that is optimal (minimum symbol error probability) under the assumption that the receiver *knows the amplitude*, a, *perfectly*. (In practice the amplitude cannot be estimated perfectly at the receiver.)

Your task is to study the average symbol error probability P_e of the signalling described above (averaged over the noise and the amplitude variation). Use the Union Bound technique to determine an upper bound to P_e as a function of the parameters d, N_0 and σ^2. The bound should be as tight as possible (i.e., do not include "unnecessary" terms).

Hint: Note that

$$\Pr(\text{error}) = \int_0^\infty \Pr(\text{error}|a) f(a) da$$

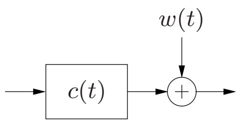
and that

$$\int_0^\infty Q(\alpha x) x e^{-\beta x^2} dx = \frac{1}{4\beta}\left[1 - \frac{\alpha}{\sqrt{\alpha^2 + 2\beta}}\right]$$

2–39 Consider the following three signal waveforms

$$s_1(t) = 2A \ , \quad s_2(t) = \begin{cases} +(\sqrt{3}-1)\,A, & 0 \leq t < 1 \\ -(\sqrt{3}+1)\,A, & 1 \leq t \leq 2 \end{cases} \ , \quad s_3(t) = \begin{cases} -(\sqrt{3}+1)\,A, & 0 \leq t < 1 \\ +(\sqrt{3}-1)\,A, & 1 \leq t \leq 2 \end{cases}$$

with $A = (2\sqrt{2})^{-1}$ and $s_i(t) = 0$, $i = 1, 2, 3$, for $t < 0$ and $t > 2$. These are used in signaling equiprobable and independent symbol alternatives over the channel depicted below,



where $c(t) = \delta(t) + a\,\delta(t - \tau)$ and $w(t)$ is AWGN with spectral density $N_0/2$. One signal is transmitted each $T = 2$ seconds. A receiver optimal in the sense of minimum symbol error probability *in the case* $a = 0$ is employed. Derive an upper bound to the resulting symbol error probability P_e when $a = 1/4$ and

(a) $\tau = 2$.

(b) $\tau = 1$.

The bounds should be non-trivial (i.e., $P_e \leq 1$ is naturally not an accepted bound), however they need not to be tight for high SNR's.

2–40 Consider the four signals in Figure 2.21.

These are used, together with an optimal receiver, in signaling four equally probable symbol alternatives over an AWGN channel with noise spectral density $N_0/2$. Let P_e be the resulting symbol error probability and let $\gamma = 1/N_0$.

(a) Determine an analytical *upper bound* to P_e as a function of γ.

(b) Determine an analytical *lower bound* to P_e as a function of γ.

The analytical bounds in (a) and (b) should be as "tight" as possible, in the sense that they should lie as close to the true P_e-curve as possible.

2–41 Consider the waveforms $s_0(t)$, $s_1(t)$ and $s_2(t)$.

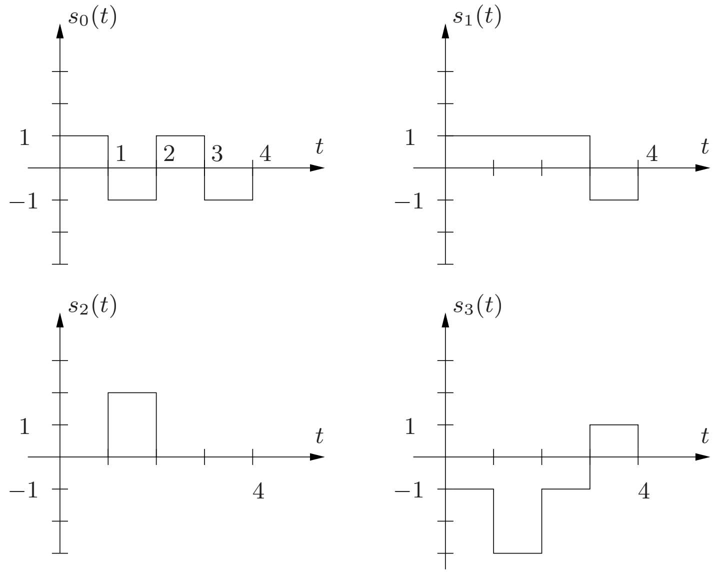
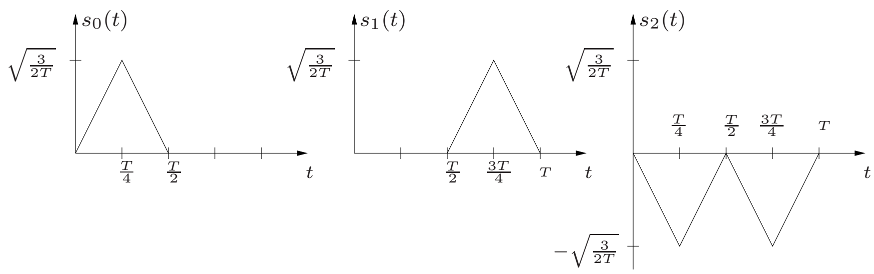
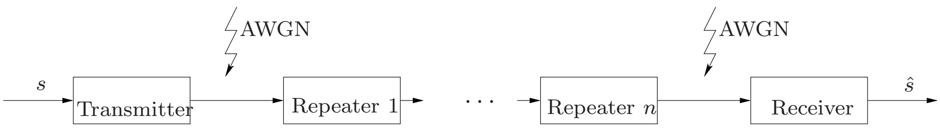


Figure 2.21: Signals



They are used to transmit symbols in a communication system with n repeaters. Each repeater estimates the transmitted symbol from the previous transmitter and retransmits it. Assume ML detection in all receivers.



Assume that $\Pr\{s_0(t) \text{ transmitted}\} = \Pr\{s_1(t) \text{ transmitted}\} = \frac{1}{2}\Pr\{s_2(t) \text{ transmitted}\} = \frac{1}{4}$ and that each link is disturbed by additive white Gaussian noise with spectral density $N_0/2$. Derive an upper bound to the total probability of error, $\Pr(\hat{s} \neq s)$.

2–42 Let three orthonormal waveforms be defined as

$$\psi_1(t) = \begin{cases} \sqrt{\frac{3}{T}}, & 0 \le t < \frac{T}{3} \\ 0, & \text{otherwise} \end{cases} \quad \psi_2(t) = \begin{cases} \sqrt{\frac{3}{T}}, & \frac{T}{3} \le t < \frac{2T}{3} \\ 0, & \text{otherwise} \end{cases} \quad \psi_3(t) = \begin{cases} \sqrt{\frac{3}{T}}, & \frac{2T}{3} \le t < T \\ 0, & \text{otherwise} \end{cases}$$

and consider the three signal waveforms

$$s_1(t) = A\left(\psi_1(t) + \frac{3}{4}\,\psi_2(t) + \frac{\sqrt{3}}{4}\,\psi_3(t)\right)$$

$$s_2(t) = A\left(-\psi_1(t) + \frac{3}{4}\,\psi_2(t) + \frac{\sqrt{3}}{4}\,\psi_3(t)\right)$$

$$s_3(t) = A\left(-\frac{3}{4}\,\psi_2(t) - \frac{\sqrt{3}}{4}\,\psi_3(t)\right)$$

Assume that these signals are used to transmit equally likely symbol alternatives over an AWGN channel with noise spectral density $N_0/2$.

(a) Show that optimal decisions (minimum probability of symbol error) can be obtained via the outputs of *two* correlators (or sampled matched filters) and specify the waveforms used in these correlators (or the impulse responses of the filters).

(b) Assume that P_e is the resulting probability of symbol error when optimal demodulation and detection is employed. Show that

$$Q\left(\sqrt{\frac{2A^2}{N_0}}\right) < P_e < 2Q\left(\sqrt{\frac{2A^2}{N_0}}\right)$$

(c) Use the bounds in (b) to upper-bound the symbol-rate (symbols/s) that can be transmitted, under the constraint that $P_e < 10^{-4}$, and that the average transmitted power is less than or equal to P. Express the bound in terms of P and N_0

2–43 Pulse duration modulation (PDM) is commonly used, with some modifications, in read-only optical storage, e.g. compact discs. In optical data storage, data is recorded by the length of a hole or "pit" burned into the storage medium. Figure 2.22 below depicts the three waveforms that are used in a particular optical storage system.

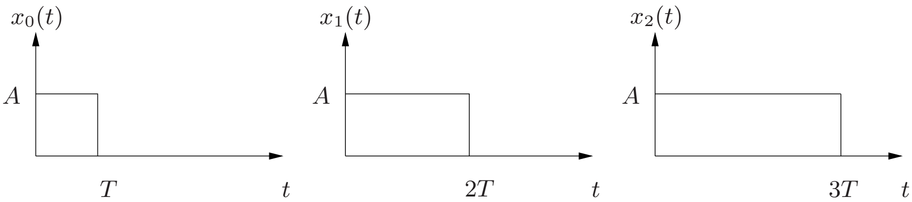


Figure 2.22: PDM Signal waveforms

The write and read process can be approximated by an AWGN channel as described in Figure 2.23. The ternary information variable $I \in \{0, 1, 2\}$ with equiprobable outcomes is directly mapped to the waveform $x(t) = x_i(t)$. In the reading device, additive white Gaussian noise with spectral density $N_0/2$ is added to the waveform. The received signal $r(t)$ is demodulated into the vector of decision variables $\mathbf{r}$. The decision variable vector is used to detect the transmitted information.

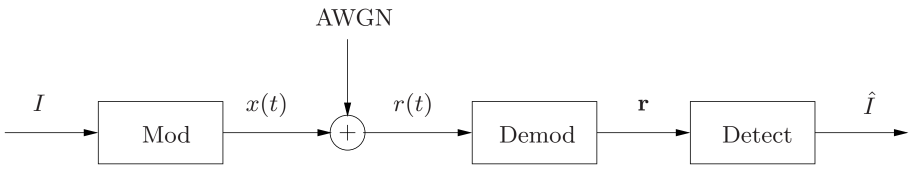


Figure 2.23: PDM system

(a) Find an optimal demodulator of $r(t)$.

(b) Find the detector that minimizes $\Pr\left(\hat{I} \neq I\right)$, given the demodulator in (a).

(c) Find tight lower and upper bounds on $\Pr\left(\hat{I} \neq I\right)$ if the demodulator in (a) and the detector in (b) are used.

2–44 The two-dimensional constellation illustrated in Figure 2.24 is used over an AWGN channel, with spectral density $N_0/2$, to convey equally probable symbols and with an optimal (minimum symbol error probability) receiver. The radius of the inner circle is a and the radius of the outer $2a$. Assume $a^2/N_0 = 10$, and let P_e be the resulting average symbol error probability. Show that $0.0085 < P_e < 0.011$.

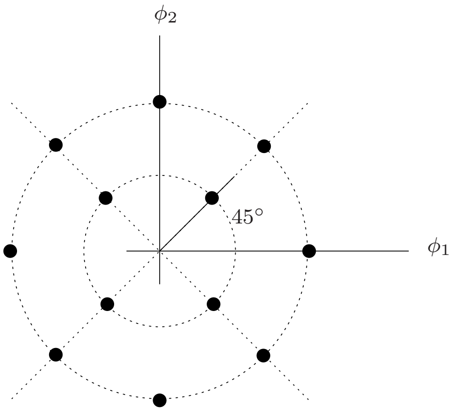


Figure 2.24: QAM constellation

(Compute the upper and lower bounds in terms of the Q-function, if you are not able to compute its numerical values.)

2–45 Consider the six signal alternatives $s_1(t), \ldots, s_6(t)$, where $s_1(t), s_2(t), s_3(t)$ are shown in Figure 2.25, and with

$$s_4(t) = -s_1(t), \quad s_5(t) = -s_2(t), \quad s_6(t) = -s_3(t)$$

These signals are employed in transmitting equally probable symbols over an AWGN channel,

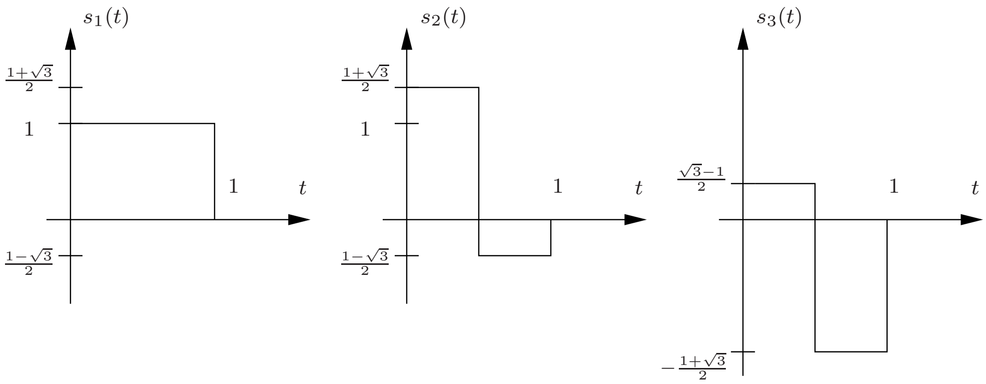


Figure 2.25: Signals

with noise spectral density $N_0/2$ and using an optimal (minimum symbol error probability) receiver.

With $P_e = \Pr(\text{symbol error})$, show that

$$Q\left(\frac{1}{\sqrt{2N_0}}\right) < P_e < 2\,Q\left(\frac{1}{\sqrt{2N_0}}\right)$$

2–46 Show that $\sin(2\pi f t)$ and $\cos(2\pi f t)$, that are defined for $t \in [0\ T)$, are approximately orthogonal if $f \gg 1/T$.

2–47 In this problem, BPSK signaling is considered. The two equally probable signal alternatives are

$$\begin{cases} \pm\sqrt{\frac{2E}{T}}\cos(2\pi f_c t) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \qquad f_c \gg \frac{1}{T}\,.$$

The system operates over an AWGN channel with power spectral density $N_0/2$ and the coherent receiver shown in Figure 2.26 is used.

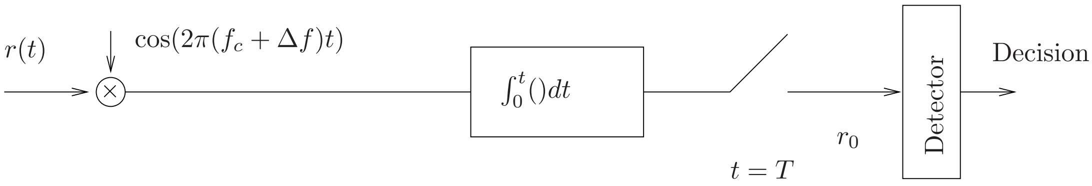


Figure 2.26: Receiver.

It is often impossible to design a receiver that operates at exactly the same carrier frequency as the transmitter. This small frequency offset between the oscillators is modeled by Δf.

The detector is designed so that the receiver is optimal when $\Delta f = 0$. Determine an exact expression for the average symbol error probability as a function of Δf. The approximation $f_c \gg \frac{1}{T}$ can of course be used. Can the receiver be worse than guessing (e.g. by flipping a coin) the transmitted symbol?

2–48 Before the advent of optical fibre, microwave links were the preferred method for long distance land telephoney. Consider transmitting a voice signal occupying 3 KHz over a microwave repeater system. Figure 2.27 illustrates the physical situation while Figure 2.28 shows the equivalent circuit. Each amplifier has automatic gain control with a fixed output power P. The noise is Gaussian white noise with a noise spectral density of 5 dB higher than kT. k is the Bolzmann constant and T is the temperature, use 290 Kelvin. For analogue voice the minimum acceptable received signal to noise ratio is 10 dB as measured in 3 KHz passband channel.

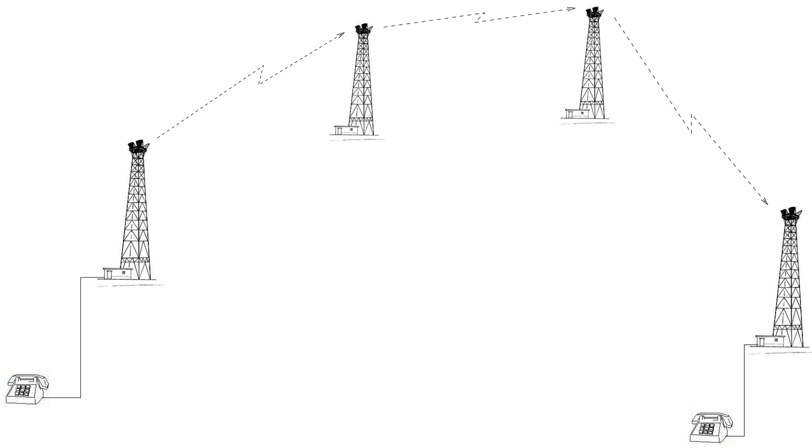


Figure 2.27: Repeater System Diagram

It is proposed that the system is converted to digital. The voice is simply passed through an analogue to digital converter creating a 64 kbit/s stream, 8 bit quantisation at 8000 samples per second. The digital voice is sent using BPSK and then passed through a digital to analogue converter at the other end. It was found that the maximum allowable error rate was 10^{-3}. The digital version uses more bandwidth but you can assume that the bandwidth is available and that the regenerative repeaters are used as shown in Figure 2.29.

Bolzman constant $k = 1.38 \times 10^{-23}\text{JK}^{-1}$.

(a) What is the minimum output power P required at each repeater by the analogue system? You may need to use approximation(s) to simplify the problem. Remember to explain all approximations used. **Hint:** Work out the gain of each amplifier if the signal to noise ratio is high.

(b) What is the minimum output power required P at each repeater by the digital system?

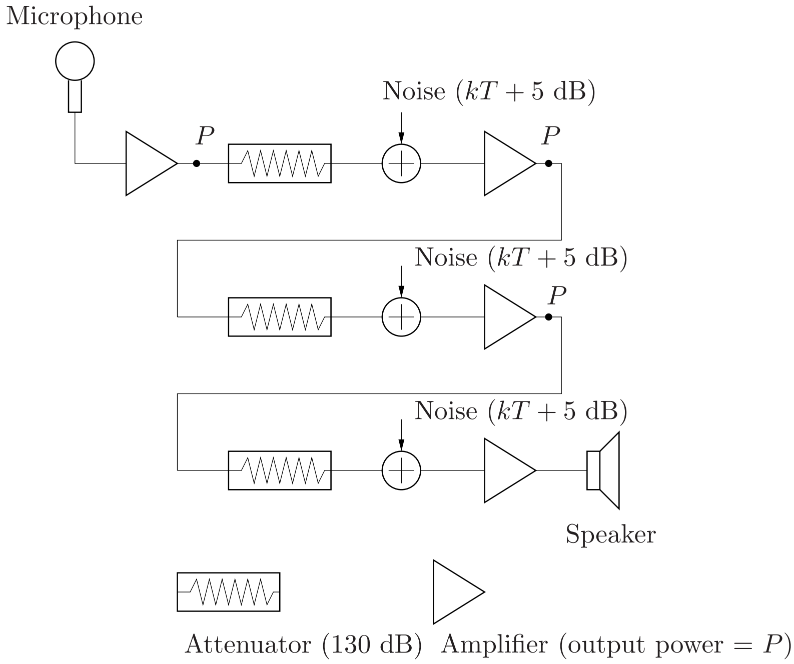


Figure 2.28: Repeater System Schematic

(c) Make atleast 2 suggestions as to how the power efficiency of the digital system could be improved.

2–49 Suppose that BPSK modulation is used to transmit binary data at the rate 10^5 bits per second, over an AWGN channel with noise power spectral density $N_0/2 = 10^{-10}$ W/Hz.

The transmitted signals can be written as

$$\begin{aligned} s_1(t) &= g(t)\cos(2\pi f_c t) \\ s_2(t) &= -g(t)\cos(2\pi f_c t) \end{aligned}$$

where $g(t)$ is a rectangular pulse of amplitude 2×10^{-2} V.

The receiver implements coherent demodulation, however with a carrier phase estimation error, $\Delta\phi = \frac{1}{7}\pi$. That is, the receiver "believes" the transmitted signal is

$$\pm g(t)\cos(2\pi f_c t + \Delta\phi)$$

(a) Derive a general expression for the resulting bit error probability, P_e, in terms of $E_g = \|g(t)\|^2$, N_0 and $\Delta\phi$.

(b) Calculate the resulting numerical value for P_e, and compare it to the case with no phase-error, $\Delta\phi = 0$.

2–50 Two equiprobable bits are Gray encoded onto a QPSK modulated signal. The signal is transmitted over a channel with AWGN with power spectral density $N_0/2$. The transmitted energy per bit is E_b. The receiver is designed to minimize the symbol error probability. What is approximately the required SNR ($\frac{2E_b}{N_0}$) in dB to achieve a *bit* error probability of 10%?

2–51 Parts of the receiver in a QPSK modulated communication system are illustrated in Figure 2.30. The system uses rectangular pulses and operates over an AWGN channel with infinite bandwidth.

(a) The local oscillator (i.e., the cos and sin in the figure) in the receiver must of course have the same frequency f_c as the transmitter for the system to work properly. However, in a wireless system, if the receiver is moving towards the transmitter the transmitter frequency

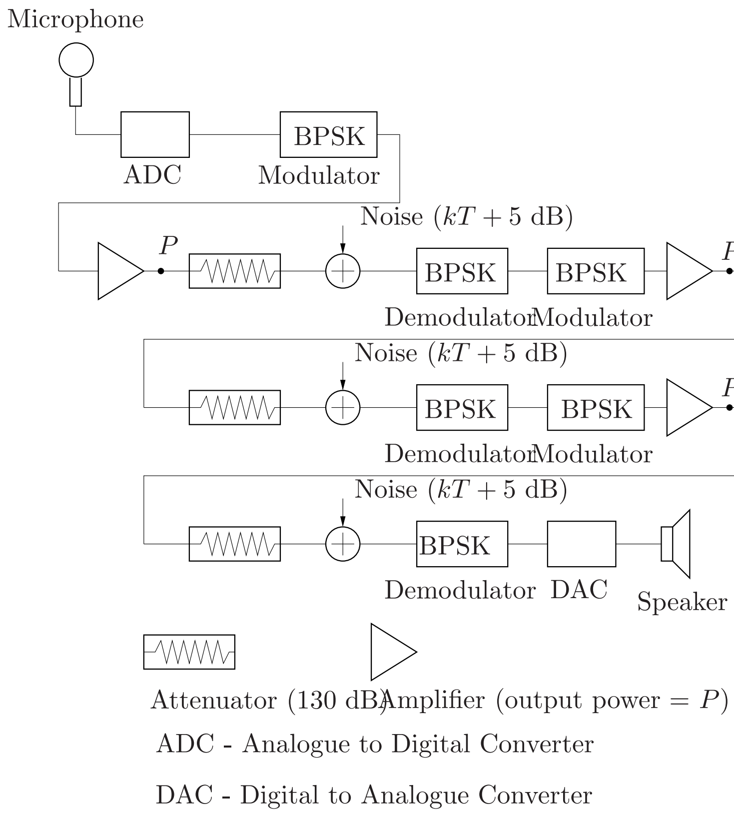


Figure 2.29: Regenerative Repeater System Schematic

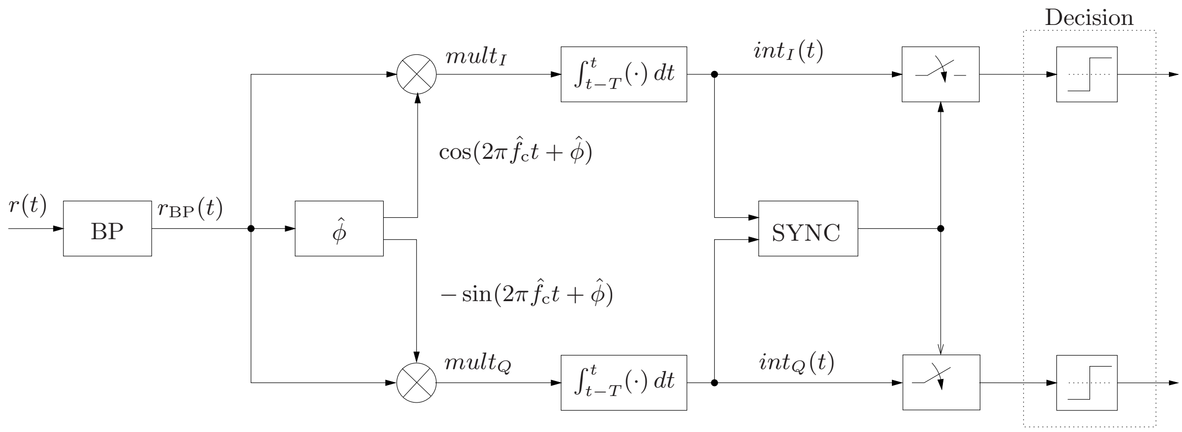


Figure 2.30: The receiver in Problem 2–51.

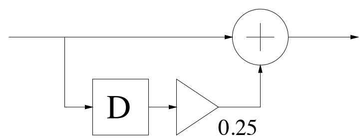


Figure 2.31: Channel model.

will appear as a higher frequency, $f_c + f_D$, than the frequency in the receiver due to the Doppler effect. How will this affect the signal constellation seen by the receiver if the carrier frequency $f_c = 1$ GHz, the Doppler frequency $f_D = 100$ Hz and the symbol time $T = 100$ μs?

(b) The communication system operates over the channel depicted in Figure 2.31. Sketch the signal $int_I(t)$ in the interval $[0, 2T]$ for all possible bit sequences (assume perfect phase estimates). Explain in words what this is called and what can be deducted from the sketch.

(c) Suggest at least one way of alleviating the reduced performance caused by the channel in the previous question and where in the block diagram your technique should be applied to the QPSK system considered.

(d) If you are asked to modify the system from QPSK to 8-PSK, which boxes in the block diagram do you need to change (or add) and how?

(e) Assume that the channel is band-limited instead of having infinite bandwidth. If you were allowed to redesign both the transmitter and the receiver for highest possible data rate, would you choose PAM or orthogonal modulation?

2–52 A modulation technique, of which a modification is used in the Japanese PDC system, is $\pi/4$-QPSK. In $\pi/4$-QPSK, the transmitter adds a phase offset to every transmitted symbol. The phase offset is increased by $\pi/4$ between every transmitted symbol. In contrast, ordinary QPSK does not add a phase offset to the transmitted symbols. Given knowledge of the phase offset, the receiver can demodulate the transmitted symbols.

(a) Assuming an AWGN channel and Gray coding of the transmitted bits, derive exact expressions for the bit and symbol error probabilities as functions of E_b and N_0!

(b) What are some of the advantages of $\pi/4$-QPSK compared to conventional QPSK? Hint: look at the phase of the transmitted signal.

2–53 Consider a QPSK system with a conventional transmitter using basis functions cos and sin, operating over an AWGN channel with noise spectral density $N_0/2$. The receiver, depicted in Figure 2.32, has a correlator-type front-end using the two functions

$$\Psi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f t)$$
$$\Psi_2(t) = \sqrt{\frac{2}{T}}\cos(2\pi f t + \pi/4)$$

How should the box marked *decision* in the figure be designed for an ML type receiver?

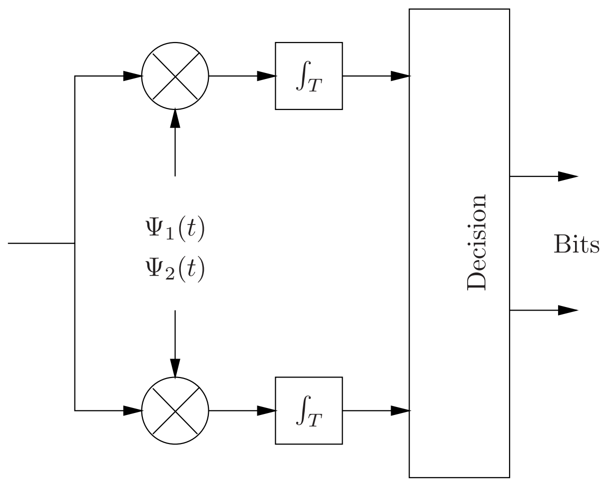


Figure 2.32: The front-end of the demodulator in Problem 2–53.

2–54 Parts of the receiver in a QPSK modulated communication system are illustrated in Figure 2.33. The system uses rectangular pulses and operates over an AWGN channel with infinite bandwidth.

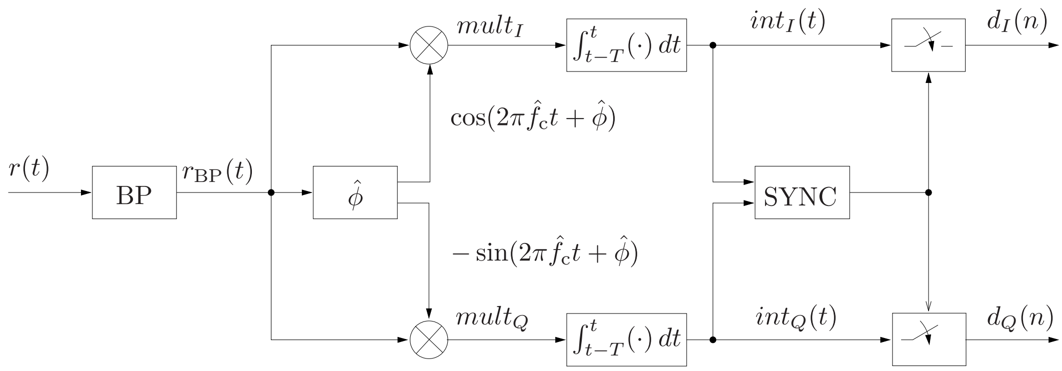


Figure 2.33: The receiver in Problem 2–54

(a) Sketch the signal $int_I(t)$ in the interval $[0, 2T]$ for all possible bit sequences (assume perfect phase estimates). Explain in words what this is called and what can be deducted from the sketch.

(b) When would you choose a coherent vs non-coherent (e.g., differential *demodulation*) receiver and why?

(c) In a QPSK system, the transmitter uses four different signal alternatives. Plot the QPSK signal constellation *in the receiver* (i.e., d_I and d_Q using the basis functions $\cos(2\pi\hat{f}_c t+\hat{\phi})$ and $-\sin(2\pi\hat{f}_c t+\hat{\phi})$) when the phase estimate in the receiver, $\hat{\phi}$, has an error of 30° compared to the transmitter phase.

(d) The phase estimator typically has problems with the noise. To improve this, the bandwidth of the existing bandpass filter (BP) can be reduced. If the bandwidth is reduced significantly, what happens then with $d_I(n)$ and $d_Q(n)$? Assume that the phase estimation and synchronization provides reliable estimates of the phase and timing, respectively.

(e) If the pulse shape is to be changed from the rectangular pulse $[0, T]$ to the pulse $h(t)$, limited to $[0, T]$, which block or blocks in Figure 2.33 should be modified for decent performance and how? Assume that reliable estimates of phase and timing are available.

2–55 In some applications, for example the uplink in the new WCDMA 3rd generation cellular standard, there is a need to dynamically change the data rate. In this problem, you will study one possible way of achieving this. Suppose we have two bit streams to transmit, one control channel with a constant bit rate and one information channel with varying bit rate, that are transmitted using (a form of) QPSK. The control bits are transmitted on the Q channel (Quadrature phase channel, the basis function $\sim \sin(2\pi ft+\varphi)$), while the information is transmitted on the I channel (In-phase channel, the basis function $\sim \cos(2\pi ft+\varphi)$). Antipodal modulation is used on both channels. The required average bit error probability of all transmitted bits, both control and information, is 10^{-3}.

Which *average* power is needed in the *transmitter* for the I and Q channels, respectively, assuming the signal power is attenuated by 10 dB in the channel and additive white Gaussian noise is present at the receiver input?

Q (control)	4 kbit/s
I (information)	0 kbit/s, 30% of the time
	8 kbit/s, 70% of the time
Channel power attenuation	10 dB
Noise PSD *at receiver*	$R_0 = 10^{-9}$ WHz

2–56 In this problem, QPSK signaling is considered. The four equally probable possible signal alternatives are given by

$$\begin{cases} \pm a\sqrt{\frac{2E}{T}}\cos(2\pi f_c t) \pm b\sqrt{\frac{2E}{T}}\sin(2\pi f_c t) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \qquad f_c \gg \frac{1}{T},$$

The system operates over an AWGN channel with power spectral density $N_0/2$ and the coherent ML-receiver shown in Figure 2.34 is used.

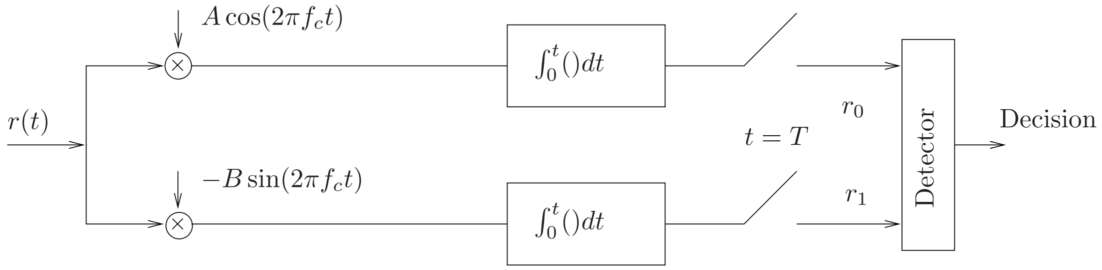


Figure 2.34: The receiver in Problem 2–56.

The detector is designed so that the receiver is optimum in the sense that the average symbol error probability is minimized for the case

$$a = b = A = B = 1\,.$$

The manufacturer cannot guarantee that $A = B = 1$ in the receiver. Suppose, however, that the transmitter is balanced, that is $a = b = 1$. Determine an *exact* expression for the average symbol error probability as a function of $A > 0$ and $B > 0$.

2–57 A communication engineer has collected field data from a wireless linear digital communication system. The receiver structure during the field trial is given in Figure 2.35. Thus, as usual, the receiver consists of bandpass filtering, down-conversion, integration and sampling. The true carrier frequency in the transmitter is f_c and the true phase in the transmitter is ϕ. The receiver estimates the phase of the carrier, $\hat{\phi}$, and also the carrier frequency, $\hat{f}_c$. All the other quantities needed in the receiver can be assumed to be known exactly. The channel from the transmitter to the receiver may introduce inter symbol interference (ISI). The outputs from the receiver, $d_I(n)$ and $d_Q(n)$, are the in-phase and quadrature-phase components, respectively. Unfortunately, the engineer has not a very good memory, so back in his office after the field trial he needs to figure out several things.

(a) He measured the spectra for $r(t)$, $r_{\text{BP}}(t)$ and $mult_{\text{Q}}(t)$. The channel between the transmitter and receiver can in this case be approximated with an additive white Gaussian noise channel (AWGN) without ISI. The spectra are shown in Figure 2.36 (next page). Determine which signal is which and, in addition, estimate the carrier frequency! Motivate your estimate of the carrier frequency.

(b) In the field trial they of course also measured the outputs $d_I(n)$ and $d_Q(n)$ under different conditions. The different conditions were:

i. An offset in the carrier frequencies between the transmitter and the receiver. That is, $f_c \neq \hat{f}_c$, $\phi = \hat{\phi}$ and no inter symbol interference (ISI).

ii. Inter symbol interference (ISI) in the channel from the transmitter to the receiver. Otherwise, $f_c = \hat{f}_c$ and the phase estimate is irrelevant when ISI is present.

iii. A combination of a frequency offset and some ISI in the channel. The phase estimate is irrelevant when ISI is present.

iv. A phase offset in the receiver. Otherwise, no ISI and $f_c = \hat{f}_c$.

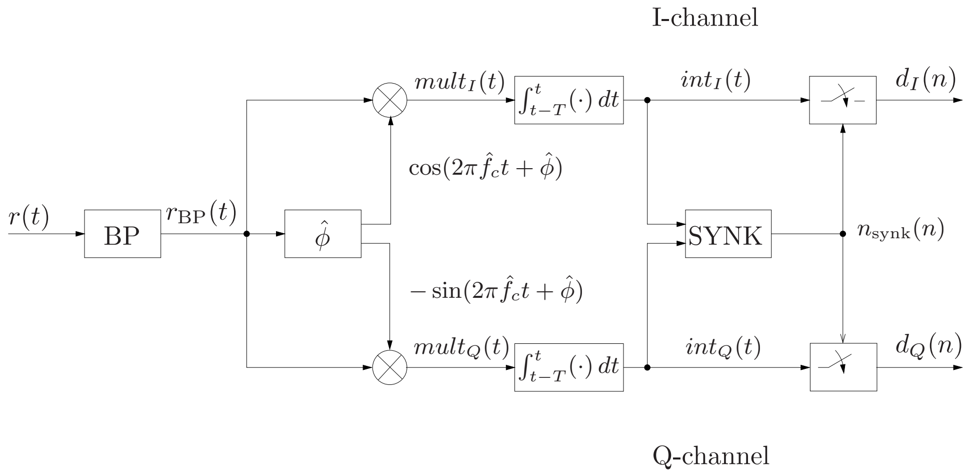


Figure 2.35: The receiver structure during the field trial.

These four cases are shown in Figure 2.37. Determine what type of signaling scheme they used in the field trial and also determine which condition in the above list is what constellation. (That is, does Condition 1 and Constellation A match, etc.)

2–58 In this problem we study the impact of phase estimation errors on the performance of a QPSK system. Consider one shot signalling with the following four possible signal alternatives.

$$s_m(t) = \begin{cases} \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \frac{2\pi}{4}m - \frac{\pi}{4} + \phi) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}, \qquad f_c \gg \frac{1}{T}, \qquad m = 1, 2, 3, 4\,,$$

where ϕ represents the phase shift introduced by the channel. The system operates over an AWGN channel with noise spectral density $N_0/2$ and the coherent ML receiver shown in Figure 2.38 is used.

The detector is designed for optimal detection in the case when $\phi = \hat{\phi}$. However, in practice the phase cannot in general be estimated without errors and therefore we typically have $\hat{\phi} \neq \phi$.

Derive an exact expression for the resulting symbol error probability, assuming $|\phi - \hat{\phi}| \le \pi/4$.

2–59 The company Nilsson, with a background in producing rubber boots, is about to enter the communication market. For a future product, they are comparing two alternatives: a single carrier QAM system and a multicarrier QPSK system. Both systems have identical bit rates, R, use rectangular pulse shaping, and operate at identical average transmitted powers, P. In the multicarrier system, the data stream, consisting of independent equiprobable bits, is split into four streams of lower rate. Each such low-rate stream is used to QPSK modulate carriers centered around f_i, $i = 1, \dots, 4$. The carrier phases are different and independent of each other. In the 256-QAM system, eight bits at a time are used for selecting the symbol to be transmitted.

(a) How should the subcarriers be spaced to avoid inter-carrier interference and in order not to consume unnecessary bandwidth?

(b) Derive and plot expressions for the symbol error probability for coherent detection in the two cases as a function of average SNR $= 2E_b/N_0$.

(c) Sketch the spectra for the two systems as a function of the normalized frequency fT, where $T = 8/R$ is the symbol duration. Normalize each spectrum, i.e., let the highest peak be at 0 dB.

(d) Which system would you prefer if the channel has an unknown (time-varying) gain? Motivate!

2–60 A common technique to counteract signal fading (time-varying amplitude and phase variations) in radio communications is based on utilizing several receiver antennas. The received signals in

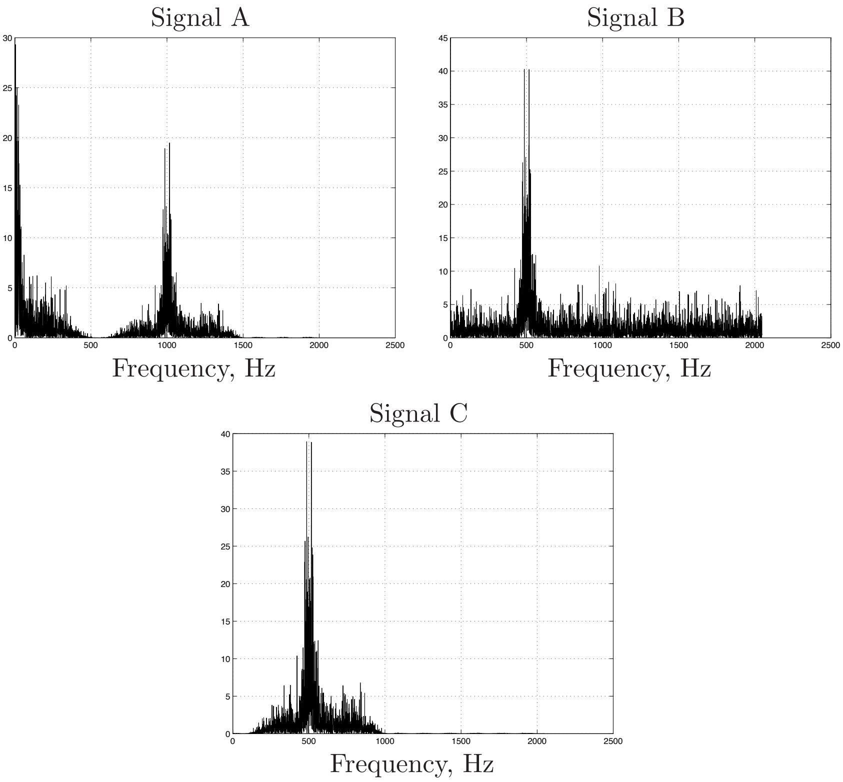


Figure 2.36: Frequency representations of the three signals $r(t)$, $r_{\mathrm{BP}}(t)$ and $mult_{\mathrm{Q}}(t)$ in the communication system (not necessarily in that order!).

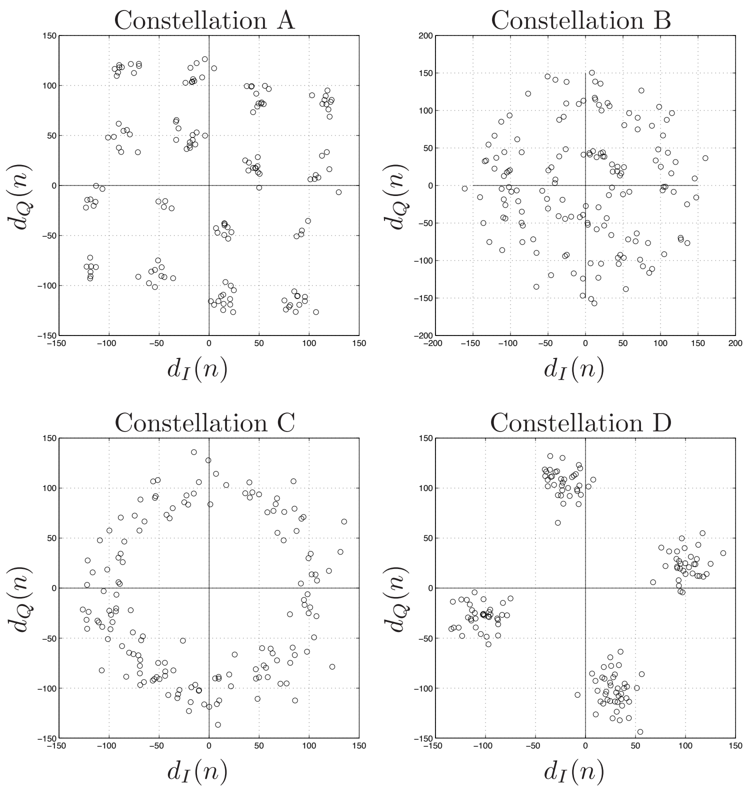


Figure 2.37: Signal constellations for each of the four conditions in Problem 2–57.

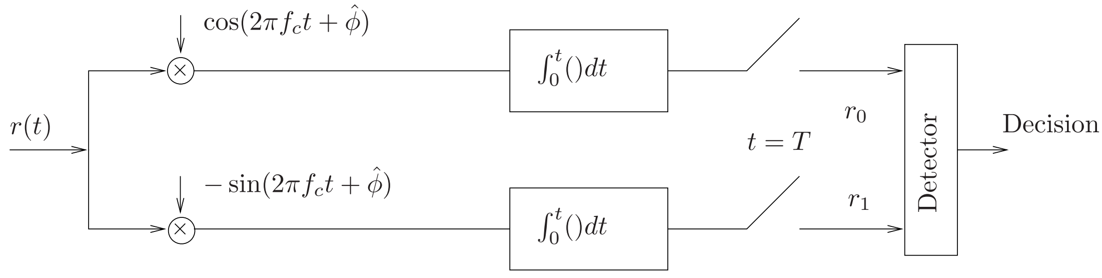
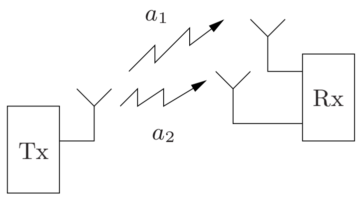


Figure 2.38: Receiver.

the different receiver antennas are demodulated and *combined* into one signal that is then fed to a detector. This technique, of using several antennas, is called *space diversity* and the process of combining the contributions from the different antennas is called *diversity combining.*

To study a very simple system utilizing space diversity we consider the figure below, using one transmitter and two receiver antennas.



Assume that the transmitted signal is a QPSK signal, chosen from the set $\{s_i(t)\}$ with

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_c t + i\pi/4), \quad 0 \le t \le T, \quad i = 1, \ldots, 4$$

where f_c = (large integer)$\cdot 1/T$ is the carrier frequency, and where the transmitted signal alternatives are equally likely and independent. The received signal in the mth receiver antenna ($m = 1, 2$) can be modeled as

$$r_m(t) = a_m s_i(t) + w_m(t)$$

where a_m is the propagation attenuation ($0 < a_m < 1$) corresponding to the path from the transmitter antenna to receiver antenna m, and where $w_m(t)$ is AWGN with spectral density $N_0/2$. Assume that the noise contributions $w_m(t)$, $m = 1, 2$, are independent of each-other, and that the attenuations a_m, $m = 1, 2$, are real-valued constants that are known to the receiver[1].

The signal $r_m(t)$, $m = 1, 2$, is demodulated into the discrete variables

$$r_{cm} = \sqrt{\frac{2}{T}} \int_0^T r_m(t) \cos(2\pi f_c t) dt, \qquad r_{sm} = -\sqrt{\frac{2}{T}} \int_0^T r_m(t) \sin(2\pi f_c t)$$

Letting $\mathbf{r}_m = (r_{cm}, r_{sm})$, $m = 1, 2$, the contributions from the two antennas are then combined into one vector $\mathbf{u}$ according to

$$\mathbf{u} = b_1 \mathbf{r}_1 + b_2 \mathbf{r}_2,$$

where b_1 and b_2 are real-valued constants. The vector $\mathbf{u}$ is then fed to an ML-detector deciding on the transmitted signal based on the observed value of $\mathbf{u}$.

How should the parameters b_1 and b_2 be chosen to minimize the resulting symbol error probability and which is the corresponding minimum value of the error probability?

[1] A more reasonable assumption, in practice, would be that the amplitudes are modeled as Rayleigh distributed random variables that are estimated, and hence not perfectly known. We also notice that the model for the received signal does not take into account potential propagation delays and/or phase distortion.

2–61 An example of a receiver in a QPSK communication system using rectangular pulse shaping is shown in Figure 2.39. The system operates over an AWGN channel with an unknown phase offset and uses a 10 bit training sequence for phase estimation and synchronization followed by 1000 information bits. A number of different signals has been measured in the receiver at various points, denoted with numbers in gray circles, and the results are shown in Figure 2.40. Eight samples were taken for each symbol, i.e., the oversampling factor is eight, and all time axes refer to the sample number (and not the symbol number). The E_b/N_0 were 10 dB when the measurements were taken unless otherwise noted.

(a) Pair the plots A–D in Figure 2.40 with the corresponding measurement points in Figure 2.39! Carefully explain you answer.

(b) Studying the plots in Figure 2.40, what do you think is the main reason for the degraded performance compared to the theoretical curve? Why? Which box in Figure 2.39 would you start improving?

(c) Explain how the plot E was obtained and what can be concluded from it. Which, one or several, of the measurement points in Figure 2.39 was used and how?

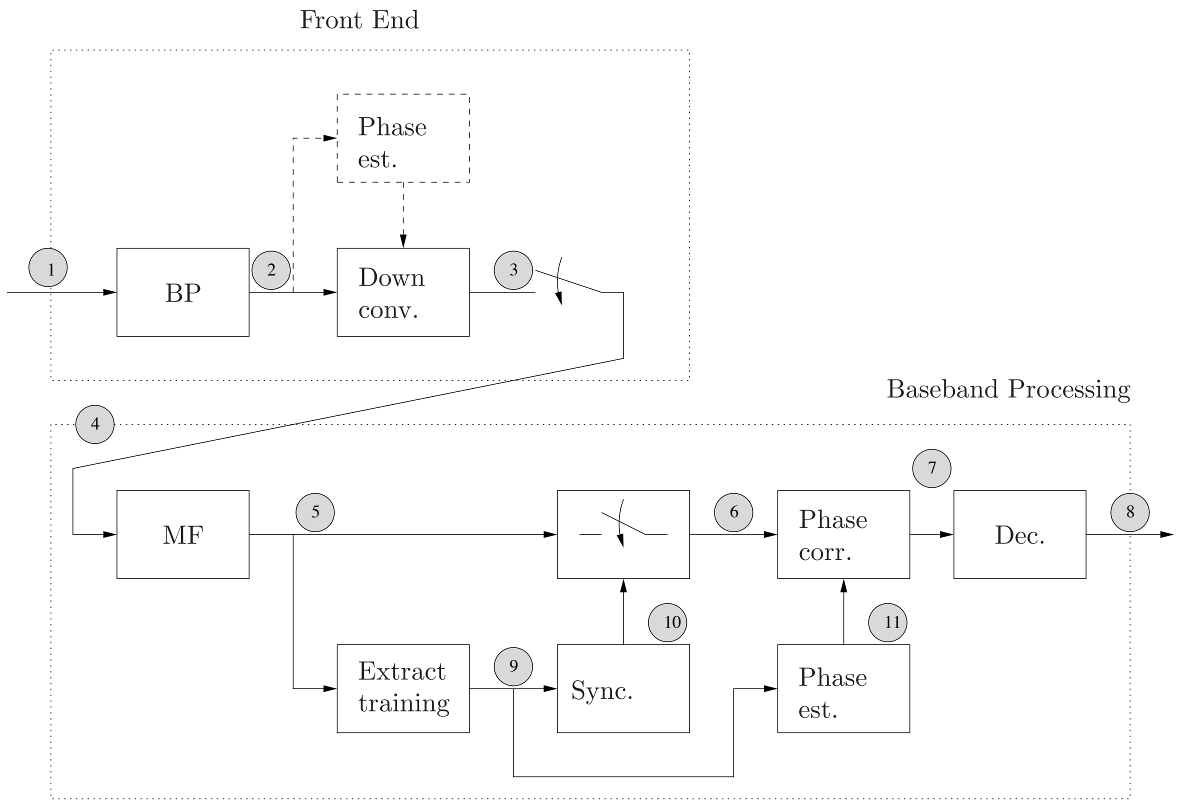


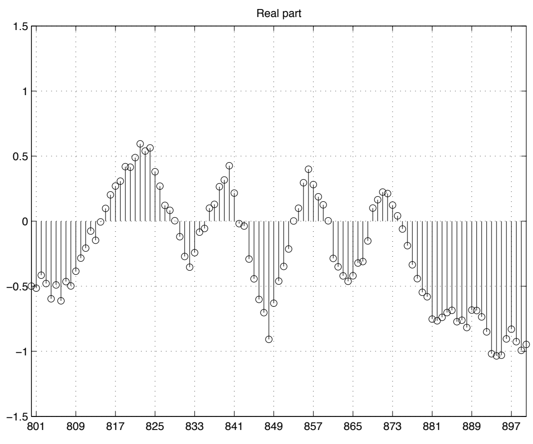
Figure 2.39: A QPSK receiver.

2–62 Two different data transmission systems A and B are based on size-L signal sets (with L a large integer). Transmitted symbols are equally likely. System A uses equidistant PAM, with signals at distance d_A, and system B uses PSK with signals evenly distributed over a circle (in signal space) and at distance d_B.

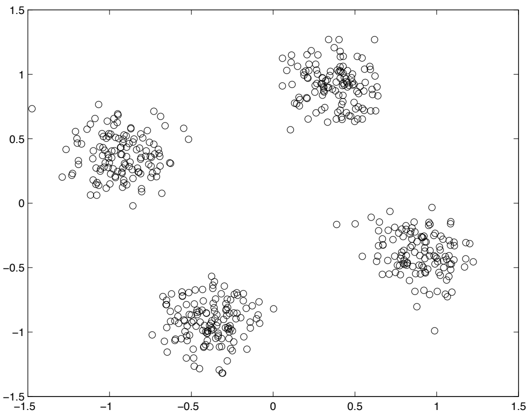
(a) The two systems will have approximately the same error rate performance when $d_A = d_B$. Determine the ratio between the *peak signal powers* of the systems when $d_A = d_B$.

(b) Determine the ratio between *average signal powers* when $d_A = d_B$.

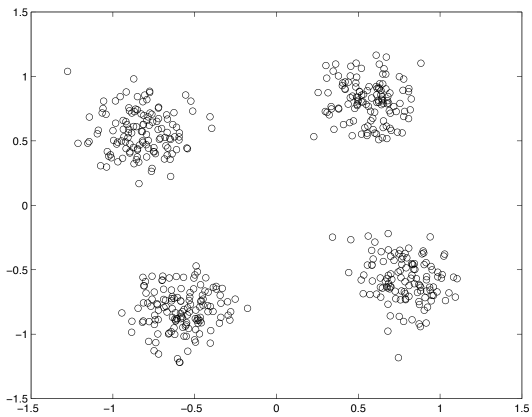
2–63 Consider an 8-PSK constellation with symbol energy E used over an AWGN channel with spectral density $N_0/2$ and with an optimum (minimum symbol-error probability) receiver. The constellation is employed in mapping three equally likely and independent bits to each of the eight PSK



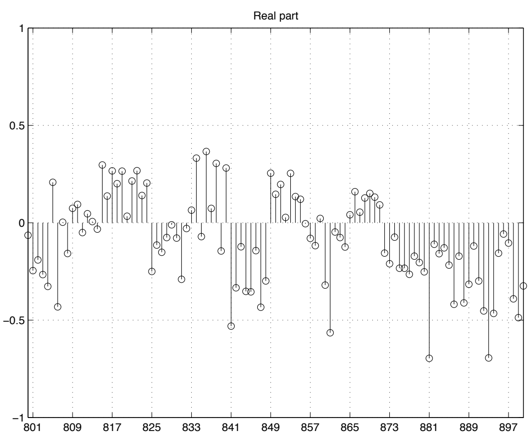
A



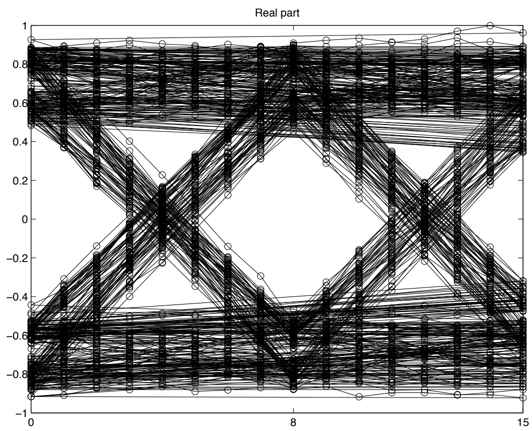
B



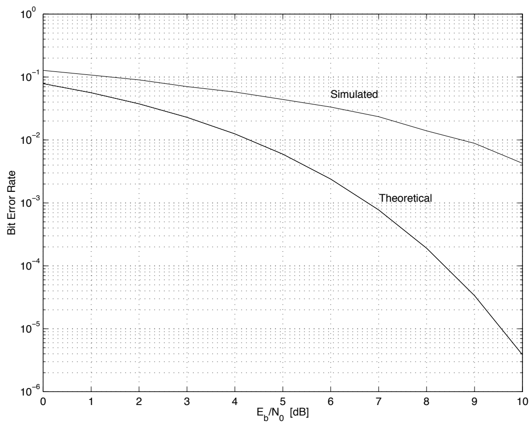
C



D



E (E_b/N_0=20 dB)



BER

Figure 2.40: The plotted signals.

symbols. Assuming a large ratio E/N_0 show that some mappings of the three bits yield a different bit-error probability for different bits. This effect is a kind of "unequal error protection" and is useful when the bits are of different "importance." Specify the mappings that yield equal error probabilities for the bits and in addition the mappings that yield the largest ratio between the maximum and minimum error probability.

2–64 Consider carrier-modulated M-PSK, on the form

$$u(t) = Re\left\{\sum_n x_n g(t-nT)e^{j2\pi f_c t}\right\}$$

with equally likely and independent (in n) symbols

$$x_n \in \{e^{\phi_0}, e^{\phi_1}, \dots, e^{j\phi_{M-1}}\}$$

where $\phi_m = 2\pi\, m/M, m = 0, \dots, M-1$, and with the pulse

$$g(t) = \sqrt{\frac{2}{T}}\sin(\pi t/T), \quad 0 \le t \le T$$

(with $g(t) = 0$ for $t < 0$ and $t > T$).

Consider now a receiver that first converts the received bandpass signal (the signal $u(t)$ in AWGN) to a complex baseband signal

$$y(t) = \sum_n x_n g(t - \tau - nT) + n(t)$$

where $\tau < T/2$ models an *unknown propagation delay*, and where $n(t)$ is complex-valued AWGN, that is

$$n(t) = n_c(t) + jn_s(t)$$

where $n_c(t)$ and $n_s(t)$ are independent real-valued AWGN processes, both with spectral density $N_0/2$. (Since the conversion from bandpass to complex baseband involves filtering, the noise $n(t)$ will not be exactly white, however we model it as being perfectly white.)

Correlation demodulation is then implemented in the complex domain on the signal $y(t)$, more precisely the receiver forms the decision variables

$$y_n = \int_{nT}^{(n+1)T} g(t-nT)y(t)dt$$

Note that the receiver *does not compensate for the unknown delay* τ.

Based on y_n a decision $\hat{x}_n$ is then computed as

$$\hat{x}_n = \arg\min_x |y_n - x|$$

over all $x \in \{e^{j\phi_0}, e^{j\phi_1}, \dots, e^{j\phi_{M-1}}\}$ (the possible values for x_n).

Assume $M = 8$ (8-PSK), and $\tau = T/4$. Compute an expression for the symbol error probability $P_e = \Pr(\hat{x}_n \neq x_n)$. You may use approximations based on the assumption $1/N_0 \gg 1$.

2–65 Consider a coherent binary FSK modulator and demodulator. The mth transmitted bit is denoted $d_m \in \{0, 1\}$ and the two alternatives are equally probable. The transmitted signal as a function of time t is $x(t)$,

$$x(t) = \sum_{\forall m} x_m(t)$$

$$x_m(t) = \begin{cases} \sqcap(1000t - m)e^{j2\pi 500t} & \text{if } d_m = 0 \\ \sqcap(1000t - m)e^{j2\pi 1500t} & \text{if } d_m = 1, \end{cases}$$

where the rectangular function $\sqcap(\cdot)$ is defined as

$$\sqcap(\alpha) = \begin{cases} 1 & \text{if} \quad -\frac{1}{2} < \alpha < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

The communications channel adds complex AWGN $n(t)$ with auto correlation

$$E[n(t)n^*(t-\tau)] = N_0\,\delta(t-\tau)$$

as well as carrier wave interference $e^{j2\pi 2000t}$ to the signal and the received signal is

$$r(t) = x(t) + n(t) + e^{j2\pi 2000t} \quad .$$

The precise phase and frequency of the interference is known.

The FSK demodulator has perfect timing, phase and frequency synchronization. The first step of the demodulation process is to generate 2 decision variables, $v_{0,m}$ and $v_{1,m}$,

$$v_{0,m} = \int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} r(t)e^{-j2\pi 500t}\,dt$$

$$v_{1,m} = \int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} r(t)e^{-j2\pi 1500t}\,dt \quad .$$

(a) Find an expression for $v_{0,m}$ and $v_{1,m}$ in terms of d_m and m!

(b) Find the optimum decision rule for estimating d_m using $v_{0,m}$ and $v_{1,m}$! The optimum decision rule is the one that minimizes the probability of bit error.

2–66 Consider a binary FSK system over an AWGN channel with noise variance $N_0/2$. Equally probable symbols are transmitted with the two signal alternatives

$$s_1(t) = \begin{cases} \sqrt{\frac{2E}{T}}\sin\frac{2\pi t}{T}, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases}$$

$$s_2(t) = \begin{cases} \sqrt{\frac{2E}{T}}\sin\frac{4\pi t}{T}, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases}$$

A coherent receiver is used that is optimized for the signals above. Using this receiver, the transmitted signals are changed to

$$u_1(t) = \begin{cases} \sqrt{\frac{2E}{T}}\sin\frac{2\pi t}{T}, & 0 \le t < \rho T \\ 0, & \text{otherwise} \end{cases}$$

$$u_2(t) = \begin{cases} \sqrt{\frac{2E}{T}}\sin\frac{4\pi t}{T}, & 0 \le t < \rho T \\ 0, & \text{otherwise} \end{cases}$$

where $0 \le \rho \le 1$. Thus, the signals are zero during part of the symbol interval.

(a) Determine the Euclidean distance in the vector model between the signal alternatives $u_1(t)$ and $u_2(t)$.

(b) Is the receiver constructed for $s_1(t)$ and $s_2(t)$ also optimal when $u_1(t)$ and $u_2(t)$ are transmitted?

2–67 In this problem we compare coherent and non-coherent detection of binary FSK. A coherent receiver must know the signal phase. In situations where the phase is badly estimated or unknown a non-coherent receiver typically gives better performance.

Consider one-shot signalling using the following two signal alternatives.

$$s_m(t) = \begin{cases} \sqrt{\frac{2E_b}{T}} \cos(2\pi f_c t + 2\pi m \triangle f t) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \qquad m = 0, 1$$

$$\triangle f = \frac{1}{T} \ll f_c \,.$$

The signals are equally likely, and transmission takes place over an AWGN channel. Figure 2.41 illustrates a coherent receiver. The detector decides that $s_0(t)$ was transmitted if $r_0 \ge r_1$.

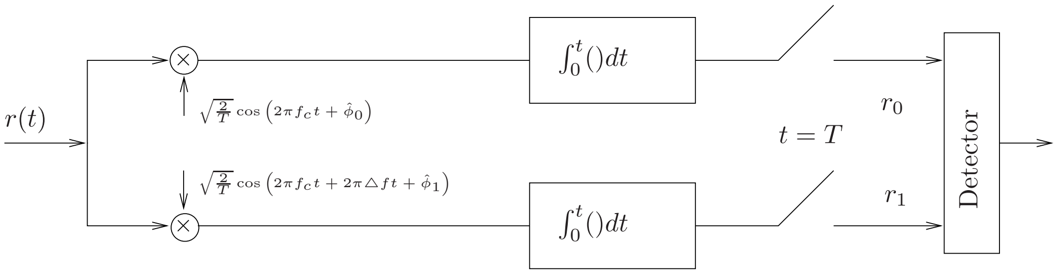


Figure 2.41: Coherent receiver.

Furthermore, a non-coherent receiver is shown in Figure 2.42. In this case the receiver decides $s_0(t)$ if $d = (r_{0c}^2 + r_{0s}^2) - (r_{1c}^2 + r_{1s}^2) \ge 0$ Now assume that $s_0(t)$ was transmitted. The received

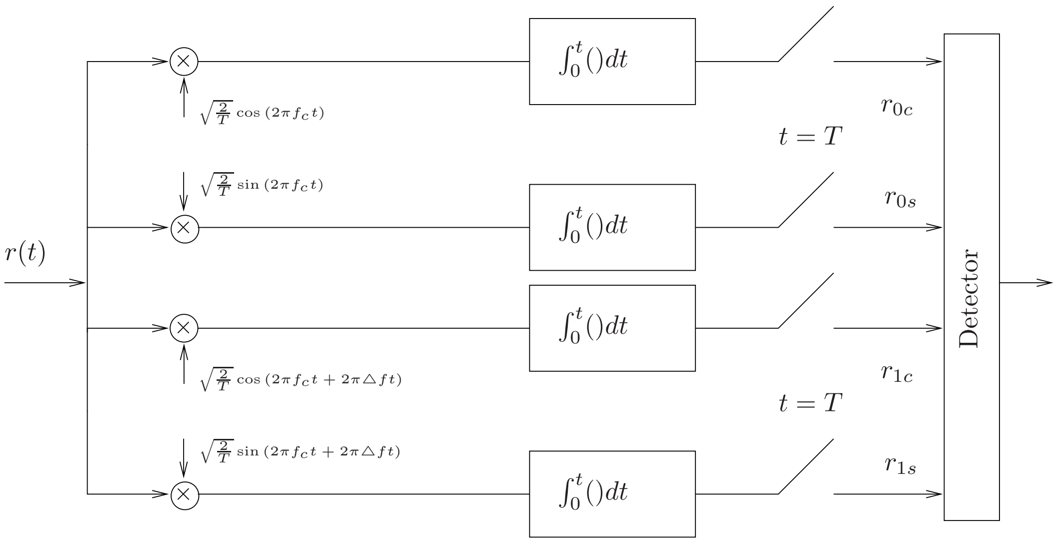


Figure 2.42: Non-coherent receiver.

signal is then obtained as

$$r(t) = \sqrt{\frac{2E_b}{T}} \cos(2\pi f_c t + \phi_0) + n(t)$$

for $0 \le t \le T$, where the AWGN $n(t)$ has spectral density $N_0/2$. Assume that $10 \log_{10}(2E_b/N_0) =$ 10 dB. How large an estimation error $|\phi_0 - \hat{\phi}_0|$ can at maximum be tolerated by the coherent receiver to maintain better performance than the non-coherent receiver?

2–68 Your future employer has heard that you have taken a class in Digital Communications. She tells you that she considers the design of a new communication system to transmit and receive carrier modulated digital information over a channel where AWGN with power spectral density $N_0/2$

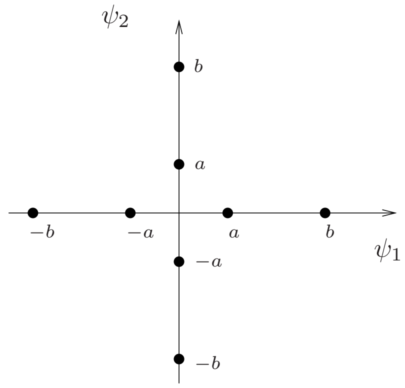


Figure 2.43: Signal constellation.

is added. She wants to be able to transmit $\log_2(M)$ bits every T seconds, so she considers two different M-ary carrier modulated signal sets that are to be used every T seconds. She mentions that a phase-coherent receiver will be used that minimizes the probability that the wrong signal is detected.

The signal sets under consideration are

$$\begin{aligned} s_i^1(t) &= \sqrt{\frac{2E_s}{T}}\cos(2\pi f_c t + 2\pi i/M) \\ s_i^2(t) &= \sqrt{\frac{2E_s}{T}}\cos(2\pi f_c t + 2\pi i \Delta f\, t) \end{aligned}$$

for $0 \le t < T$, where

$$\begin{aligned} i &\in \{0, \dots, M-1\} \\ \Delta f &= \frac{1}{2T} \\ E_s/N_0 &= 16 \\ f_c &\gg \frac{1}{T} \end{aligned}$$

The value of M is still open.

(a) Your employer's task for you is to find which of the signal sets 1 and 2 that is preferable from a symbol error probability perspective for different values of M.

(b) Discuss the feasibility of the two signal sets for different values of M.

2–69 Consider the signal constellation depicted in Figure 2.43. The signals are used over an AWGN channel. An optimal receiver is employed. Assume that the SNR in the transmission is high and that all signals are used with equal probability. Assume also that $0 < a < b$.

(a) Assume a fixed b. How should a be chosen to minimize the symbol error probability?

(b) How much higher/lower average transmit energy is needed in order for the system to give the same error rate performance as an 8PSK system (at high SNRs)? Assume $b = 3a$.

2–70 Consider the 16-PSK, and 16-Star signal constellations as shown in Figure 2.44.

(a) Find an optimal ratio R_1/R_2 for the 16-Star constellation. Here "optimal" means the ratio resulting in the lowest probability of symbol error.

(b) Derive an equation for the bit error probability versus E_b/N_0 for both constellations. It is very difficult to get an exact solution to this problem. Approximations are acceptable however the approximations used must be stated. Use the optimum R_1/R_2 as determined in Part a.

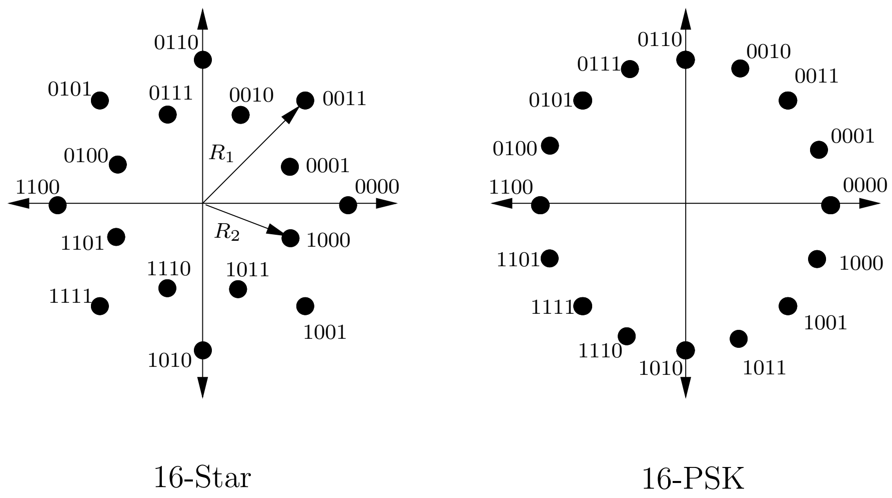


Figure 2.44: Constellations.

2–71 Consider the 16-QAM signal constellation depicted in Figure 2.45. With this constellation, where the mapping is Gray-coded, different bits will have different error probabilities.

(a) Find an asymptotic relation (high E_b/N_0, AWGN channel) **between** the error probabilities for the four bits carried by each transmitted symbol! Note that you don't need to explicitly compute the values of the bit error probabilities, i.e. a solution where the bit error probabilities are compared relative to each other is fine.

(b) Consider a scenario where the 16QAM system of above is used for conveying information from four different users. As illustrated in Figure 2.46, the data bits of the four users are first multiplexed into one bit stream which, in turn, is mapped into 16QAM symbols and then transmitted (i.e., each user transmits on every fourth bit). At the receiver, the 16QAM symbols are decoded and demultiplexed into the bit sequences for the different users.

If all the users are to transmit a large number of bits, how can the system be modified to give the users approximately the same bit error probability, averaged over time?

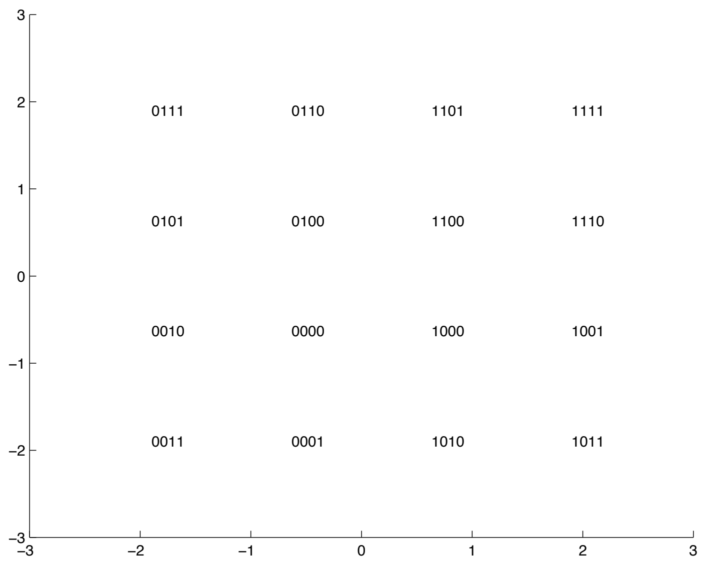


Figure 2.45: 16QAM signal constellation.

2–72 In a multicarrier system, the high-rate bitstream is split into several low-rate streams and each

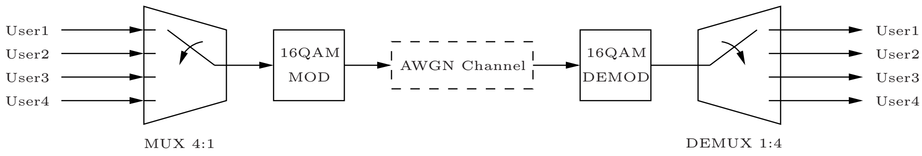


Figure 2.46: A four user 16QAM system.

low-rate stream is transmitted on a separate carrier. These systems can be designed so that the inter carrier interference is negligible. Consider two systems, a single-carrier system using 64-QAM and a multicarrier system using 3 separate QPSK carriers. Both systems have a bitrate of R, a total average transmit power of P and operate over an AWGN channel with two-sided noise spectral density of $N_0/2$.

Which system is to prefer from a symbol error probability point of view and how large is the gain in dB at an error probability of 10^{-2}? For the multicarrier QPSK (and the 64-QAM) system a symbol is considered to contain 6 bits.

2–73 Equally likely and independent information bits arrive at a constant rate of 1000 bits per second. These bits are divided into blocks of L bits and each L-bit block is transmitted based on linear memoryless carrier modulation at carrier frequency $f_c = 10^9$ Hz. The transmitted signal is hence of the form

$$u(t) = Re\left\{\sum_n g(t-nT)x_n e^{j2\pi f_c t}\right\}$$

where $g(t)$ is a rectangular pulse

$$g(t) = \begin{cases} A, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

and $\{x_n\}$ are complex information symbols, each carrying L bits of information. (The signaling rate $1/T$ thus depends on L.) The signal $u(t)$ is transmitted over an AWGN channel, producing a received signal

$$r(t) = u(t) + w(t)$$

where $w(t)$ is AWGN of spectral density $N_0/2$, with $N_0 = 0.004$ V^2/Hz. The receiver implements optimal ML demodulation/detection, producing symbol estimates $\hat{x}_n$. Find *one* modulation format out of the following nine formats

- Uniform 4-ASK, 8-ASK or 16-ASK
- Uniform 4-PSK, 8-PSK or 16-PSK
- Rectangular 4-QAM, 16-QAM or 64-QAM

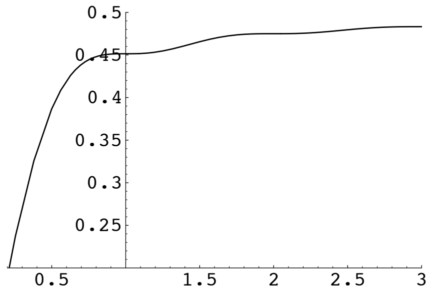
that will result in a transmission that fulfills all of the following criteria

- Symbol error probability $P_e = \Pr(\hat{x}_n \neq x_n) < 0.01$
- Average transmitted power < 100 V^2
- At least 90% of the transmitted power within the frequency range $|\pm f_c \pm B|$ with $B = 250$ Hz

Useful result: The value of

$$f(x) = \int_0^x \left[\frac{\sin \pi\tau}{\pi\tau}\right]^2 d\tau$$

in the range $0.2 < x < 3$ is plotted below



and it holds that $f(0.5) \approx 0.387, f(1) \approx 0.451, f(2) \approx 0.475, f(3) \approx 0.483, f(\infty) = 0.5$.

2–74 Consider the carrier-modulated signal, on the form

$$u(t) = Re\left\{\sum_n x_n g(t - nT)e^{j2\pi f_c t}\right\}$$

with equally likely and independent (in n) symbols

$$x_n \in \{-3a, -a, a, 3a, -j3a, -ja, ja, j3a\}$$

and with the pulse

$$g(t) = \sqrt{\frac{2}{T}}\,\sin(\pi t/T), \quad 0 \le t \le T$$

(with $g(t) = 0$ for $t < 0$ and $t > T$).

Consider now a receiver that first converts the received bandpass signal (the signal $u(t)$ in AWGN) to a complex baseband signal

$$y(t) = \sum_n x_n e^{j\phi} g(t - nT) + n(t)$$

where ϕ models an *unknown phase shift*, and where $n(t)$ is complex-valued AWGN, that is

$$n(t) = n_c(t) + jn_s(t)$$

where $n_c(t)$ and $n_s(t)$ are independent real-valued AWGN processes, both with spectral density $N_0/2$. (Since the conversion from bandpass to complex baseband involves filtering, the noise $n(t)$ will not be exactly white, however we model it as being perfectly white.)

Correlation demodulation is then implemented in the complex domain on the signal $y(t)$, more precisely the receiver forms the decision variables

$$y_n = \int_{nT}^{(n+1)T} g(t - nT)y(t)dt$$

Note that the receiver *does not compensate for the phase-shift* ϕ.

Based on y_n a decision $\hat{x}_n$ is then computed as

$$\hat{x}_n = \arg\min_x |y_n - x|$$

over all x (the possible values for x_n).

(a) Compute the spectral density of the transmitted signal.

(b) Compute an expression for the symbol error probability $P_e = \Pr(\hat{x}_n \ne x_n)$ in the case that the phase-shift is $\phi = \pi/8$. You may use approximations based on the assumption $1/N_0 \gg 1$.

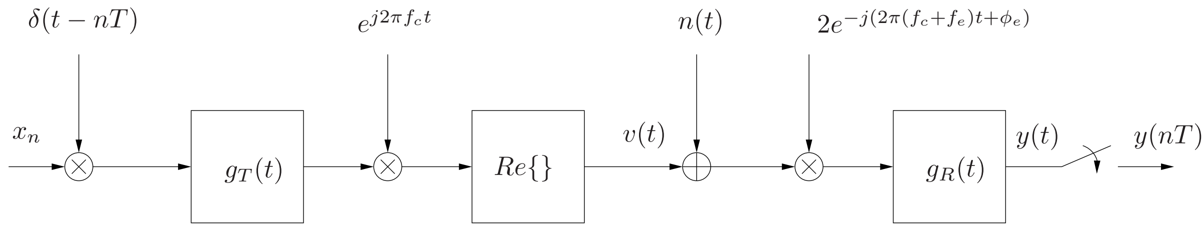


Figure 2.47: Carrier modulation system.

2–75 Consider the communication system depicted in Figure 2.47.

The independent and equiprobable information symbols $x_n = e^{j\phi_n}$, where $\phi_n \in \{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$, are transmitted on a carrier with frequency f_c. The resulting signal is transmitted over a channel with AWGN $n(t)$ with power spectral density $N_0/2$. In the receiver, the received signal is imperfectly down-converted to baseband, due to a frequency error f_e and phase error ϕ_e. Assume $f_e < \frac{1}{2T} \ll f_c$.

The impulse responses of the transmitter and receiver filters are

$$\begin{aligned} g_T(t) &= \frac{\sin(\pi t/T)}{\pi t} \\ g_R(t) &= \frac{\sin(2\pi t/T)}{\pi t} \end{aligned}$$

for $-\infty < t < \infty$.

(a) Derive and plot the power spectral density of the transmitted signal $v(t)$.

(b) Derive an expression for $y(nT)$ as a function of f_e, ϕ_e, ϕ_n and $n(t)$.

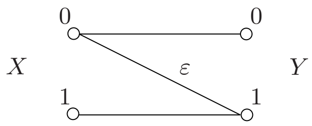
(c) Find the autocorrelation function for the noise in $y(t)$. Is this noise additive, white and Gaussian?

(d) The transmitted information symbols x_n are estimated from the sampled complex baseband signal $y(nT)$ using a symbol-by-symbol detector. The detector is optimal when $f_e = \phi_e = 0$, in terms of symbol error probability. Derive the symbol error probability assuming $f_e = 0$ and $0 < |\phi_e| < \frac{\pi}{4}$.

Chapter 3

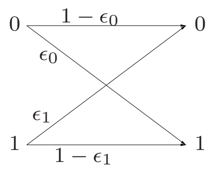
Channel Capacity and Coding

3–1 Consider the binary memoryless channel illustrated below.



With $\Pr(X = 1) = \Pr(X = 0) = 1/2$ and $\varepsilon = 0.05$. Compute the mutual information $I(X;Y)$.

3–2 Consider the binary memoryless channel



(a) Assume $p_0 = \Pr\{0 \text{ is transmitted}\} = \frac{1}{4}$ and $\epsilon_0 = 3\epsilon_1$. Find the average error probability $\Pr(\,\text{output} \neq \text{input}\,)$.

(b) Apply a negative decision rule, that is, interpret a received 0 as 1 and a received 1 as 0. Find the error probability as a function of ϵ_0. When will this error probability be less than the one in (a)?

3–3 Consider the binary OOK communication system depicted in Figure 3.1, where a pulse represents a binary "1" and no pulse represents a "0". The two independent transmitters share the same bandwidth for transmitting equiprobable binary symbols over the AWGN channel. Of course the two transmitters will interfere with each other. However, a multiuser decoder can, to some extent, separate the two users and reliably detect the information coming from each user.

The detector, which only has access to hard decisions, 0 or 1, is characterized by three probabilities:

$P_{\mathrm{d},1}$ the detection failure when *one* transmitter is transmitting "1".

$P_{\mathrm{d},2}$ the detection failure when *two* transmitters are transmitting "1".

P_f the false alarm probability.

A *false alarm* is when the output of the threshold device indicates that at least one pulse was transmitted when no pulse was transmitted. A *detection failure* is when the output of the threshold device indicates that no pulse was present on the channel despite that one or both were transmitters were ON. The scrambler in the figure is only used to make it possible for the receiver to separate the two users and is not important for the problem posed.

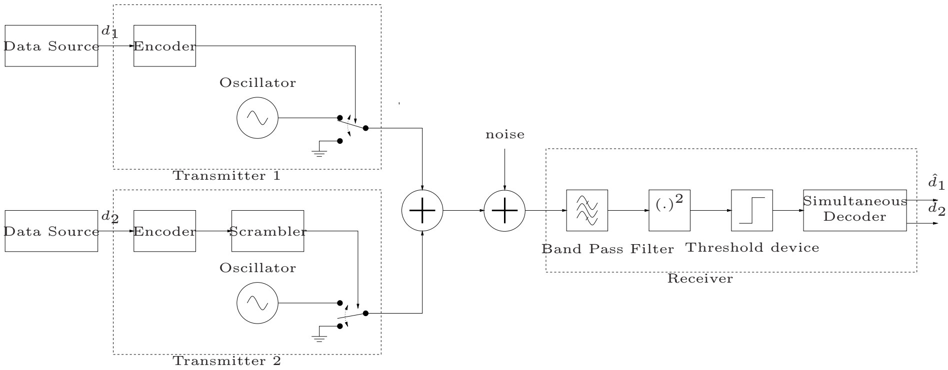


Figure 3.1: Binary OOK system.

(a) Determine the amount of information (bits/channel use) that is possible to transmit from the two transmitters to the receiver as a function of $P_{\mathrm{d},1}$, $P_{\mathrm{d},2}$, and P_f!

(b) The best possible detector of course has $P_{\mathrm{d},\,1} = P_{\mathrm{d},2} = P_f = 0$. Which capacity in terms of bits/channel use will this result in?

3–4 A space probe, called Moonraker, is planned by Drax Companies. It is to be launched for investigating Jupiter, $6.28 \cdot 10^8$ km away from the earth. The probe is to be controlled by digital BPSK modulated data, transmitted from the earth at 10 GHz and 100 W EIRP. The receiving antenna at the space probe has an antenna gain of 5 dB and the bit rate required for controlling the space probe is 1 kbit/s.

Based on these (simplistic) assumptions, is it possible to succeed with the mission?

3–5 Consider the channel depicted in Figure 3.2 where the input and output variables are denoted

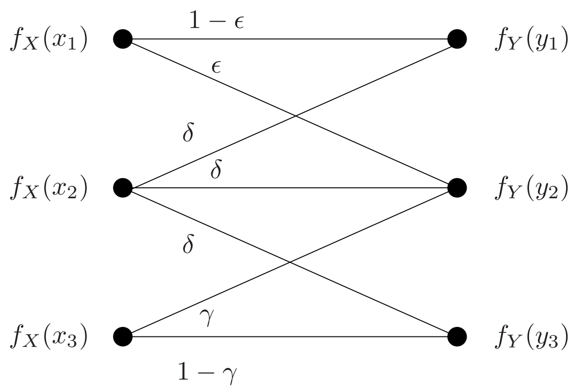


Figure 3.2: Channel.

with X and Y, respectively. The transition probabilities are given by $\epsilon = \delta = \gamma = 1/3$. Determine the following properties for this channel:

(a) the maximal entropy of the output variable Y. Show that this is achieved when

$$f_X(x_1) = f_X(x_3) = \frac{1 - f_X(x_2)}{2}.$$

(b) the entropy of the output variable given the input variable, that is, $H(Y|X)$. Express your answer in terms of the input probabilities of the channel and the binary entropy function

$$H_b(u) = -u \log_2 u - (1-u)\log_2(1-u)$$

(c) the channel capacity. *Hint*: The channel capacity is by definition given by

$$C = \max_{f_X(x)} I(X;Y) = \max_{f_X(x)} (H(Y) - H(Y|X))$$

An upper bound on the channel capacity is

$$C \leq \max_{f_X(x)} H(Y) - \min_{f_X(x)} H(Y|X).$$

Find the capacity by showing that this upper bound is actually attainable!

3–6 Figure 3.3 below illustrates two discrete channels

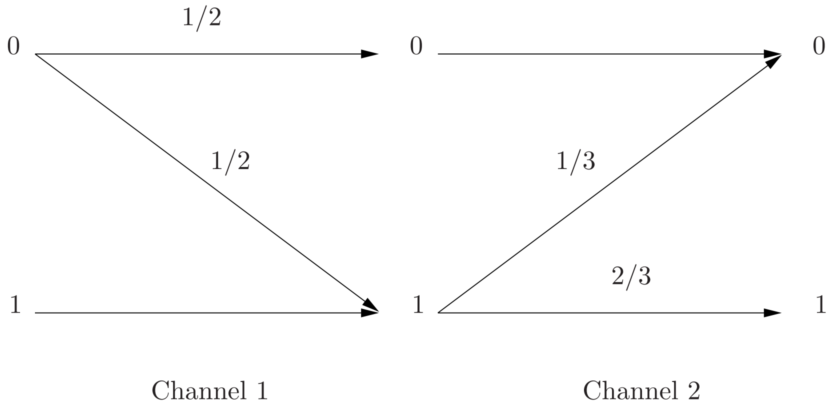


Figure 3.3: Two discrete channels.

(a) Determine the capacity of channel 1.

(b) Channel 1 and channel 2 are concatenated. Determine the capacity of the resulting overall channel.

3–7 Samples from a memoryless random process, having a marginal probability density function according to Figure 3.4, are quantized using 4-level scalar quantization.

The thresholds of the quantizer are $-b$, 0, b, where $0 < b < 1$. The 4 levels of the quantizer output are represented using 2 bits. The stream of quantizer output bits are then coded using a source code that converts the bitstream into a more efficient representation. Before transmission the output bits of the source code are subsequently coded using a block channel code, and the coded bits are then transmitted over a memoryless binary symmetric channel having crossover probability 0.02. What values for b can be allowed if the combination of the source code and the channel code is to be able to provide errorfree transmission of the quantizer output bits in the limit as the block lengths of the codes go to infinity?

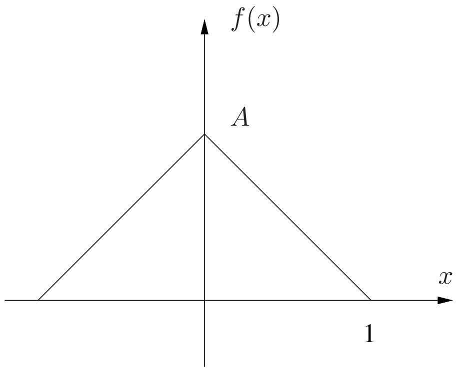
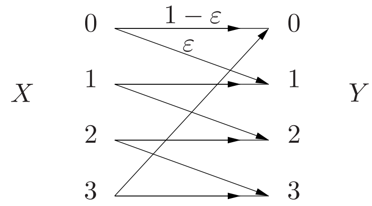


Figure 3.4: A pdf.

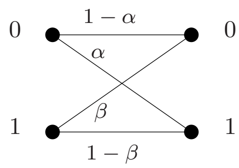
3–8 A binary memoryless source, with output $X_n \in \{0, 1\}$ and with $p = \Pr(X_n = 1)$, generates one symbol X_n every T_s seconds. The symbols from the source are to be conveyed over a time- and amplitude-continuous AWGN channel with noise spectral density $N_0/2$ [W/Hz], via separate source and channel coding. The channel is a baseband channel with bandwidth W [Hz], and the maximum allowed transmit power is P [W]. Assume that $T_s = 0.001$, $W = 1000$ and $P/N_0 = 700$. Then, for which values of p is it *impossible* to reconstruct X_n at the receiver without errors?

3–9 Consider the discrete memoryless channel depicted below.



As illustrated, any realization of the input variable X can either be correctly received, with probability $1-\varepsilon$, or incorrectly received, with probability ε, as one of the other Y-values. (Assume $0 \le \varepsilon \le 1$.) What is the capacity of the channel in bits per channel use?

3–10 The discrete memoryless binary channel illustrated below is know as the *asymmetric* binary channel.



Compute the capacity of this channel in the case $\beta = 2\,\alpha$.

3–11 Consider the block code with the following parity check matrix:

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(a) Write down the generator matrix $\mathbf{G}$ and explain how you derived it.

(b) Write down the complete *distance profile* of the above code. What is its minimum Hamming distance?

(c) How many errors can this code **detect**?

(d) Can it be used in correction and detection modes simultaneously. If so how many errors can it correct while also detecting errors? Explain your answer.

(e) Write down the syndrome table for this code explaining how you derived it. The syndrome table shows the error vector associated with all possible syndromes.

3–12 Determine the minimum distance of the binary linear block code defined by the generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

3–13 Consider the (n, k) binary cyclic block code specified by the following generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

(a) List the codewords of the code, and specify the parameters n (block-length), k (number of information bits), R (rate), and $d_{\min}$ (minimum distance).

(b) Specify the generator polynomial $g(p)$ and parity check polynomial $h(p)$ of the code.

(c) Derive generator and parity check matrices in systematic form.

(d) The code obtained when *using the parity check matrix* of the code specified above *as generator matrix* is called the *dual code*. Assume that the dual code is used over a binary symmetric channel with cross-over probability $\varepsilon = 0.01$, and with hard decision decoding based on the standard array. Derive an *exact* expression for the resulting block error probability p_e, and evaluate it numerically.

3–14 Three friends A, B and C agree to play a game. The game is done using 16 different species of flowers that happen to grow in abundance in the neighborhood. First, C picks four flowers in any combination and puts them in a row. Then, A is allowed to add three flowers of his choice. The third step in the game is to let C replace any of the seven flowers with a new flower of any of the 16 species. Finally, A claims that B always can tell the original sequence of seven flowers by looking at the new sequence (B had his eyes closed while C and A did their previous steps)! C thinks this is impossible and the three friends therefore bet a chocolate bar each. With this at stake, A and B simply cannot afford to loose!

Of course A and B have done this trick several times before, agreeing on a common technique based on their profound knowledge of communication theory. Winning the chocolate bars is therefore easy for them. Describe at least one technique A and B could have used!

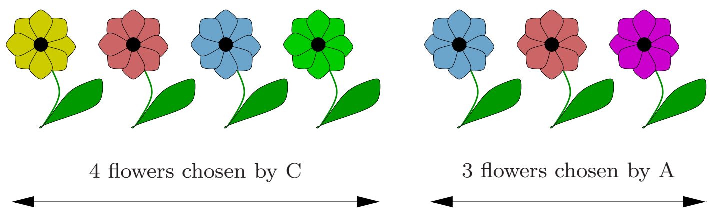


Figure 3.5: Flowers.

3–15 Consider the concatenated coding scheme depicted below.

The outer encoder is a $(7, 4)$ Hamming code and the interleaver is ideal (i.e., no correlation between the bits coming out of the interleaver). The inner encoder is a linear $(3, 2)$ block encoder with generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} .$$

The coded bits are transmitted using antipodal signaling over an AWGN channel with the signal to noise ratio $2E_c/N_0 = 10$ dB, where E_c is the energy per bit on the *channel* (these are of course the coded bits, not the information bits). In the receiver, the inner code is decoded using soft decision decoding. The decoded bits (2 for each 3 channel bits) out from the inner decoder (note, these bits are either 0 or 1) are passed through the deinterleaver and decoded in the outer decoder.

(a) Compute the *word* error probability for the inner decoder that uses *soft* decision decoding as a function of $2E_c/N_0$.

(b) Express the bit error probability of the decoded bits out from the inner decoder as a function of $2E_c/N_0$.

(c) Derive an expression of the *word* error probability for the outer decoder as a function of $2E_c/N_0$.

Reasonable approximation are allowed as long as they are clearly motivated.

Hint: Study the code words for the inner encoder and the decision rule used.

3–16 Consider the memoryless discrete channel depicted in Figure 3.6.

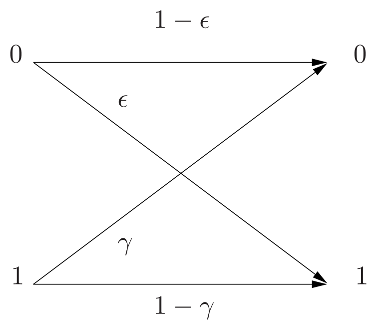


Figure 3.6: Discrete channel.

A linear code with generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

is used in coding independent and equally likely information bits over the given channel. A receiver fully utilizes the ability of the code to *detect* errors. Determine the probability that an error pattern is not detected.

3–17 Consider a cyclic binary $(7, 4)$ block channel code having the generator polynomial

$$g(z) = 1 + z + z^3. \tag{3.1}$$

The code is to be used for error correction in transmission over a non-symmetric binary memoryless channel, with crossover probabilities $P(1|0) = \varepsilon$ and $P(0|1) = \delta$, as illustrated in Figure 3.7.

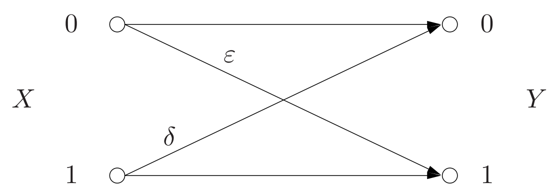


Figure 3.7: The channel in Problem 3–17.

(a) For the code, determine the generator matrix $\mathbf{G}$ and the parity check matrix $\mathbf{H}$. Both are to be given in systematic form.

(b) Assume that $\Pr(X = 0) = p$. Express the mutual information, $I(Y; X)$, as a function of p, ε, δ and the function h defined by

$$h(u) = -u \log(u) - (1 - u) \log(1 - u). \tag{3.2}$$

State the *definition* of the capacity of the channel in terms of the derived expression for the mutual information. (You do not need to compute the capacity).

(c) Suppose that $\delta = \varepsilon$ (that is, that the channel is symmetric). Determine the maximal ε that can be allowed if the probability of block-error employing maximum likelihood decoding for the described code, is to be less than 0.1 %. (*Hint*: the given code is a, so called, *Hamming code*. It has the special property that it can correct *exactly* $(d_{\min} - 1)/2$ errors, where $d_{\min}$ is the minimum distance of the code, irrespectively of which codeword is transmitted.)

3–18 (a) Consider the 4-AM constellation with Gray coding as shown in Figure 3.8. The probability of the bit1 being in error is different from the error probability of bit2. Write down the expressions for the probability of error for bit1 (P_{b1}) and bit2 (P_{b2}) versus energy per symbol over noise spectral density E_s/N_0. Of-course coherent detection is used.

(b) An (n, k, t) block encoder encodes k information bits into n channel bits and has the ability to correct t errors per block. Write down an expression for the bit error probability P_d after decoding given a pre-decoder bit error probability P_{cb}. Assume that the block coder is systematic and that whenever the number of channel bits in error exceeds t then the systematic bits are passed straight through the decoder. i.e. the output error probability equals the input error probability for that block.

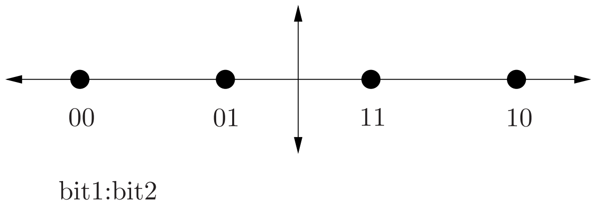


Figure 3.8: 4-AM constellation

3–19 Codes can be used either for error correction, error detection or combinations thereof. A set of commonly used error detection codes are the so-called CRC codes (cyclic redundancy check). CRC codes are popular as they offer near-optimum performance and are very easy to generate and decode.

Consider a system employing a (n, k) CRC code for error detection. Hard decision decoding is employed and the code length $n = 128$ is fixed. Find the maximum value of k that guarantees a probability of false detection P_{fd} of less than 10^{-10}. False detection occurs when the codeword received is different from the codeword transmitted yet the error detector accepts it as correct.

(*Hint*: Consider the worst case, where all word errors are equally likely)

3–20 Consider the communication system depicted below.

Independent and equally likely information bits are blocked into k-bit blocks and coded into n-bit blocks (where $n > k$) by a channel encoder. The output bits from the channel encoder are transmitted over an AWGN channel, with noise spectral density $N_0/2$, using QPSK. The codeword bits are mapped onto the QPSK constellation using Gray labeling, and the energy per transmitted QPSK symbol is denoted E_s. It holds that $E_s/N_0 = 7$ dB. The received signal is demodulated optimally and the transmitted symbols are detected using ML detection. The resulting detected symbols are mapped back into bits and then fed to a channel decoder that decodes the received n-bit blocks into k-bit information blocks.

(a) Let p_e denote the probability that the output k-bit block from the decoder is not equal to the input k-bit block of information bits. At what rates $R_c = k/n$ do there exist channel codes that are able to achieve $p_e \to 0$ as $n \to \infty$?

(b) Assume that the channel code is a linear block code specified by the generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Derive an *exact* expression for the corresponding error probability p_e, when this code is used for error correction in the system depicted in the figure, and evaluate it numerically for $E_s/N_0 = 7$ dB. Assume that the decoding is based on the standard array of the code.

(c) When using the code as in part (b) the energy E_s is employed to transmit one QPSK symbol, and one such symbols corresponds to *two coded bits*. Hence the energy spent *per information bit* is higher than $E_s/2$. With $E_s/N_0 = 7$ dB, as before, for the coded system make a fair comparison with a system that transmits information bits using QPSK with Gray labeling *without coding* (and spends as much energy per information bit as the coded system does) in terms of the corresponding probabilities of conveying k information bits without error. Is the coded system better than the uncoded (is there any *coding gain*)?

3–21 Consider a cyclic code of length $n = 15$ and with generator polynomial

$$g(x) = 1 + x^4 + x^6 + x^7 + x^8$$

For this code, derive

(a) the generator matrix in systematic form.

(b) the parity check matrix in systematic form and the minimum distance $d_{\min}$.

Assume that the code is used over a binary symmetric channel with bit-error probability $\varepsilon = 0.05$ and with hard-decision decoding based on the standard array at the receiver.

(c) Compute an upper bound to the resulting block error probability.

Now assume instead that the code is used over a binary *erasure channel*. That is, a binary memoryless channel with input symbols $\{0, 1\}$ and output symbols $\{0, 1, e\}$ where the symbol e means "bit was lost." Such erasure information is often available, e.g. when transmitting over the Internet. Letting $P(y|x)$ denote $\Pr(\text{output} = y|\,\text{input} = x\,)$ the statistics of the transmission are further specified as

$$P(0|0) = P(1|1) = 1 - \alpha, \quad P(e|0) = P(e|1) = \alpha, \quad P(1|0) = P(0|1) = 0$$

That is, if the receiver sees 0 or 1 it knows that the received bit is correct.

(d) Assuming maximum likelihood decoding over the erasure channel as specified, and with $\alpha = 0.05$, derive an upper bound to the block error probabilty.

3–22 Consider the communication system depicted Figure 3.9.

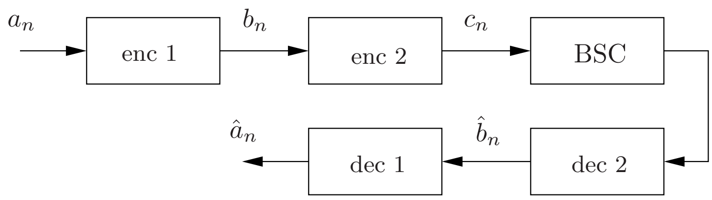


Figure 3.9: Communication system

Independent and equally likely bits $a_n \in \{0, 1\}$, $n = 1, \ldots, 4$, are blocked into length-4 blocks $\mathbf{a} = (a_1, \ldots, a_4)$ and are encoded by the encoder of a $(7, 4)$ binary cyclic block code with generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

The output-bits $b_n \in \{0, 1\}$, $n = 1, \ldots, 7$, are then blocked into a length-7 block $\mathbf{b} = (b_1, \ldots, b_7)$ and encoded by the encoder of a $(15, 7)$ binary cyclic code with generator polynomial

$$g(p) = x^8 + x^7 + x^6 + x^4 + 1,$$

producing the transmitted bits c_n, $n = 1, \ldots, 15$, in a block $\mathbf{c} = (c_1, \ldots, c_{15})$. Assume that both encoders are *systematic* (corresponding to the systematic forms of the generator matrices).

The output bits $c_n \in \{0,1\}$ are transmitted over a BSC with bit-error probability 0.1. The recieved bits are first decoded by a decoder for the $(15,7)$ code and then by a decoder for the $(7,4)$ code. Both these decoders implement maximum likelihood decoding ("choose the nearest codeword").

(a) The four information bits $\mathbf{a} = (0,0,1,1)$ are encoded as described. Specify the corresponding output codeword $\mathbf{c}$ from the second encoder.

(b) Assume that the binary block

$$000101110111100$$

is received at the output of the BSC. Specify the resulting estimates $\hat{a}_n \in \{0,1\}$, $n = 1,\ldots,4$, of the corresponding transmitted information bits.

(c) Notice that the concatenation described above results in an equivalent block code, with 4 information bits and 15 codeword bits. Since both the $(7,4)$ and the $(15,7)$ codes are cyclic, the overall code will also be cyclic. Specify the generator polynomial and the generator matrix of the equivalent concatenated code.

3–23 Consider a (7,3) cyclic block code with generator polynomial $g(p) = p^4 + p^2 + p + 1$.

(a) Find the generator matrix in systematic form.

(b) Find the minimum distance of the code.

(c) Suppose the generator matrix in (a) is used for encoding 9 independent and equiprobable bits of information $\mathbf{x}$ for transmission over a BSC with $\epsilon < 0.5$. Find the maximum likelihood sequence estimate of $\mathbf{x}$ if the output from the BSC is

$$\mathbf{y} = (101111011011011111010) \tag{3.3}$$

3–24 Consider a $(10,5)$ binary block code with parity check matrix

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Find the generator matrix $\mathbf{G}$.

(b) What is the coset leader of the syndrome [00010]?

(c) Give one received vector of weight 5 and one received vector of weight 7 that both correspond to the same syndrome [00010].

(d) Compute the syndrome of the received word $\mathbf{y} = [0011111100]$ and the corresponding coset leader.

(e) Is there a syndrome for which the weight of the corresponding coset leader is 3? If so, find such a syndrome.

3–25 A binary Hamming code has length $n = 2^m - 1$ and number of information bits $k = 2^m - m - 1$, for $m = 2,3,4,\ldots$, and a parity check matrix can be obtained by using as columns all the $2^m - 1$ different m-bit words except the all-zero word. For example a parity check matrix for the $m = 3, n = 7, k = 4$ Hamming code is

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

A Hamming code of any size always has minimum distance $d_{\min} = 3$.

An *extended* Hamming is obtained by taking the parity check matrix of any Hamming code and adding one new row containing only 1's, and also the new column $[0\ 0\ \cdots\ 0\ 1]^T$. That is, in the case of the $(7,4)$ Hamming code, the resulting new parity check matrix of the extended code is

$$\mathbf{H}_{\text{ext}} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

corresponding to an $(8,4)$ code.

(a) What is the minimum distance of the $(8,4)$ extended Hamming code?

(b) Show that an extended Hamming code has the same minimum distance regardless of its size, that is the same $d_{\min}$ as for the $(8,4)$ code.

3–26 Consider a (5,2) block code with the generator matrix given by

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The information $\mathbf{x} \in \{0,1\}^k$ is encoded using the above code and the elements of the resulting code word $\mathbf{c} \in \{0,1\}^n$ are BPSK modulated and transmitted over an AWGN channel. The BPSK modulator is designed such that a 0 input results in the transmission of $\sqrt{E}$ and 1 results in $-\sqrt{E}$ (for simplicity, E can be normalized to 1). The received codeword (sequence) is denoted $\mathbf{r}$.

As you know, soft decoding is superior to hard decoding on an AWGN channel. Despite this, block codes are commonly decoded using hard decoding due to simplicity, especially for long blocks. However, there has lately been progress within the coding community on soft decoding using the Viterbi algorithm on block codes as well. The key problem is to find a good trellis representation of the block code. For the code considered here, a trellis representation of the code before BPSK modulation is given in Figure 3.10 where a path through the trellis represents a code word.

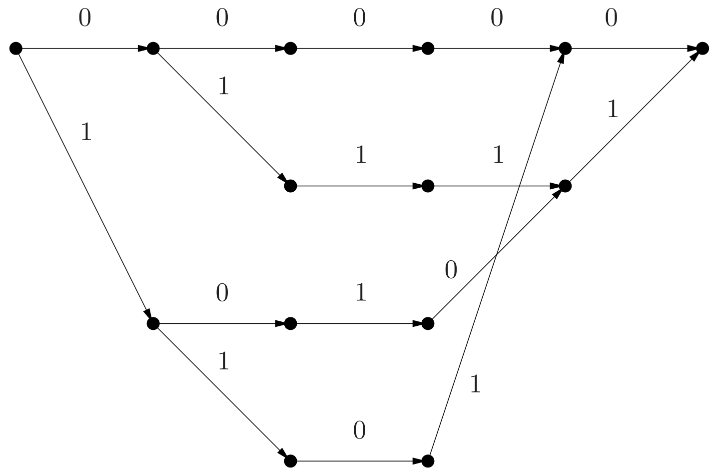


Figure 3.10: Trellis representation of block code.

Compared to a trellis corresponding to a convolutional code, the "states" in the trellis do not carry any specific meaning. Decoding is done in a similar fashion to decoding of convolutional codes, i.e., finding the best way through the trellis and the corresponding information sequence.

Decode the received sequence

$$\mathbf{r} = \sqrt{E}\big(-0.9, +1.1, -1.2, +0.6, +0.2\big)$$

using

(a) hard decisions and syndrome decoding

Figure 3.11: Communication system with block coding.

(b) soft decision decoding and the trellis in Figure 3.10.

3–27 Consider the communication system depicted in Figure 3.11.

The system utilizes a linear $(7, 3)$ block code with generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Three information bits $\mathbf{x} = [x_1, x_2, x_3]$ are hence coded into 7 codeword bits $\mathbf{c} = [c_1, c_2, \dots c_7]$, according to

$$\mathbf{c} = [c_1, c_2, \dots c_7] = \mathbf{x}\mathbf{G}\,.$$

The codeword is transmitted over the memoryless channel illustrated in Figure 3.12. That is,

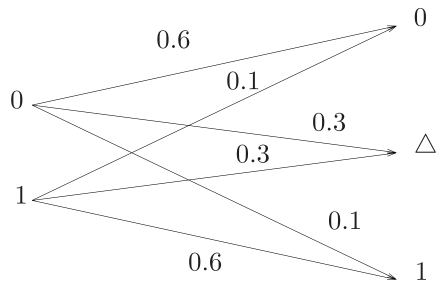


Figure 3.12: Discrete channel model.

$$\begin{array}{lll} \Pr(r_i = 0 | c_i = 0) = 0.6 & \Pr(r_i = \triangle | c_i = 0) = 0.3 & \Pr(r_i = 1 | c_i = 0) = 0.1 \\ \Pr(r_i = 1 | c_i = 1) = 0.6 & \Pr(r_i = \triangle | c_i = 1) = 0.3 & \Pr(r_i = 0 | c_i = 1) = 0.1 \end{array}$$

Assume the following block is received

$$\mathbf{r} = [r_1, r_2, \dots, r_7] = [1, 0, 1, \triangle, \triangle, 1, 0]$$

and decode this received block based on the ML criterion.

3–28 Consider a length $n = 9$ cyclic binary block code with generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

For this code:

(a) Determine the generator polynomial $g(p)$ and the parity check polynomial $h(p)$.

(b) Determine the parity check matrix $\mathbf{H}$ and the minimum distance $d_{\min}$.

Assume that the code is used to code a block of equally likely and independent information bits into a codeword $\mathbf{c} = (c_1, \dots, c_n)$ and that the coded bits are then transmitted over a time-discrete AWGN channel with received signal r_n, so that

$$r_n = (2c_n - 1)\sqrt{E} + w_n$$

where w_n is white and zero-mean Gaussian with variance $N_0/2$. The decoder uses *soft* ML decoding, meaning that based on $\mathbf{r} = (r_1, \dots, r_n)$ it decides that $\hat{\mathbf{c}}$ is the transmitted codeword if $\hat{\mathbf{c}}$ minimizes $\|\mathbf{r} - \mathbf{c}\|^2$ over all $\mathbf{c}$ in the code. Let $p_e = \Pr(\hat{\mathbf{c}} \neq \mathbf{c})$.

(c) An upper bound to p_e can be obtained as

$$p_e \leq N_1\, Q\left(\sqrt{\frac{2M_1 E}{N_0}}\right) + N_2\, Q\left(\sqrt{\frac{2M_2 E}{N_0}}\right) + N_3\, Q\left(\sqrt{\frac{2M_3 E}{N_0}}\right).$$

Specify the integer parameters $M_i,\ i = 1, 2, 3$, and $N_i,\ i = 1, 2, 3$.

3–29 Figure 3.13 shows the encoder of a convolutional code. Each information symbol $x \in \{0, 1\}$ gives

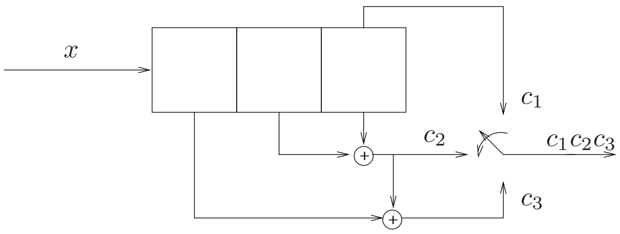


Figure 3.13: Encoder.

rise to three code symbols $c_1c_2c_3$. BPSK is utilized as modulation format to transmit the binary data over an AWGN channel. The noise power spectral density is $N_0/2$ and optimal matched-filter demodulation is employed. The sampled output of the matched filter can be described by a discrete-time model

$$r(n) = b(n) + w(n), \tag{3.4}$$

where $w(n)$ is white and Gaussian(0,$N_0/2$) and where the transmission of "0" corresponds to $b(n) = +1$ and the transmission of "1" corresponds to $b(n) = -1$. Assume that the observed sequence is (corresponding to $c_1,\ c_2,\ c_3,\ c_1,\ c_2,\ c_3, \ldots$):

$$1.51,\ 0.63,\ -0.04,\ 1.14,\ 0.56,\ -0.57,\ -0.07,\ 1.53,\ -0.9,$$

$$-1.68,\ 0.9,\ 0.98,\ -1.99,\ -0.04,\ -0.76$$

Assume, furthermore, that the encoder starts with zeroed registers and that the data is ended with a "tail" of two bits that zeroes the registers again. That is, the received sequence corresponds to 3 information bits and 2 tail-bits. At the receiver hard decisions are taken according to the sign of the received symbols. Use hard decision decoding to determine the maximum likelihood estimate of the transmitted information bits, based on the hard decisions on the received sequence.

3–30 Determine whether the convolutional encoder illustrated in Figure 3.14 is catastrophic or not.

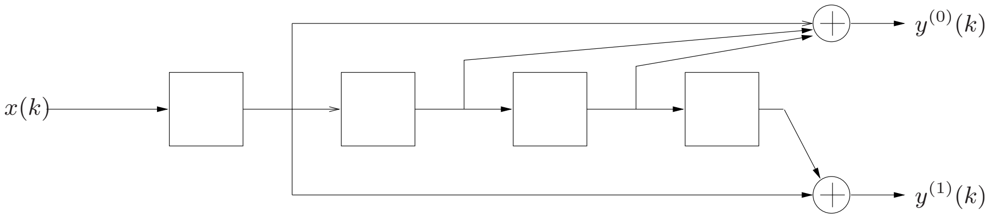


Figure 3.14: Encoder.

3–31 In this problem we consider a communication system which uses a $1/k$ rate convolutional code for error control coding over the channel. The channel is assumed to be a binary symmetric channel with cross over probability γ. Denote with $\mathbf{r}$ a vector with the hard inputs to the decoder in the receiver.

$$\mathbf{r} = \begin{bmatrix} r(0), & r(1), & \ldots, & r(N) \end{bmatrix}$$

A code word in the convolutional code is denoted with $\mathbf{c}$:

$$\mathbf{c} = \begin{bmatrix} c(0), & c(1), & \ldots, & c(N) \end{bmatrix}$$

The path metric for code word $\mathbf{c}^i$ is with these notations

$$\log p(\mathbf{r}|\mathbf{c}^i)$$

Show that, independent of γ, the ML estimate of the information sequence is simply the path corresponding to the code word $\mathbf{c}^i$ that minimizes the Hamming distance between $\mathbf{c}$ and $\mathbf{r}$.

3–32 Consider the encoder shown in Figure 3.15.

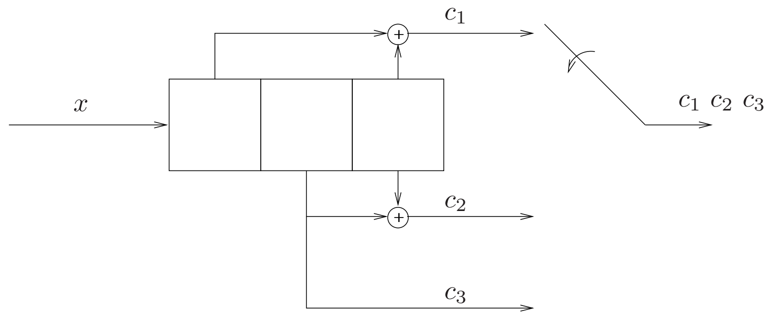


Figure 3.15: Encoder.

The encoder corresponds to a convolutional code of rate 1/3. Each information bit $x(n) \in \{0, 1\}$ is coded into three codeword bits c_1, c_2 and c_3, where c_1 is transmitted first. The system operates over a BSC with crossover probability 0.01.

(a) Determine the free distance of the code.

(b) The code starts and ends in the zero state. Determine the ML estimate of the transmitted information bits when the received sequence is

$$0\,0\,1\,1\,1\,0\,1\,0\,1\,1\,1\,1\,.$$

3–33 Figure 3.16 illustrates the encoder of a convolutional code.

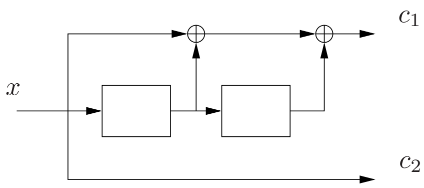


Figure 3.16: Encoder.

(a) Determine the free distance d_{free} of the code.

(b) Assume that the encoder starts in state 00 and consider coding input sequences of the form

$$\mathbf{x} = (x_1, x_2, x_3, 0, 0)$$

That is, three information bits and two 0's that garantee that the encoder returns to state 00. (Assume time goes "from left to right," i.e. x_1 comes before x_2.) This will produce output sequences

$$\mathbf{c} = (c_{11}, c_{12}, c_{21}, c_{22}, c_{31}, c_{32}, c_{41}, c_{42}, c_{51}, c_{52})$$

where c_{i1} and c_{i2} correspond to c_1 and c_2 produced by the ith input bit. Letting $\tilde{\mathbf{x}} = (x_1, x_2, x_3)$ (skip the two 0's in $\mathbf{x}$), the mapping from $\tilde{\mathbf{x}}$ to $\mathbf{c}$ describes a $(10, 3)$ linear binary block code. For this code: Determine a generator matrix $\mathbf{G}$ and the corresponding parity check matrix $\mathbf{H}$.

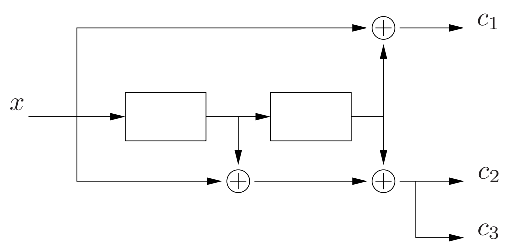


Figure 3.17: Encoder.

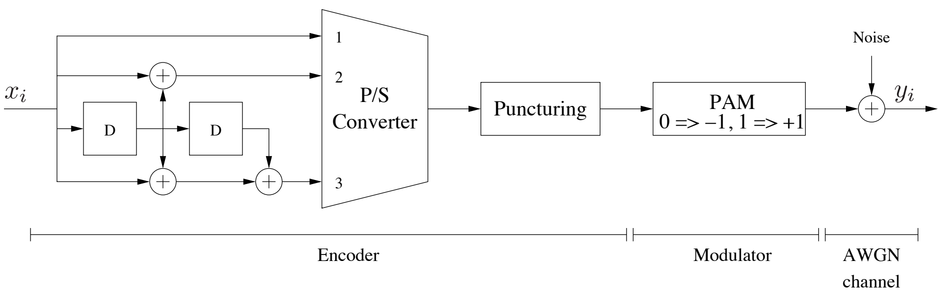
3–34 Consider the rate $R = 1/3$ convolutional code defined by the linear encoder shown in Figure 3.17. For each information bit x the three coded bits c_1, c_2, c_3 are produced. Note that $c_3 = c_2$. For this code:

(a) Determine the free distance d_{free}.

(b) Compute the transfer function in terms of the variable "D" that represents output distance (use $N = J = 1$ in the general N, J, D form discussed in class).

(c) Based on the result in (b), determine the number of different sequences of output weight 14 that can be produced by leaving state zero and returning to state zero at some point later in time.

3–35 Puncturing is a simple technique to change the rate of an error correcting code. In the communication system depicted below, binary data symbols are fed to a convolutional encoder with rate 1/3. The output bits from the encoder is parallel-to-serial (P/S) converted and punctured by removing every sixth bit from the output bit stream. For example, if the P/S device output data is $\ldots abcdefghijklmn \ldots$, the output from the puncture device is $\ldots abcdeghijkmn \ldots$. After being punctured, the bit stream is BPSK modulated and sent over an AWGN channel prior to being received.



(a) What is the rate of the punctured code?

(b) Assuming that sequence detection based on the observed data y_i, using the Viterbi algorithm, is considered to recover the sent data, draw the trellis diagram of the decoder. In addition, indicate in your graph what observed data y_i that is used in each time step of the trellis. Also, state what distance measure that the Viterbi algorithm should use.

(c) The concatenation of the convolutional rate 1/3 encoder and the puncturing device considered previously can be viewed as a conventional convolutional encoder with constraint length $L = 4$. Note, for this to be true, the new encoder must for each set of output bits clock 2 input bits into its shift register. Show this by drawing the block diagram of this new conventional encoder. Neglect any problems with how to initially load the different memory elements. Also, draw the associated trellis diagram of the new encoder.

(d) Which trellis representation, the one in (b) or the one in (c), is more advantageous to consider in order to recover the source data. Motivate!

3–36 One way of changing the code rate of an error correcting code is so-called *puncturing*. In the system depicted in Figure 3.18, a simple convolutional code is punctured by removing every fourth output bit (illustrated by the puncturing pattern 1110 in the figure below) before transmission. For example, if the output from the convolutional encoder is *abcdefgh*..., the bit sequence

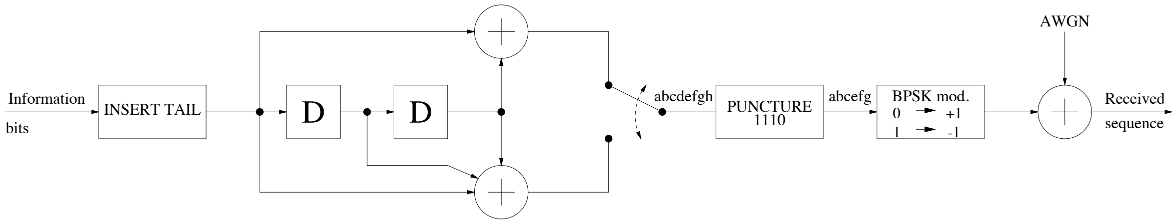


Figure 3.18: Puncturing.

abcefg... is transmitted, while bits *d,h,*... have been *punctured*. In the receiver, the punctured bits are treated as “lost” on the channel, i.e., nothing is known about the punctured bits in the receiver. This depuncturing process can be accomplished by inserting a suitable value before conventional decoding at the positions where the punctured bit should have appeared in absence of puncturing. For example, if *abcefg*... is received, the decoder operates on *abcxefgx*..., where x is a value representing “no knowledge” (i.e., neither logical “1”, nor logical “0”) to the decoder.

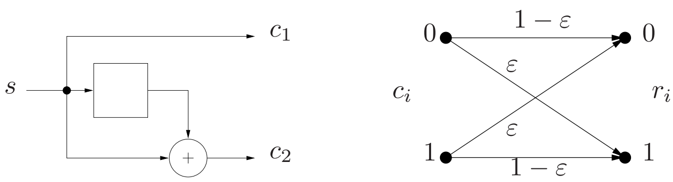
(a) What is the resulting code rate of the convolutional encoder followed by the puncturer?

(b) Assume the received sequence is $1.1, 0.7, -0.8, 0.7, 0.1, 0.8, 0.8, -0.2, -0.3$, which is the most likely information sequence?

3–37 After successfully completing your Master and PhD degrees at KTH, specializing in communications, you have been appointed assistant professor at the Department of Signals, Sensors and Systems. Your current task is to grade the exam in communication theory. The problem is given as:

Problem:

Consider the convolutional encoder and the Binary Symmetric Channel (BSC) below. The information bits are independent with probabilities $p_0 = \Pr\{s = 0\} = 0.3$ *and* $p_1 = \Pr\{s = 1\} = 0.7$*, respectively. Due to the coder, the encoded bits are independent as well, and from this the probabilities for the code bits can be derived. The error probability of the channel is* $\varepsilon = 0.1$*. Determine the coded sequence and the information sequence in an optimal way (lowest probability of sequence error) given the received sequence* $\mathbf{r} = 10\,01\,11\,00$*! The encoder starts, and by transmitting the appropriate tail, terminates in state zero.*

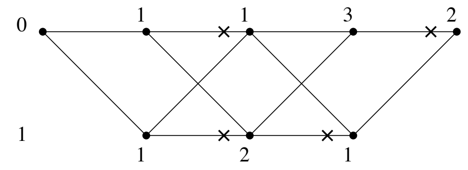


Two students, Alice and Bob, solve the problem in two different ways. Grade (0–5 points) and, if necessary, correct their answers. (Explain what is correct and incorrect in each of the answers.)

Alice's answer:

Use Viterbi. Decoded information: $\mathbf{s} = 1010$

$\mathbf{r}_i = (r_{i,1}, r_{i,2}) =$ 10 01 11 00

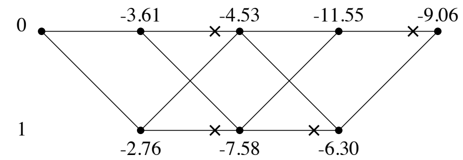


$s_i =$ 1 0 1 0

Bob's answer:

MAP decoding. Maximize
$\Pr\{\mathbf{r}|\mathbf{c}\} \Pr\{\mathbf{c}\} = \prod_{i=1}^{4} p(r_{i,1}|c_{i,1}) p(c_{i,1}) p(r_{i,2}|c_{i,2}) p(c_{i,2})$
$\Rightarrow$
$\max \sum (\log p(c_{i,1}) p(c_{i,2}) + \log p(r_{i,1}|c_{i,1}) p(r_{i,2}|c_{i,2}))$

$\mathbf{r}_i = (r_{i,1}, r_{i,2}) =$ 10 01 11 00



$s_i =$ 1 0 1 0

3–38 Consider the combination of a convolutional encoder and a PAM modulator shown in Figure 3.19.

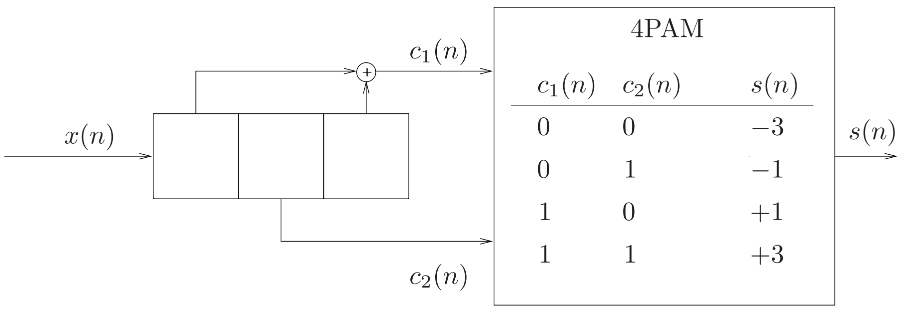


Figure 3.19: Combined encoding and modulation.

The information bits $x(n) \in \{0, 1\}$ are coded at the rate 1/2. Each output pair, $c_1(n), c_2(n)$, then determines which modulation symbol is to be fed to the channel. The encoder starts and ends in the zero state.

The system is used over an AWGN with noise spectral density $N_0/2$. The demodulated received signal is

$$r(n) = s(n) + e(n),$$

where $\{e(n)\}$ is a white Gaussian process.

The following sequence is received

$$-2.0,\ 2.5,\ -4.0,\ -0.5,\ \underbrace{1.5,\ 0.0}_{\text{"tail"}}\ .$$

Determine an ML-estimate of the transmitted sequence based on the received data.

3–39 Figure 3.20 shows the encoder of a convolutional code.

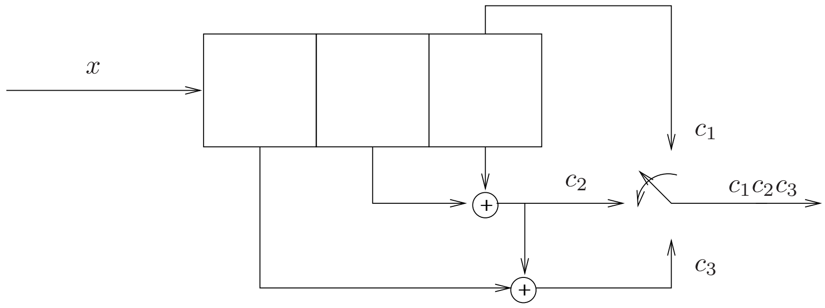


Figure 3.20: Convolutional encoder.

Each information symbol $x \in \{0,1\}$ gives rise to three code symbols $c_1c_2c_3$. BPSK is utilized as modulation format to transmit the binary data over an AWGN channel. The noise power spectral density is $N_0/2$ and optimal matched-filter demodulation is employed. The sampled output of the matched filter can be described by a discrete-time model

$$r(n) = b(n) + r(n), \tag{3.5}$$

where $w(n)$ is white and zero-mean Gaussian with variance $N_0/2$, and where the transmission of "0" corresponds to $b(n) = +1$ and the transmission of "1" corresponds to $b(n) = -1$. Assume that the observed sequence is (corresponding to $c_1,\ c_2,\ c_3,\ c_1,\ c_2,\ c_3, \ldots$):

$$1.5,\ 0.5,\ 0,\ 1,\ 0.5,\ -0.5,\ 0,\ 1.5,\ -1,\ -1.5,\ 0.5,\ 1,\ -1.5,\ -1,\ -1$$

Assume, furthermore, that the encoder starts with zeroed registers and that the data is ended with a "tail" of two bits that zeroes the registers again. That is, the received sequence corresponds to 3 information bits and 2 tail-bits. Use soft decision decoding to determine the maximum likelihood estimate of the transmitted information bits, based on the received sequence.

3–40 Consider the simple rate 1/2 convolutional encoder depicted in Figure 3.21 (the encoder is not necessarily a good one). The resulting encoded stream, c_{i1} followed by c_{i2}, are transmitted using

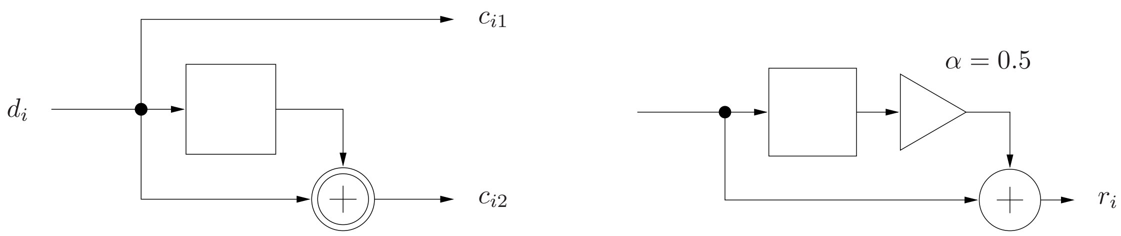


Figure 3.21: Encoder (left) and channel (right).

on-off keying with unit symbol energy over the ISI channel shown in the figure. Assuming that the receiver has perfect knowledge of the channel as well as of the encoder, ML sequence decoding using the Viterbi is possible. Find the ML estimate of the received sequence

$$\mathbf{r} = (r_1, \ldots, r_{10}) = (1, 1.5, .5, 1, 1.5, 1.5, .5, 1, 0.5, 0)\ .$$

The channel is silent before transmission and the transmission is ended by transmitting a tail of zeros (which are included in the sequence above).

Hint: write r_i as a function of d_i and view the encoder followed by the channel as a composite encoder. Note that the adder in the encoder operates modulo 2. Remember that two symbols (r_{i1} and r_{i2}) must be received for each information bit and that the transmitted symbol stream is $(\ldots c_{i-1,1}, c_{i-1,2}, c_{i,1}, c_{i,2})$.

3–41 Figure 3.22 shows the encoder of a convolutional code. Each information bit x produces three coded bits c_1, c_2, c_3. The coded bits are transmitted based on antipodal signalling, where the symbol "0" is mapped to "−1" and "1" to "+1." The transmission takes place over an AWGN channel. The encoder can be assumed to start and end in the zero state.

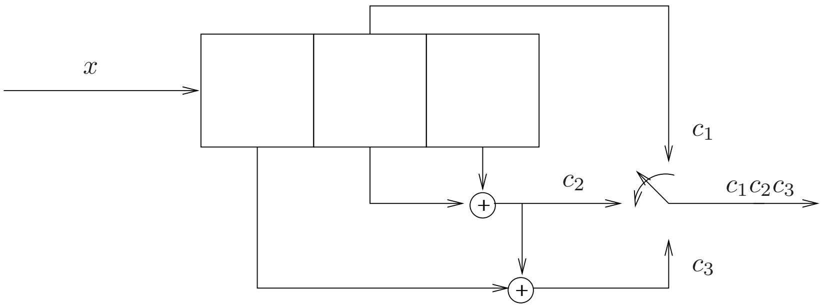


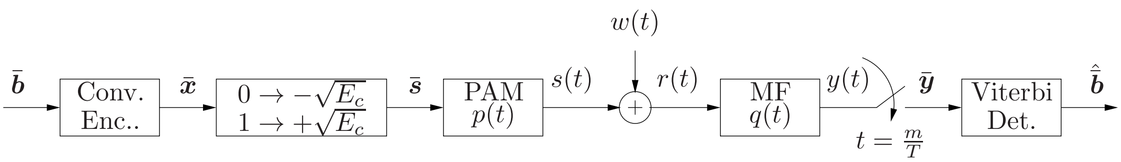
Figure 3.22: Encoder.

Consider a received sequence (ended by a "tail"):

$$-1.5,\ 0.5,\ 0,\ 1,\ 0.5,\ -0.5,\ 0,\ 1.5,\ -1,\ -1,\ 0.5,\ 1,\ 1,\ 1.5,\ 0,\ -1,\ 0,\ 0.5$$

Decode the sequence using soft ML decoding.

3–42 Consider the communication system model shown below



The binary convolutional encoder is described by the generator sequences:

$$\boldsymbol{g}_1 = [1\,0\,0], \quad \boldsymbol{g}_2 = [1\,0\,1], \quad \boldsymbol{g}_3 = [1\,1\,1],$$

and the impulse responses of the pulse amplitude modulator and the matched filter are given by,

$$p(t) = q(t) = \begin{cases} \frac{1}{\sqrt{T}} & -\frac{T}{2} \le t < \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

where T denotes the channel symbol period.

(a) Draw the encoder. What is the rate and the free distance of the code?

(b) Let $\bar{\boldsymbol{x}} = [\ldots, x_i, \ldots]$ and $\bar{\boldsymbol{y}} = [\ldots, y_i, \ldots]$ denote sequences of binary symbols in $\{0, 1\}$ to be sent and real valued received symbol samples respectively. If $w(t)$ is assumed Gaussian distributed with zero mean and spectral density $R_w(f) = \frac{N_0}{2}$, then we may relate y_i and x_i via the transformation:

$$y_i = \alpha(2x_i - 1) + e_i$$

where e_i is zero-mean Gaussian with variance β. Calculate α and β. Is e_i a temporally white sequence?

(c) Assume that the message $\bar{\boldsymbol{b}}$ consists of 7 binary symbols out of which the two last are known at the receiver. For reasonable SNR values we may approximate the sequence error probability i.e., the probability that $\hat{\bar{\boldsymbol{b}}} \neq \bar{\boldsymbol{b}}$, as

$$P_{\text{seq. error}} \leq \sum_{i \in \mathcal{I}} \Pr\{\bar{\boldsymbol{b}}_i \text{ detected} \,|\, \bar{\boldsymbol{b}}_0 \text{ sent}\}$$

where $\mathcal{I} = \{i : d_H(\bar{\boldsymbol{x}}_i, \bar{\boldsymbol{x}}_0) = d_{\text{free}}\}$, $\bar{\boldsymbol{b}}_0$ is an arbitrary message sequence and $\bar{\boldsymbol{x}}_i$ is the output sequence from the convolutional encoder corresponding to the input sequence $\bar{\boldsymbol{b}}_i$.

i. What is the size of the set $\mathcal{I}$ for the given message length?

ii. Relate the sequence error probability to the error probability of the encoded symbols passing the AWGN channel.
Hint: The bound $Q(x) \leq e^{-x^2/2}$ may be useful.

3–43 Consider a rate 1/2 convolutional code with generator sequences $\mathbf{g}_1 = (111)$ and $\mathbf{g}_2 = (110)$. Assume that BPSK modulation is used to transmit the coded bits.

(a) Draw a shift register for the encoder. Draw also the state diagram for this code.

(b) **Viterbi hard decoding**: The decoder always starts at all-zero state and ends up at all-zero state. Draw a trellis for a code terminated to a length of 10 bits. By inspection of the trellis, what could you say about the free distance d_{free}.

(c) **Viterbi soft decoding**: The decoder still starts from and ends up at all-zero state. The received sequence $\mathbf{r}$ is first fed to a 3 bits uniform quantizer. The maximum reconstruction value of this quantizer is 1.75, while the minimum reconstruction value is -1.75. The quantized sequence is then fed into a Viterbi soft decoder, which employs the minimum Euclidean distance algorithm. Suppose that the received vector $\mathbf{r}$ is given by:

$$\mathbf{r} = \{1.38 \;\; 0.68 \;\; 0.05 \;\; 1.04 \;\; -0.33 \;\; -0.81 \;\; 0.12 \;\; -0.41 \;\; -0.93 \;\; 0.21\}$$

Estimate the information bits corresponding to $\mathbf{r}$.

3–44 Consider the communication system shown in Figure 3.23.

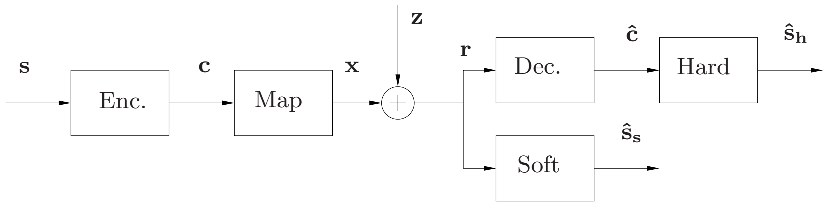


Figure 3.23: System

The i.i.d. information bits in the vector $\mathbf{s} = [s_1 s_2 s_3]$ are encoded into the codewords $\mathbf{c} = [c_1 \dots c_8]$ using the half-rate convolutional encoder depicted in Figure 3.24. The encoded bits are antipodal-modulated into $\mathbf{x} = [x_1 \dots x_8]$ via the mapping: $c_i = 0 \Rightarrow x_i = -1$ and $c_i = 1 \Rightarrow x_i = 1$. The signal vector $\mathbf{x}$ is transmitted over an AWGN channel with noise vector $\mathbf{z} = [z_1 \dots z_8]$. The received vector is $\mathbf{r} = \mathbf{x} + \mathbf{z}$.

The receiver is equipped with both hard and soft decoding units. Preceding the hard decoding is a decision unit that minimizes the decision error probability. Both the hard and soft decoding units perform maximum likelihood decoding.

The encoder starts in an all-zero state and returns to the all-zero state after the encoding of $\mathbf{s}$. The codeword $\mathbf{c}$ is constructed as $\mathbf{c} = [a_1 b_1 \dots a_4 b_4]$. The output bits $a_i b_i$ correspond to the input bit s_i and $s_4 = 0$ is the bit that resets the encoder.

(a) Determine the free distance of the code.

(b) Assume that $\mathbf{s} = [101]$ and $\mathbf{r} = [1.1 \;\; 1.3 \;\; -0.7 \;\; 0.7 \;\; -0.1 \;\; 1.8 \;\; -1.4 \;\; -0.2]$.
Find $\hat{s}_h$ and $\hat{s}_s$.

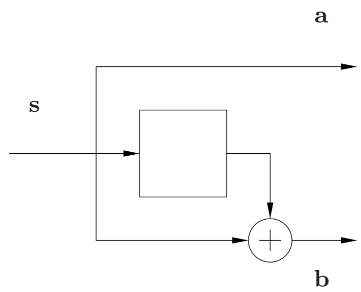


Figure 3.24: Encoder

(c) Is it possible that the hard decoding unit corrects an error that the soft decision unit does not correct? Motivate thoroughly, for instance by providing an **s** and a **z** where it happens.

3–45 The fancy IT company "MultiMediaFusion" has promised to deliver a packet data communication system. As the company had a background in producing fireworks and not communication equipment (which is not supposed to blow up), they have acquired two supposedly competent engineers, Larry and Irwin. Of course the company is running in the red, so the engineers are paid in stock options.

The communication system is supposed to transfer blocks of 100 information bits with BPSK modulation and coherent detection as fast as possible over an AWGN channel to the receiver with a block error probability less than 10%. Of course more information bits can be transmitted in a given time without coding, but at the price of a higher bit error probability. Hence, there is no reason for using more coding than necessary. Error-free feedback of any parameters from the receiver to the transmitter is available and the transmitter power is fixed. The E_b/N_0 of the AWGN channel has a distribution according to the pdf in Figure 3.25.

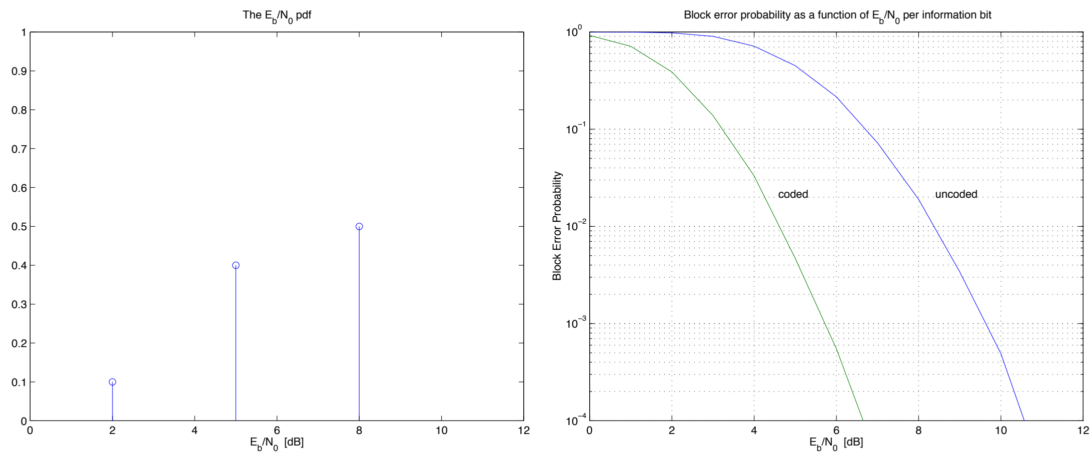


Figure 3.25: Pdf and error probabilities.

Larry proposed a system where the receiver measures the current signal-to-noise ratio, E_b/N_0, and feeds back this information to the transmitter. Depending on the measured E_b/N_0, the transmitter chooses either no coding or rate 1/2 convolutional coding for transmitting the information block. The receiver knows which coding scheme that is used and decodes it using soft decoding. The block error probability for both uncoded and coded transmission is found in the right plot in Figure 3.25.

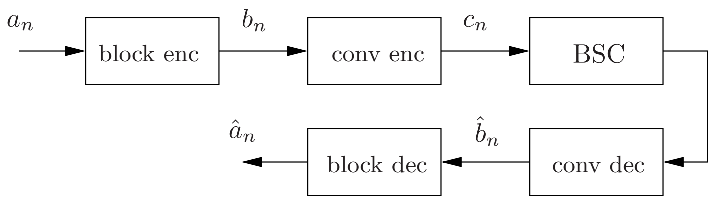
Irwin takes another approach. In his scheme, coding is never used. Instead, he has a genie in the receiver, informing the transmitter if the block was correctly received or not. In case of an incorrectly received block, the transmitter retransmits the incorrect block. The receiver has stored the soft values from the previous transmission (one sample per transmitted symbol) and add these stored values to the corresponding soft values received during the retransmission. Only one retransmission is allowed, i.e., a maximum of two transmissions in total for a single block. The channel is unchanged between the transmission and a possible retransmission. Again, the block error probability is found in the right plot.

(a) Which coding scheme should Larry's system use for different values of E_b/N_0?

(b) On average, how many blocks of 100 bits each *on the channel* are transmitted per 100-bit information block for the two schemes?

(c) What is the resulting block error probability for the two schemes?

3–46 Consider the communication system depicted below.



The independent and equally likely bits $a_n \in \{0, 1\}$ are encoded by the encoder of a $(5, 2)$ linear binary block code with generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The output-bits $b_n \in \{0, 1\}$ from the block code are then encoded by the encoder of a convolutional code. The convolutional code has parameters $k = 1$, $n = 2$, constraint-length 2 and generator sequences

$$\mathbf{g}_1 = (11), \quad \mathbf{g}_2 = (10)$$

The 5 output bits per codeword from the block code and one additional zero are fed to the convolutional encoder. The extra zero (tail bit) is added in order to force the state of the encoder back to the zero state.

The output bits $c_n \in \{0, 1\}$ from the convolutional encoder are transmitted over a BSC with bit-error probability 0.1, and the received bits are then decoded by the decoders of the convolutional code and block code, respectively. Both decoders implement maximum likelihood decoding.

(a) The four information bits 0011 are encoded by the block encoder and then fed to the convolutional encoder. Specify the corresponding output bits c_n from the convolutional encoder.

(b) Assume that the sequence

$$101101110111$$

is received at the output of the BSC. Specify the resulting estimates $\hat{a}_n \in \{0, 1\}$ of the corresponding transmitted information bits.

(c) Notice that the concatenation described above of the block and convolutional encoders specifies the encoder of an equivalent block code, with 2 information bits and 12 codeword bits.

i. Specify the generator matrix of the equivalent concatenated code.

ii. Show that the minimum distance of the concatenated code is equal to the product of the minimum distance of the $(5, 2)$ code and the free distance of the convolutional code.

Answers, Hints and Solutions

Answers and hints (or incomplete solutions) are provided to all problems. Most problems are provided with full solutions.

---

1 Information Sources and Source Coding

1–1 The possible outcomes of a game are

V	X	S	*Prob.*	V	X	S	Prob.
A	AAA	3	0.1754	A	BAABA	5	0.0305
B	BBB	3	0.0850	A	ABBAA	5	0.0305
A	BAAA	4	0.0660	A	ABABA	5	0.0331
A	ABAA	4	0.0717	A	AABBA	5	0.0359
A	AABA	4	0.0777	B	AABBB	5	0.0359
B	ABBB	4	0.0563	B	ABABB	5	0.0331
B	BABB	4	0.0519	B	ABBAB	5	0.0305
B	BBAB	4	0.0479	B	BAABB	5	0.0305
A	BBAAA	5	0.0259	B	BABAB	5	0.0281
A	BABAA	5	0.0281	B	BBAAB	5	0.0259

Based on the table above we conclude the following probabilities

	Winner V		
	Alice	Bob	Sum
1st set Alice ($x_1 = $ A)	0.4242	0.1558	0.58
1st set Bob ($x_1 = $ B)	0.1506	0.2694	0.42
Sum	0.5748	0.4252	

	Winner V		
	Alice	Bob	Sum
$s = 3$ set	0.1754	0.0850	0.2604
$s = 4$ set	0.2154	0.1562	0.3716
$s = 5$ set	0.1840	0.1840	0.3680

(a) We need to determine if $I(V, X_1)$ is greater or less than $I(V, S)$. It holds that $I(V, X_1) = H(V) - H(V|X_1)$ and $I(V, S) = H(V) - H(V|S)$. From the table above we conclude

$$H(V) = -\sum_v p_V(v) \log_2 p_V(v) \approx 0.9838 \text{ bit}$$

We also see that

$$H(V|X_1) = -\sum p_{VX_1}(v, x_1) \log_2 p_{V|X_1}(v|x_1) = -\sum p_{VX_1}(v, x_1) \log_2 \frac{p_{VX_1}(v, x_1)}{p_{X_1}(x_1)} \approx 0.8823 \text{ bit.}$$

$$H(V|S) = -\sum p_{VS}(v, s) \log_2 p_{V|S}(v|s) = -\sum p_{VS}(v, s) \log_2 \frac{p_{VS}(v, s)}{p_S(s)} \approx 0.9701 \text{ bit.}$$

Finally we get $I(V, X_1) = H(V) - H(V|X_1) \approx 0.1015$ bit and $I(V, S) = H(V) - H(V|S) \approx 0.0137$ bit. Knowing the winner of the first set hence gives more information.

(b) The number of bits required (on average) is $H(X|S) = H(X, S) - H(S) \approx 4.0732 - 1.5669 \approx 2.5$ bits.

1–2 (a) The average number of bits is (if we assume that we transmit the results from many matches in one transmission) equal to the entropy of K:

$$\begin{aligned} H(K) &= -\sum_{k=1}^{3} f_K(k)\log_2(f_K(k)) \\ &= 0.26040\log_2(0.26040) + 0.37158\log_2(0.37158) + 0.36802\log_2(0.36802) \\ &= 1.5669 \approx 1.567 \end{aligned}$$

(b) To answer the question, the conditional entropy of K given W is needed. The formula for this quantity is

$$H(K|W) = -\sum_{k=1}^{3}\sum_{w=A}^{B} f_{KW}(k,w)\log_2 f_{K|W}(k|w)$$

The values for $f_{KW}(k,w)$ are given in one of the tables to the problem and with $f_W(A) = 0.57480$ and $f_W(B) = 0.42520$, the conditional probabilities are easily calculated to

$f_{K\|W}(k\|w)$		W	
		A	B
k	3	0.30513	0.19993
	4	0.37474	0.36731
	5	0.32013	0.43276

The conditioned entropy is thus given by

$$\begin{aligned} H(K|W) = &-\big[0.17539\log_2(0.30513) + 0.21540\log_2(0.37474) + 0.18401\log_2(0.32013) \\ &+ 0.08501\log_2(0.19993) + 0.15618\log_2(0.36731) + 0.18401\log_2(0.43276)\big] \\ =&1.5532 \approx 1.553 \end{aligned}$$

(c) First, the mutual information of K and W is computed via the formula

$$I(K;W) = H(K) - H(K|W) = 1.5669 - 1.5532 \approx 0.01370$$

Now, the mutual information of K and S_3 is determined. If the same approach as in b is used ($f_{S_3}(A) = 0.54$, $f_{S_3}(B) = 0.46$) the conditional probabilities

$f_{K\|S_3}(k\|s_3)$		s_3	
		A	B
k	3	0.32479	0.18480
	4	0.34370	0.40428
	5	0.33148	0.41091

are useful. Thus, the conditional entropy is

$$\begin{aligned} H(K|S_3) &= -\sum_{k=1}^{3}\sum_{s_3=A}^{B} f_{KS_3}(k,s_3)\log_2 f_{K|S_3}(k|s_3) \\ &= -\big[0.17539\log_2(0.32479) + 0.18560\log_2(0.34370) + 0.17900\log_2(0.33148) \\ &\quad + 0.08501\log_2(0.18480) + 0.18597\log_2(0.40428) + 0.18902\log_2(0.41091)\big] \\ &=1.5483 \end{aligned}$$

Thus, the mutual information of K and S_3 is

$$I(K;S_3) = H(K) - H(K|S_3) = 1.5669 - 1.5483 = 0.01860 \approx 0.0186$$

The winner of the third set thus gives most information of the number of sets in a match. An alternative (simpler?) is to use the relation $I(K;W) = H(W) + H(K) - H(W,K)$ and the upper left table.

1–3 Let $p_{i,\mathrm{A}}$ denote the probability of the ith symbol for source A and n_{A} the number of possible outcomes for a source symbol. For the source B the notation $p_{i,\mathrm{B}}$ and n_{B} is used, and similarly $p_{i,\mathrm{S}}$, and $n_{\mathrm{S}} = n_{\mathrm{A}} + n_{\mathrm{B}}$ for the resulting source S. Using the definition of entropy

$$\begin{aligned}
H(\mathrm{S}) &= -\sum_{i=1}^{n_{\mathrm{S}}} p_{i,\mathrm{S}} \log p_{i,\mathrm{S}} \\
&= -\sum_{i=1}^{n_{\mathrm{A}}} \lambda p_{i,\mathrm{A}} \log \lambda p_{i,\mathrm{A}} - \sum_{i=1}^{n_{\mathrm{B}}} (1-\lambda) p_{i,\mathrm{B}} \log (1-\lambda) p_{i,\mathrm{B}} \\
&= -\lambda \sum_{i=1}^{n_{\mathrm{A}}} p_{i,\mathrm{A}} (\log p_{i,\mathrm{A}} + \log \lambda) - (1-\lambda) \sum_{i=1}^{n_{\mathrm{B}}} p_{i,\mathrm{B}} (\log p_{i,\mathrm{B}} + \log(1-\lambda)) \\
&= \lambda H_{\mathrm{A}} + (1-\lambda) H_{\mathrm{B}} - \lambda \log \lambda - (1-\lambda)\log(1-\lambda) \\
&= \lambda H_{\mathrm{A}} + (1-\lambda) H_{\mathrm{B}} + H_{\lambda}
\end{aligned}$$

Hence, the entropy of the resulting source S is the weighted average of the two entropies plus some extra uncertainty as to which source was chosen.

1–4 (a) The entropy of the source is

$$H(S) = -0.04 \log_2 0.04 - 0.25 \log_2 0.25 \ldots - 0.05 \log_2 0.05 = 2.58 \text{ bits/symbol}$$

Hence, it is *not* possible to compress the source (lossless compression) at a rate below 2.58 bits per source symbol.

(b) We need to calculate $H(S) - I(S;U)$.

$$I(S;U) = H(U) - H(U|S) = \{H(U|S) = 0\} = H(U)$$

since U is uniquely determined by S. Hence

$$I(S;U) = H(U) = -\sum_{i=0}^{2} f_U(u_i) \log_2(f_U(u_i)) = \ldots = 1.47 \text{ bits/symbol}$$

Compare this with $H(S) = 2.58$. Hence 1.11 bits/symbol are lost.

1–5 The entropy

$$\begin{aligned}
H &= -\sum p_i \log p_i \\
&= -0.2 \log 0.2 - 0.8 \log 0.8 \\
&= 0.722\text{bits}
\end{aligned}$$

For $m = 3$, the probabilities:

$0.2 \times 0.2 \times 0.2$	0.008
$0.2 \times 0.2 \times 0.8$	0.032
$0.2 \times 0.8 \times 0.2$	0.032
$0.2 \times 0.8 \times 0.8$	0.128
$0.8 \times 0.2 \times 0.2$	0.032
$0.8 \times 0.2 \times 0.8$	0.128
$0.8 \times 0.8 \times 0.2$	0.128
$0.8 \times 0.8 \times 0.8$	0.512

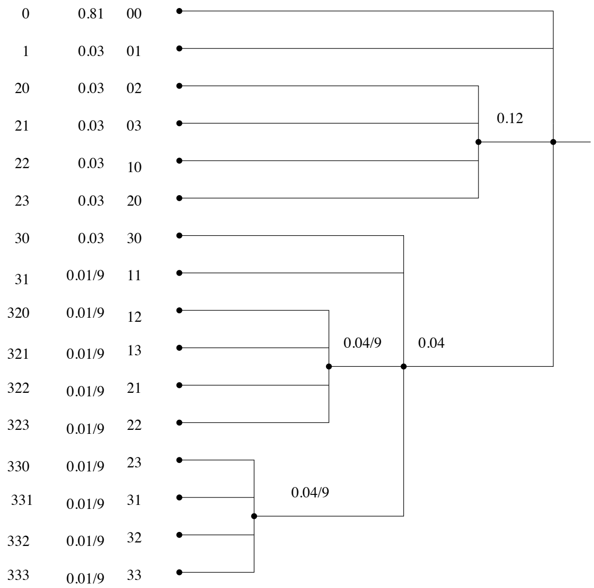
The corresponding codeword lengths are $\{5, 5, 5, 5, 3, 3, 3, 1\}$.

$$\begin{aligned}
L &= \frac{1}{3} \sum p_i l_i \\
&= 0.728 \text{ bits} \\
&> H_p
\end{aligned}$$

1–6 (a) The lowest possible number of coded symbols per source symbol is

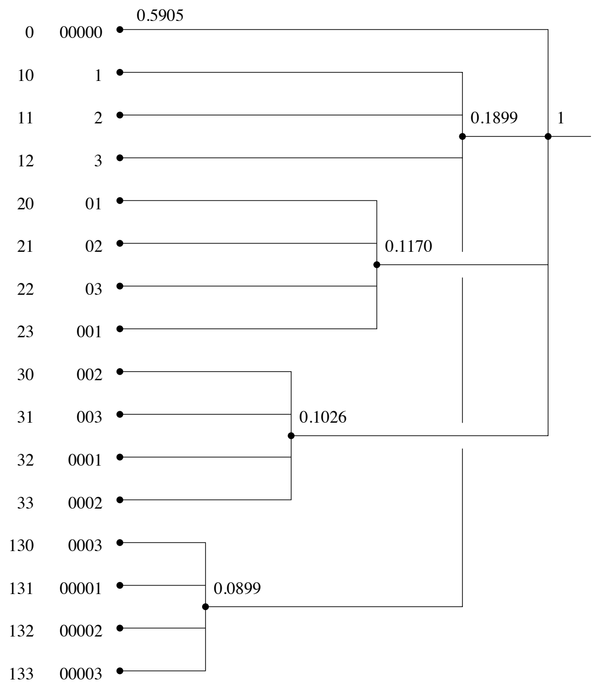
$$H(x) = -(1-p)\log_4(1-p) - 3\,\frac{p}{3}\log_4\frac{p}{3}\ ,$$

(b) The Huffman code is shown below.



The rate is 0.584 symbols/source symbol.

(c) The code is shown below.



The corresponding rate is 0.544 symbols/source symbol.

(d) The sequence is divided according to 1, 0, 3, 10, 00, 000, 03 , 001, 0000, 0002, 00000 , 000001, 003, 00020, 30, 000000. The code (x, y), where x is the row in the table and y the new symbol, is (0,1), (0,0), (0,3), (1,0), (2,0), (5,0), (2,3), (5,1), (6,0), (6,2), (9,0), (11,1), (5,3), (10,0), (3,0), (11,0). The corresponding dictionary is shown below

Row	**Content**	**Row**	**Content**
0	–	8	001
1	1	9	0000
2	0	10	0002
3	3	11	00000
4	10	12	000001
5	00	13	003
6	000	14	00020
7	03	15	30

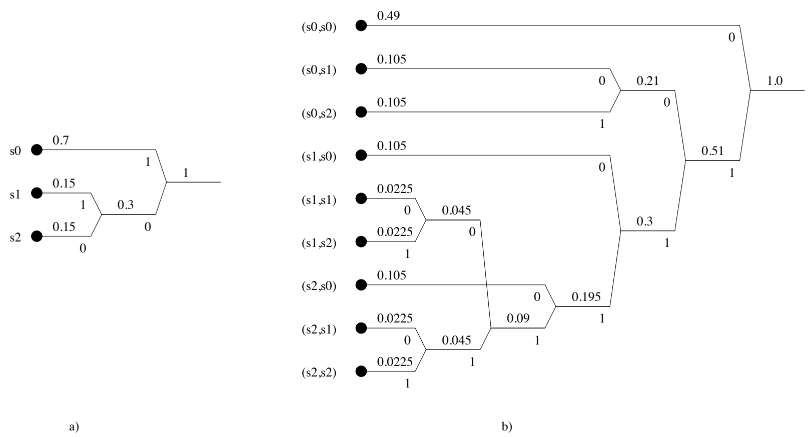
Note that it is given that the code shall use symbols from the set $\{0, 1, 2, 3\}$. The "new symbols" are from this set while the dictionary indices are from $\{0, \ldots, 15\}$ but these can be coded using two symbols from $\{0, 1, 2, 3\}$.

(e) The LZ algorithm requires $16 \cdot (2 + 1) = 48$ symbols (from $\{0, 1, 2, 3\}$) to code the given string. This gives a compression ratio 48/50, corresponding to a minor improvement in terms of compression. The Huffman codes, on the other hand, give the ratios 0.68, 0.52, respectively.

1–7 The smallest number of bits required on average is given by the entropy,

$$H(X) = -\sum_{i=\mathrm{A}}^{\mathrm{Z}} p_i \log_2 p_i \approx 4.3 \text{ bits/character}$$

1–8 (a) Applying the Huffman algorithm to the given source is a standard procedure described in the textbook yielding the code tree shown below (a to the left).



The average codeword length in bits per source symbol is

$$\bar{L} = \sum_{k=1}^{3} l_k P(s_k) = 1 \times 0.7 + 2 \times 0.15 + 2 \times 0.15 = 1.3,$$

where l_k is the number of bits in the codeword representing the source symbol s_k and $P(s_k)$ is the corresponding probability of s_k.

(b) The output alphabet of the extended source consist of all combinations of 2 symbols belonging to $\mathcal{S}$ i.e., $\mathcal{S}_{\text{ext}} = \{(s_0, s_0), (s_0, s_1), \ldots, (s_2, s_2)\}$. One example of the corresponding Huffman code tree is shown in the figure (b to the right). The average codeword length given in bits/extended code symbol is thus

$$\bar{L}_{\text{ext}} = \sum_{j=1}^{3} \sum_{k=1}^{3} l_{j,k} P\left((s_j, s_k)\right) = 2.3950,$$

where $l_{j,k}$ is the number of bits in the codeword representing the extended source symbol (s_j, s_k) and $P\left((s_j, s_j)\right)$ is the corresponding probability of the extended symbol. The average codeword length $\bar{L}$ in bits/source symbol is 1.1975.

(c) According to the what is often referred to as source coding theorem II, for a discrete memoryless source with finite entropy, it is possible to construct a code that satisfies the prefix condition and has an average length $\bar{L}$ that satisfies the inequalities

$$H(X) \leq \bar{L} < H(X) + 1.$$

However, instead of encoding on a symbol-by-symbol basis, a more efficient procedure is to encode blocks of N symbols at a time. In case of independent source symbols, the bound of the source coding theorem becomes

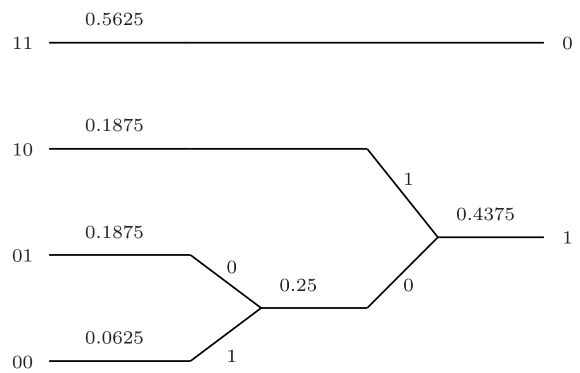
$$H(X^N) = N\,H(X) \leq \bar{L}_{\text{ext}} < N\,H(X) + 1 = H(X^N) + 1$$

which indicates that $\bar{L} = \bar{L}_{\text{ext}}/N$ can be made arbitrarily close to $H(X)$ by selecting the blocksize N large enough. For the considered source it is easily verified that the entropy of the source alphabet $H(X)$ is equal to 1.1813. Hence, it is readily verified that the inequalities holds for both $N = 1$ and 2. It can further be observed that in the considered example, a block length $N = 2$ will improve the efficiency $(H(x)/\bar{L})$ of the Huffman code from 91% to 99% i.e., a relative gain of 8.6%.

1–9 (a) The probabilities for the 2-bit symbols are given in the table below.

Symbol	Probability
00	$0.25 \cdot 0.25 = 0.0625$
01	$0.25 \cdot 0.75 = 0.1875$
10	$0.75 \cdot 0.25 = 0.1875$
11	$0.75 \cdot 0.75 = 0.5625$

The following tree diagram describes the Huffman code based on these symbols:



Thus, the codewords are

Symbol	Codeword
00	101
01	100
10	11
11	0

Using these codewords, the encoded sequence becomes

$$\text{ENC}\{x_1^{24}\} = 101, 100, 11, 100, 0, 0, 0, 100, 11, 0, 0, 0\,.$$

This sequence is 22 bits long so two bits are saved by the Huffman encoding. The entropy per output bit of the source is

$$H(X) = -0.25\log_2(0.25) - 0.75\log_2(0.75) \approx 0.8113 \text{ bits}$$

and the average codeword length is obtained as

$$\text{E}[l] = 3 \cdot 0.0625 + 3 \cdot 0.1875 + 2 \cdot 0.1875 + 1 \cdot 0.5625 = 1.6875 \text{ bits}\,.$$

To compare these quantities, we convert to bits per 2-bit symbol and bits for the entire sequence and summarize our results in the table below.

Entity	Bits Per 2-bit Symbol	Bits Per 24-bit Sequence
Encoded sequence	$22/12 \approx 1.833$	22
Entropy	$0.8113 \cdot 2 \approx 1.623$	$0.8113 \cdot 24 \approx 19.47$
Average compressed length	1.6875	$1.6875 \cdot \approx 20.25$

We see that the average length of a coded 24-bit sequence is 20.25 bits which is somewhat shorter than the 22 bits required for the particular sequence considered here. Of course, depending on the realization of the sequence, the compressed sequence can be significantly shorter. For example, 24 consecutive 1:s are compressed into a 12 bit long sequence of 0:s. It is also seen that the average codeword length is slightly larger than the corresponding entropy measure (1.6875 vs 1.623). In particular, these entities satisfy

$$1.623 \leq 1.6875 \leq 1.623 + 1 = 2.623\,,$$

which agrees well with the theory saying that

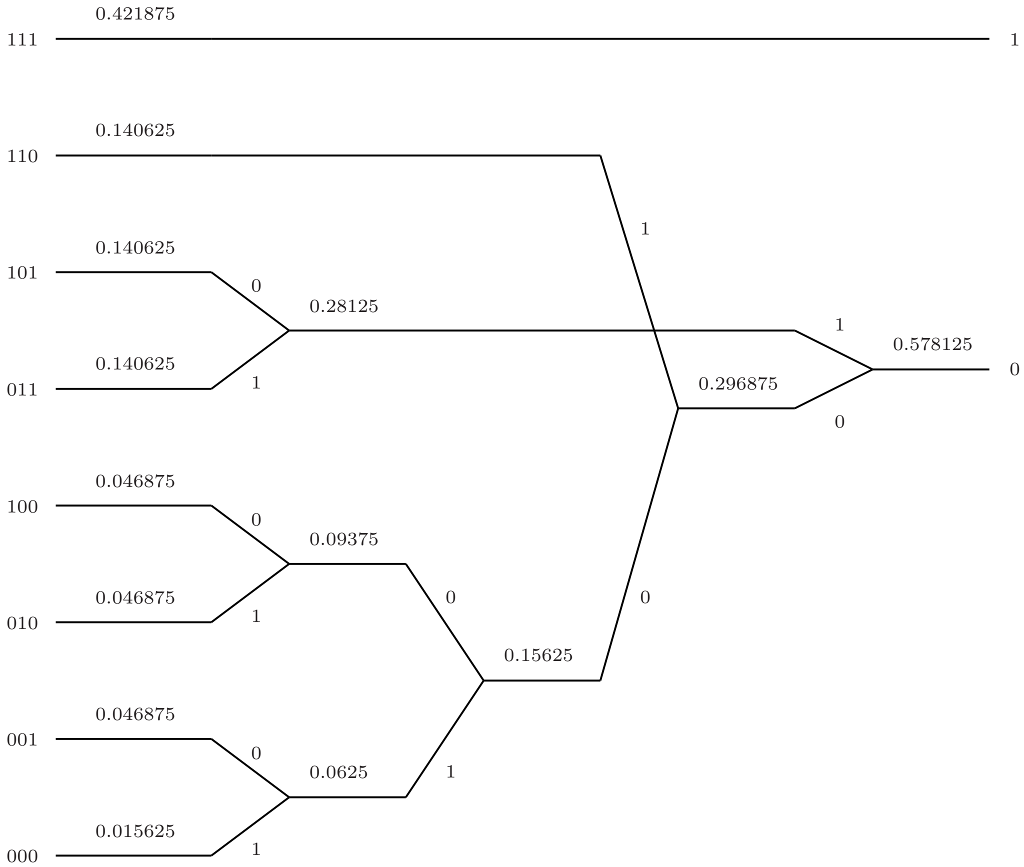
$$H(X^n) \leq \mathrm{E}[l] \leq H(X^n) + 1\,, \tag{3.6}$$

where $H(X^n)$ denotes the entropy of n-bit symbols (here $n = 2$).

(b) The probabilities for the 3-bit symbols are given by

Symbol	Probability
000	$0.25 \cdot 0.25 \cdot 0.25 = 0.015625$
001	$0.25 \cdot 0.25 \cdot 0.75 = 0.046875$
010	$0.25 \cdot 0.75 \cdot 0.25 = 0.046875$
011	$0.25 \cdot 0.75 \cdot 0.75 = 0.140625$
100	$0.75 \cdot 0.25 \cdot 0.25 = 0.046875$
101	$0.75 \cdot 0.25 \cdot 0.75 = 0.140625$
110	$0.75 \cdot 0.75 \cdot 0.25 = 0.140625$
111	$0.75 \cdot 0.75 \cdot 0.75 = 0.421875$

with the corresponding tree diagram illustrated below.



Thus, the codewords are

Symbol	Codeword
000	00011
001	00010
010	00001
011	011
100	00000
101	010
110	001
111	1

and the encoded sequence is therefore

$$\mathrm{ENC}\{x_1^{24}\} = 00011, 001, 011, 1, 001, 001, 1, 1\,,$$

which is seen to be 20 bits long. Moreover, the entropy per source bit is the same as before while the average codeword length is obtained as

$$\mathrm{E}[l] = 5 \cdot 0.0625 + 5 \cdot 0.09375 + 3 \cdot 0.28125 + 3 \cdot 0.140625 + 1 \cdot 0.421875 = 2.46875 \text{ bits}\,.$$

Similarly as was in (a), a table comparing these quantities is shown below.

Entity	Bits per 3-bit symbol	Bits per entire source sequence
Encoded sequence	$22/12 \approx 1.833$	20
Entropy	$0.8113 \cdot 3 \approx 2.4339$	$0.8113 \cdot 24 \approx 19.47$
Average code length	2.46875	$2.46875 \cdot 8 \approx 19.75$

We see that the average length of the coded sequence has decreased to 19.75 bits as opposed to the 20.25 bits for the 2-bit symbol case. This is also true in general, i.e. the average length decreases when more source bits are used for forming the symbols. Dividing (3.6) by n (here $n = 3$) and utilizing that $H(X^n) = nH(X)$, since the source is memoryless, shows that

$$H(X) \leq \frac{\mathrm{E}[l]}{n} \leq H(X) + \frac{1}{n}.$$

Hence, in the limit, as the number of symbol bits increases, the average codeword length approaches the entropy of the source.

However, the advantage of a better compression ratio when more bits are grouped together is offset by an exponential increase in complexity of the algorithm. Not only must one estimate the statistics of more symbols, but the size of the tree diagram becomes much larger.

(c) The first step in the Lempel-Ziv algorithm is to identify the phrases. For the problem at hand, they are

$$0, 00, 1, 10, 01, 11, 111, 101, 1011, 1111.$$

Since there are ten phrases, four bits are needed for describing the location in the dictionary. The dictionary is designed as

Dict. pos.	Contents	Codeword
0. 0000	–	–
1. 0001	–0	00000
2. 0010	00	00010
3. 0011	–1	00001
4. 0100	10	00110
5. 0101	01	00011
6. 0110	11	00111
7. 0111	111	01101
8. 1000	101	01001
9. 1001	1011	10001
10. 1010	1111	01111

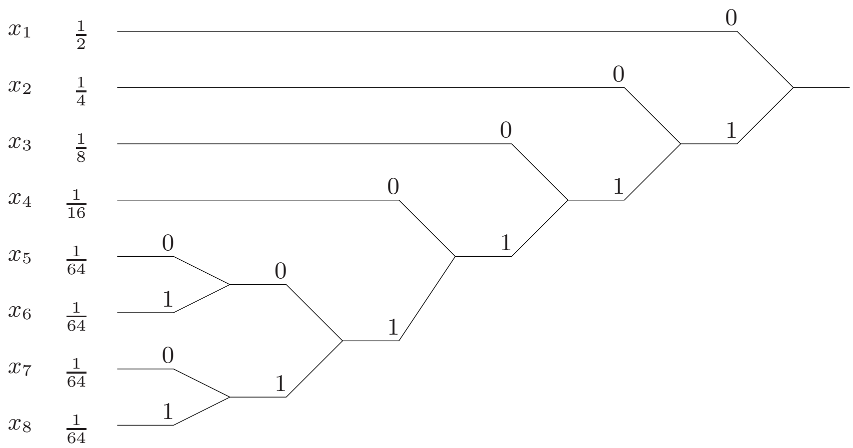
where '–' means 'empty'. The encoded sequence then becomes

$$00000, 00010, 00001, 00110, 00011, 00111, 01101, 01001, 10001, 01111,$$

which is 50 bits long. So, Lempel-Ziv coding has in this case expanded the sequence and produced a result much longer than what the entropy of the source predicts! The original sequence is simply too short for compression to take place. However, as the original sequence grows longer, it can be shown that the number of bits in the compressed sequence approaches the corresponding entropy measure, thus leading to good compression.

1–10 Let's call the different output values of the source $x_1, \ldots, x_8$, corresponding to the given probabilities $\frac{1}{2}, \ldots, \frac{1}{64}$.

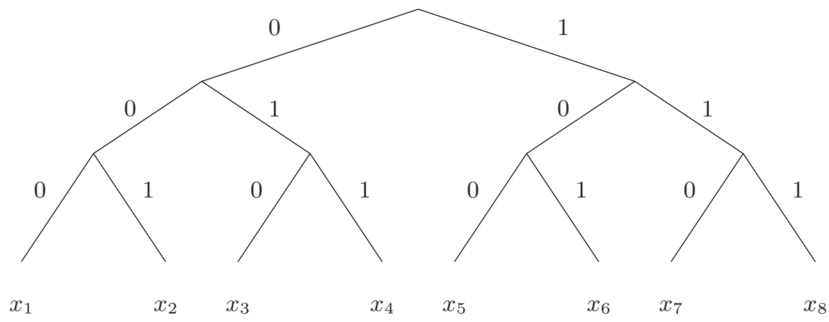
(a) The instantaneous code that minimizes the expected code-length is the Huffman code. One Huffman code is specified below.



Symbol	Codeword
x_1	0
x_2	10
x_3	110
x_4	1110
x_5	111100
x_6	111101
x_7	111110
x_8	111111

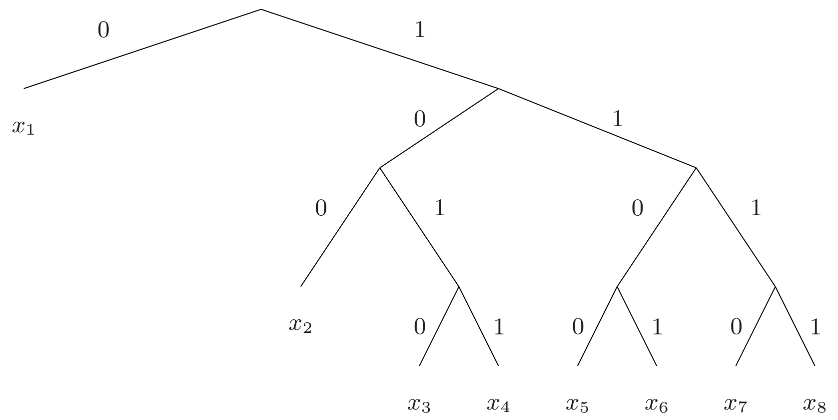
(b) The entropy of the source is 2 bits per source symbol. The expected length of the Huffman code is also 2 bits per source symbol. From the source-coding theorem, we know that a lower expected length than the entropy of the source can not be obtained by any code. If the expected length of the code would have been greater than the entropy, coding length-N blocks of source symbols would have yielded a lower expected length.

(c) To find instantaneous codes, the code-tree can be used. The code-tree that minimizes the length of the longest codeword is obviously as flat as possible, which gives the longest code-word length 3 bits per source symbol.



Symbol	Codeword
x_1	000
x_2	001
x_3	010
x_4	011
x_5	100
x_6	101
x_7	110
x_8	111

(d) In this problem, code-trees with a maximum depth of four bits should be considered. To minimize the expected length, the most probable symbols should be assigned to the nodes represented by fewer bits. This yields a few candidates to the optimal code-tree. Calculation of the expected length of the different candidates gives the optimal one that is shown below. Its expected code-word length is 2.25 bits.



Symbol	Codeword
x_1	0
x_2	100
x_3	1010
x_4	1011
x_5	1100
x_6	1101
x_7	1110
x_8	1111

1–11 (a) Obviously, there are 12 equiprobable outcomes of the pair (X, Y):

$$(X,Y) \in \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}$$

Therefore, $K = 1/12$. The marginal distributions for X and Y are

$$\Pr(X = x) = \Pr(Y = y) = 3K = \frac{1}{4} \quad \forall x, y \in \{1,2,3,4\}$$

The entropies for X and Y are therefore equal.

$$H(X) = H(Y) = -\sum_x \Pr(X = x)\log\Pr(X = x) = \log 4 = 2 \text{ bits}$$

From the definition of joint entropy:

$$H(X,Y) = -\sum_{x,y} \Pr(X = x, Y = y)\log\Pr(X = x, Y = y) = \log 12 \approx 3.58 \text{ bits}$$

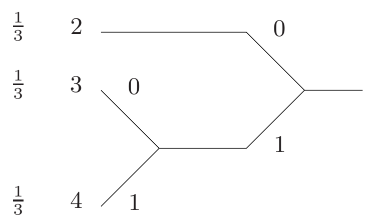
The mutual information can be written as:

$$I(X;Y) = H(X) + H(Y) - H(X,Y) = 2\log 4 - \log 12 = \log\frac{4}{3} \approx 0.415 \text{ bits}$$

(b) The probability mass function of X, if it is known that $Y = 1$, is

$$\Pr(X = x|Y = 1) = \begin{cases} \frac{1}{3} & x = 2,3,4 \\ 0 & x = 1 \end{cases}$$

Using this probability distribution to design a Huffman code yields the tree:



with the codewords:

x	Codeword
2	0
3	10
4	11

(c) The rate of the code in (b) is $R = \frac{1}{3}(1+2+2) = \frac{5}{3} \approx 1.67$ bits per input symbol. The entropy of X, given that Y is known, is

$$H(X|Y=1) = -\sum_x \Pr(X=x|Y=1)\log \Pr(X=x|Y=1) = \log 3 \approx 1.58$$

According to the lossless source coding theorem, lossless uniquely decodable codes with rate $R > H$ exist. Therefore it is possible to construct such a code with rate lower than 1.67.

1–12 (a) The minimum rate (code-bits per source-bit) is given by the entropy rate

$$H = -p\log p - (1-p)\log(1-p) \approx 0.29$$

(b) The algorithm compresses variable-length source strings into 3-bit strings, as summarized in the table below.

source string	probability	codeword
0	$1-p = 0.05$	000
10	$p(1-p) = 0.0475$	001
110	$p^2(1-p) = 0.045125$	010
1110	$p^3(1-p) = 0.0428688$	011
11110	$p^4(1-p) = 0.0407253$	100
111110	$p^5(1-p) = 0.038689$	101
1111110	$p^6(1-p) = 0.0367546$	110
1111111	$p^7 = 0.698337$	111

Hence the average rate is

$$R = 3\cdot(1-p) + \frac{3}{2}\cdot p(1-p) + 1\cdot p^2(1-p) + \frac{3}{4}\cdot p^3(1-p) + \cdots + \frac{3}{7}\cdot p^7 \approx 0.657$$

in code-bits per source-bit.

1–13 The step-size is $2/2^4 = 1/8$. Uniform variable, uniform distribution $\Rightarrow D = \Delta^2/12 = 1/768$

1–14 The resulting quantization levels are $\pm 1/16$, $\pm 3/16$, $\pm 3/8$ and $\pm 3/4$. The quantization distortion is

$$D = \sum_i p_i\, D_i$$

where D_i is the contribution to the distortion from the ith quantization interval, and p_i is the probability that the input variable belongs to the ith interval. Since the quantization error is uniformly distributed in each interval, we get

$$D = \frac{1}{12}\sum_i p_i\,\Delta_i = \ldots = \frac{1}{264}$$

where Δ_i is the length of the ith interval. Without companding the corresponding distortion is $D = 1/192$. That is, the improvement over linear quantization is $264/192 \approx 1.375$ times.

1–15 *Granular distortion*: Since there are $N = 2^8 = 256$ quantization levels (N is a "large" number), we get

$$D_g \approx \frac{\Delta^2}{12}\Pr(|X| < V)$$

where $\Delta = 2V/N = V/128$, and

$$\Pr(|X| < V) = \int_{-V}^{V} f(x)dx = 1 - \exp(-\sqrt{2}\,V/\sigma)$$

That is, $D_g \approx 8.1 \cdot 10^{-5}\,\sigma^2$.

Overload distortion: We get

$$D_o = 2\int_{V}^{\infty} (x - V)^2 f(x)dx = \ldots = \exp(-4\sqrt{2})\,\sigma^2 \approx 3.5 \cdot 10^{-3}$$

Hence, $D_o > D_g$.

1–16 Large number of levels $\Rightarrow$ linear quantization gives $D_{\text{lin}} \approx \Delta^2/12$. Optimal companding gives

$$D_{\text{opt}} \approx \frac{\Delta^2}{12}\int_{-1}^{1} \frac{f(x)}{[g'(x)]^2}dx$$

That is

$$\frac{D_{\text{opt}}}{D_{\text{lin}}} \approx \int_{-1}^{1} \frac{f(x)}{[g'(x)]^2}dx = \ldots = \frac{1}{2}$$

1–17 The signal-to-quantization-noise ratio is

$$\text{SQNR} = \frac{V^2/3}{\Delta^2/12}$$

with $\Delta = 2V/2^b$, where b is the number of bits per sample. Hence,

$$\text{SQNR} = 2^{2b} > 10^4 \Rightarrow b \geq 7$$

(noting that only an integer b is possible).

1–18 Let $[-V, V]$ be the granular region of the quantizer and let $b = 64000/8000 = 8$ be the number of bits per sample used for quantization. We get

$$D_q \approx \frac{(2V/2^b)^2}{12} = \frac{V^2}{3 \cdot 2^{16}}$$

Also, letting X denote a source sample and $\hat{X}$ its reconstructed value, we get

$$D_e = E[(X - \hat{X})^2|\,\text{error}] = \frac{V^2}{2} + \frac{V^2}{3} = \frac{5}{6}V^2$$

since X is a sample from a sinusoid, with $E[X] = 0$ and $E[X^2] = V^2/2$, and since $\hat{X}$ is uniformly distributed and independent of X (when an error has occurred). Finally, P_e is obtained as

$$P_e = 1 - \Pr(\text{no error}) = 1 - \left[1 - Q\left(\frac{A}{\sqrt{R\,N_0/2}}\right)\right]^b \approx b\,Q\left(\frac{A}{\sqrt{R\,N_0/2}}\right)$$

We hence get,

$$P_e D_e < (1 - P_e)D_q \Rightarrow P_e < \frac{2}{5}\,2^{-16} \Rightarrow A > 3.8$$

1–19 Let b be the number of bits per sample in the quantization. Then $R = 6000 \cdot b$ bits are transmitted per second. The probability of one bit error, in any of the 21 links, is

$$p = Q\left(\sqrt{\frac{2P}{N_0 R}}\right)$$

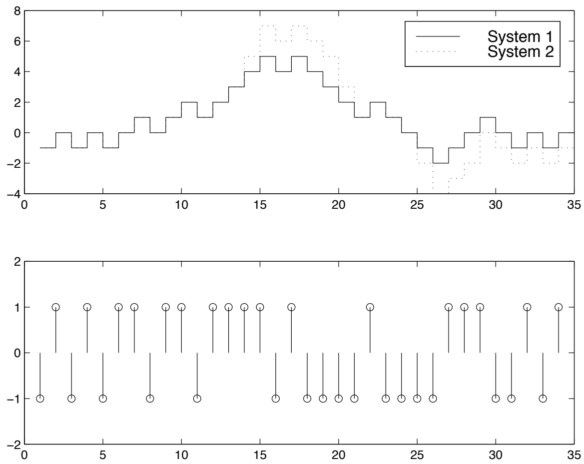
where $P = 10^{-4}$ W and $N_0/2 = 10^{-10}$ W/Hz. The probability that the transmitted index corresponding to one source sample is in error hence is

$$\Pr(\text{error}) = 1 - \Pr(\text{no error}) = 1 - (1-p)^{21b} \approx 21\,pb = 21b\,Q\left(\sqrt{\frac{10^6}{6000b}}\right)$$

and to get $\Pr(\text{error}) < 10^{-4}$ the number of bits can thus be at most $b = 8$. Since maximizing b minimizes the distortion D_q due to quantization errors, the minimum D_q is hence

$$D_q = (2/2^8)^2/12 \approx 5 \cdot 10^{-6}$$

1–20 (a) The lower part of the figure below shows the sequence ($0 \to -1$ and $1 \to +1$) and the upper part shows the decoded signal.



(b) *System 1*:

$$\frac{P_{\max}}{P_{\min}} = \left(\frac{f_d}{2\pi f_m}\right)^2$$

System 2:

$$\frac{P_{\max}}{P_{\min}} = \left(\frac{f_d}{\pi f_m}\right)^2$$

1–21 We have

$$P\left(\hat{X} = \frac{3}{4}a\right) = P\left(\hat{X} = -\frac{3}{4}a\right) = P\left(X > \gamma a\right) = \frac{1-\gamma}{2}$$
$$P\left(\hat{X} = \frac{1}{4}a\right) = P\left(\hat{X} = -\frac{1}{4}a\right) = P\left(0 < X \le \gamma a\right) = \frac{\gamma}{2}$$

The variable $\hat{X}$ cannot be coded below the rate $H(\hat{X})$ bits per symbol. Therefore

$$H(\hat{X}) = -2\frac{\gamma}{2}\log_2\frac{\gamma}{2} - 2\frac{1-\gamma}{2}\log_2\frac{1-\gamma}{2} = 1 + H_b(\gamma)\,,$$

where $H_b(\gamma) = -\gamma\log_2\gamma - (1-\gamma)\log_2(1-\gamma)$ is the binary entropy function. This gives $H(\hat{X}) > 1.5$ for $0.11 < \gamma < 0.89$.

1–22 (a) We begin with defining the sets $A_0 = (-\infty, -\sigma]$, $A_1 = (-\sigma, 0]$, $A_2 = (0, \sigma]$ and $A_3 = (\sigma, \infty)$. The average quantization distortion is

$$E[(X - \hat{X})^2] = E[X^2] - 2E[X\hat{X}] + E[\hat{X}^2].$$

Studying these three averages on-by-one, we have $E[X^2] = \sigma^2$,

$$E[X\hat{X}] = \sum_{i=0}^{3} \Pr(X \in A_i) E[X|X \in A_i] E[\hat{X}|X \in A_i],$$

and

$$E[\hat{X}^2] = \sum_{i=0}^{3} \Pr(X \in A_i) E[(\hat{X})^2 | X \in A_i].$$

We note that because of symmetry we have

$$E[X|X \in A_0] = -E[X|X \in A_3], \;\; E[X|X \in A_1] = -E[X|X \in A_2]$$

Also

$$E[\hat{X}|X \in A_0] = -E[\hat{X}|X \in A_3] = -3\sigma/2, \;\; E(\hat{X}|X \in A_1) = -E(\hat{X}|X \in A_2) = \sigma/2$$

Furthermore we note that $p_0 \triangleq \Pr(X \in A_0) = \Pr(X \in A_3)$ and $p_1 \triangleq \Pr(X \in A_1) = \Pr(X \in A_2)$. So we need (at most) to find p_0, p_1, $E[X|X \in A_0]$ and $E[X|X \in A_1]$. Since the source is Gaussian, we have

$$E[X|X \in A_0] = \frac{1}{p_0\sqrt{2\pi\sigma^2}} \int_{-\infty}^{-\sigma} x \exp(-\frac{1}{2\sigma^2}x^2)dx = \ldots = -\frac{\sigma}{p_0\sqrt{2\pi}} e^{-1/2}.$$

Similarly

$$E[X|X \in A_1] = \frac{1}{p_1\sqrt{2\pi\sigma^2}} \int_{-\sigma}^{0} x \exp(-\frac{1}{2\sigma^2}x^2)dx = \ldots = -\frac{\sigma}{p_1\sqrt{2\pi}} (e^{-1/2} - 1).$$

Hence we get

$$E[X\hat{X}] = 2p_0 \frac{\sigma}{p_0\sqrt{2\pi}} e^{-1/2} \frac{3\sigma}{2} + 2p_1 \frac{\sigma}{p_1\sqrt{2\pi}} (e^{-1/2} - 1)\frac{\sigma}{2} = \frac{1}{\sqrt{2\pi}} \sigma^2 (1 + 2e^{-1/2}).$$

We also have $E[(\hat{X})^2] = 2p_0(\frac{3\sigma}{2})^2 + 2p_1(\frac{\sigma}{2})^2$. Consequently, the average total distortion is

$$E[(X - \hat{X})^2] = \sigma^2 - \sqrt{\frac{2}{\pi}} \sigma^2 (1 + 2e^{-1/2}) + 2p_0(\frac{3\sigma}{2})^2 + 2p_1(\frac{\sigma}{2})^2.$$

Recognizing that $p_0 = Q(1) = 0.1587$ and that $p_1 = \frac{1}{2} - Q(1) = 0.3413$ (where $Q(x) = (2\pi)^{-1/2} \int_x^\infty \exp(-2^{-1}t^2)dt$), we finally have that

$$E[(X - \hat{X})^2] = \left(1.8848 - \sqrt{\frac{2}{\pi}}(1 + 2e^{-1/2})\right) \sigma^2 = 0.1190\sigma^2.$$

(b) The entropy is obtained as $H(\hat{X}) = -\sum_{i=0}^{3} \Pr(X \in A_i) \log \Pr(X \in A_i)$. Consequently $H(\hat{X}) = -2p_0 \log p_0 - 2p_1 \log p_1$ (where p_0 and p_1 were defined in part (a)). That is, $H(\hat{X}) = 1.9015$ bits.

(c) The entropy, $H(\hat{X})$, is maximum iff all values for $\hat{X}$ are equiprobable, that is iff $\Pr(X \le -a\sigma) = 1/4 \Leftrightarrow Q(a) = 1/4$. This gives $a \approx 0.675$.

(d) For simplicity, let $I \in \{0, 1, 2, 3\}$ correspond to the transmitted two-bit codeword, and let $J \in \{0, 1, 2, 3\}$ correspond to the received codeword. Then, the average mutual information is $I(X, \hat{X}) = I(I, J) = H(J) - H(J|I)$, where

$$H(J) = -\sum_{j=0}^{3} \Pr(J = j) \log \Pr(J = j),$$

$$H(J|I) = \sum_{i=0}^{3} \Pr(I = i) H(J|I = i).$$

Hence we need to find $P(j)$, $P(i)$ and $H(J|i)$. The probabilities $\Pr(I = i)$ are known from part (a), where we found that $\Pr(I = 0) = \Pr(I = 3) = p_0 = 0.1587$ and $\Pr(I = 1) = \Pr(I =$

$2) = p_1 = 0.3413$. The probabilities, $P(j)$, at the channel output are $P(j) = \sum_{i=0}^{3} \Pr(I = i)P(j|i)$. That is,

$$\Pr(J = 0) = \Pr(J = 3) = p_0\left((1-q)^2 + q^2\right) + 2p_1 q(1-q) = 0.1623$$
$$\Pr(J = 1) = \Pr(J = 2) = 2p_0 q(1-q) + p_1\left((1-q)^2 + q^2\right) = 0.3377$$

where the notation $P(j|i)$ means "probability that j is received when i was transmitted." Consequently $H(J) = \ldots = 1.9093$ bits. Furthermore,

$$\begin{aligned} H(J|I=0) &= H(J|I=1) = H(J|I=2) = H(J|I=3) \\ &= -\sum_{j=0}^{3} P(j|0)\log P(j|0) = \ldots = 0.1616 \text{ bits.} \end{aligned}$$

Hence, $H(J|I) = 0.1616$ and consequently $I(I, J) = 1.9093 - 0.1616 = 1.7477$ bits.

1–23 The quantization distortion can be expressed as

$$\begin{aligned} D &= \int_{-\infty}^{-a} (x+b)^2 f_X(x)dx + \int_{-a}^{a} x^2 f_X(x)dx + \int_{a}^{\infty} (x-b)^2 f_X(x)dx \\ &= \int_0^a x^2 e^{-x}dx + \int_a^\infty (x-b)^2 e^{-x}dx \end{aligned}$$

(a) Continuing with solving the expression for D above we get

$$D = \left(2e^a + b^2 - 2(a+1)b\right) e^{-a}$$

(b) Differentiating D wrt b and setting the result equal to zero gives

$$2e^{-a}(b - a - 1) = 0,$$

that is, $b = a + 1$. Differentiating D wrt a and setting the result equal to zero then gives

$$(2a - b)be^{-a} = 0,$$

that is, $a = b/2$. Hence we get $b = 2$ and $a = 1$, resulting in

$$D = 2(1 - 2e^{-1}) \approx 0.528$$

Note that another approach to arrive at $b = 2$ and $a = 1$ is to observe that according to the textbook/lectures a necessary condition for optimality is that b is the centroid of its encoding region, i.e

$$b = E[X|X > a] = \ldots = a + 1$$

and in addition a should define a nearest-neighbor partition of the real line, i.e.

$$a = b/2$$

(since there is a code-level at 0, the number a should lie halfway between b and 0).

1–24 (a) The zero-mean Gaussian variable X with variance 1, has the pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

The distortion of the quantizer is a function of the step-size Δ, and the reproduction points $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ and $\hat{x}_4$.

$$\begin{aligned} D(\Delta, \hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) &= E[(X - \hat{X})^2] \\ &= \int_{-\infty}^{-\Delta} (x + \hat{x}_1)^2 f_X(x)dx + \int_{-\Delta}^{0} (x + \hat{x}_2)^2 f_X(x)dx \\ &+ \int_0^\Delta (x - \hat{x}_3)^2 f_X(x)dx + \int_\Delta^\infty (x - \hat{x}_4)^2 f_X(x)dx \end{aligned}$$

In the (a) problem, it is given that $\hat{x}_1 = \hat{x}_4$ and $\hat{x}_2 = \hat{x}_3$. Since $f_X(x)$ is an even function, this is valid also in the (b)-problem. Further, using the fact that the four integrands above are all even functions gives

$$\begin{aligned}
D(\Delta, \hat{x}_1, \hat{x}_2) &= 2\int_{\Delta}^{\infty} (x-\hat{x}_1)^2 f_X(x)dx + 2\int_0^{\Delta} (x-\hat{x}_2)^2 f_X(x)dx \\
&= \underbrace{2\int_{\Delta}^{\infty} x^2 f_X(x)dx}_{I} \underbrace{+2\hat{x}_1^2 \int_{\Delta}^{\infty} f_X(x)dx}_{II} \underbrace{-4\hat{x}_1 \int_{\Delta}^{\infty} x f_X(x)dx}_{III} \\
&+ \underbrace{2\int_0^{\Delta} x^2 f_X(x)dx}_{IV} \underbrace{+2\hat{x}_2^2 \int_0^{\Delta} f_X(x)dx}_{V} \underbrace{-4\hat{x}_2 \int_0^{\Delta} x f_X(x)dx}_{VI}
\end{aligned}$$

Using the hints provided to solve or express the six integrals in terms if the Q-function yields

$$\begin{aligned}
I &= \tfrac{2}{\sqrt{2\pi}} \textstyle\int_{\Delta}^{\infty} x^2 e^{\frac{-x^2}{2}} dx &&= \sqrt{\frac{2}{\pi}}\Delta e^{\frac{-\Delta^2}{2}} + 2Q(\Delta) \\
II &= \tfrac{2\hat{x}_1^2}{\sqrt{2\pi}} \textstyle\int_{\Delta}^{\infty} e^{\frac{-x^2}{2}} dx &&= 2\hat{x}_1^2 Q(\Delta) \\
III &= -\tfrac{4\hat{x}_1}{\sqrt{2\pi}} \textstyle\int_{\Delta}^{\infty} x e^{\frac{-x^2}{2}} dx &&= -\frac{4\hat{x}_1}{\sqrt{2\pi}} e^{\frac{-\Delta^2}{2}} \\
IV &= \tfrac{2}{\sqrt{2\pi}} \textstyle\int_0^{\Delta} x^2 e^{\frac{-x^2}{2}} dx &&= (1-2Q(\Delta)) - \sqrt{\frac{2}{\pi}}\Delta e^{\frac{-\Delta^2}{2}} \\
V &= \tfrac{2\hat{x}_2^2}{\sqrt{2\pi}} \textstyle\int_0^{\Delta} e^{\frac{-x^2}{2}} dx &&= \hat{x}_2^2(1-2Q(\Delta)) \\
VI &= -\tfrac{4\hat{x}_2}{\sqrt{2\pi}} \textstyle\int_0^{\Delta} x e^{\frac{-x^2}{2}} dx &&= -\frac{4\hat{x}_2}{\sqrt{2\pi}}(1-e^{\frac{-\Delta^2}{2}})
\end{aligned}$$

Adding these six parts together gives after some simplification

$$D(\Delta, \hat{x}_1, \hat{x}_2) = 2Q(\Delta)(\hat{x}_1^2 - \hat{x}_2^2) + 1 + \hat{x}_2^2 + \frac{4e^{\frac{-\Delta^2}{2}}}{\sqrt{2\pi}}(\hat{x}_2 - \hat{x}_1) - \frac{4\hat{x}_2}{\sqrt{2\pi}}$$

Using MATLAB to find the minimum $D(\Delta)$, using $\hat{x}_1 = \frac{3\Delta}{2}$ and $\hat{x}_2 = \frac{\Delta}{2}$, gives $\Delta^* \approx 0.9957$ and $D(\Delta^*) \approx 0.1188$, which is in agreement with the table in the textbook. The minimum can be found by calculating $D(\Delta)$ for, for instance 10^6, equally spaced values in a feasible range of Δ, for instance 0-5, or by some more sophisticated numerical method.

(b) Because $f_X(x)$ is even and the quantization regions are symmetric, the optimal reproduction points are symmetric; $\hat{x}_1^* = \hat{x}_4^*$ and $\hat{x}_2^* = \hat{x}_3^*$. The optimal reproduction points are the *centroids* of the encoder regions, i.e.

$$\begin{aligned}
\hat{x}_1^* = \hat{x}_4^* = \quad & E[X|\Delta^* < X \leq \infty] = \frac{\int_{\Delta^*}^{\infty} x f_X(x)dx}{P(\Delta^* < X \leq \infty)} \\
\hat{x}_2^* = \hat{x}_3^* = \quad & E[X|0 < X \leq \Delta^*] = \frac{\int_0^{\Delta^*} x f_X(x)dx}{P(0 < X \leq \Delta^*)}
\end{aligned}$$

Since X is Gaussian, zero-mean and unit variance,

$$\begin{aligned}
P(0 < X \leq \Delta^*) &= Q(0) - Q(\Delta^*) \\
P(\Delta^* < X \leq \infty) &= Q(\Delta^*)
\end{aligned}$$

Using the hints gives

$$
\begin{aligned}
\int_{\Delta^*}^{\infty} x f_X(x) dx &= \frac{1}{\sqrt{2\pi}} e^{\frac{-(\Delta^*)^2}{2}} \\
\int_0^{\Delta^*} x f_X(x) dx &= \int_0^{\infty} x f_X(x) dx - \int_{\Delta^*}^{\infty} x f_X(x) dx = \frac{1}{\sqrt{2\pi}}(1 - e^{\frac{-(\Delta^*)^2}{2}})
\end{aligned}
$$

which finally gives

$$
\begin{aligned}
\hat{x}_1^* = \hat{x}_4^* &= \frac{1}{\sqrt{2\pi}} \frac{e^{\frac{-(\Delta^*)^2}{2}}}{Q(\Delta^*)} \approx 1.5216 \\
\hat{x}_2^* = \hat{x}_3^* &= \frac{1}{\sqrt{2\pi}} \frac{(1 - e^{\frac{-(\Delta^*)^2}{2}})}{(Q(0) - Q(\Delta^*))} \approx 0.4582
\end{aligned}
$$

Using the formula for the distortion from the (a)-part gives $D(\Delta^*, \hat{x}_1^*, \hat{x}_2^*) \approx 0.11752$

(c) The so-called Lloyd-Max conditions are necessary conditions for optimality of a scalar quantizer:

1. The boundaries of the quantization regions are the midpoints of the corresponding reproduction points.
2. The reproduction points are the centroids of the quantization regions.

From table 6.3 in the 2nd edition of the textbook, we find the following values for the optimal 4-level quantizer region boundaries, $-a_1 = a_3 = 0.9816$ and $a_2 = 0$. The optimal reproduction points are given as $-\hat{x}_1 = \hat{x}_4 = 1.510$ and $-\hat{x}_2 = \hat{x}_3 = 0.4528$. These values are of course not exact, but rounded.

The first condition is easily verified:

$$
\begin{aligned}
\frac{1}{2}(\hat{x}_1 + \hat{x}_2) &= -0.9814 \approx a_1 \\
\frac{1}{2}(\hat{x}_2 + \hat{x}_3) &= 0 = a_2 \\
\frac{1}{2}(\hat{x}_3 + \hat{x}_4) &= 0.9814 \approx a_3
\end{aligned}
$$

The second condition is similar to problem (b).

$$
\begin{aligned}
-\hat{x}_1 = \hat{x}_4 &= \frac{1}{\sqrt{2\pi}} \frac{e^{\frac{-a_3^2}{2}}}{Q(a_3)} \approx 1.510 \\
-\hat{x}_2 = \hat{x}_3 &= \frac{1}{\sqrt{2\pi}} \frac{(1 - e^{\frac{-a_3^2}{2}})}{(Q(0) - Q(a_3))} \approx 0.4528
\end{aligned}
$$

Again, using the formula for the distortion from the (a)-part gives $D(a_3, \hat{x}_4, \hat{x}_3) \approx 0.11748$ which is slightly better than in (b), and also in accordance with table 6.3.

1–25 A discrete memoryless channel with 4 inputs and 4 outputs, and a 4-level quantizer.

(a) We have

$$D = E[(Z - \hat{Z})^2]$$

with Z and $\hat{Z}$ defined as in the problem. For $Y = y$, $\hat{Z}$ takes on the value

$$\hat{z}_y = -\frac{3}{2} + \frac{y}{2}$$

We can write

$$D = E[Z^2] - 2\sum_{x=0}^{3} E[Z|X=x]E[\hat{Z}|X=x]\Pr(X=x) + E[\hat{Z}^2]$$

Here

$$E[Z^2] = \frac{1}{2}\int_{-1}^{+1} z^2 dz = \frac{1}{3}, \quad E[Z|X=x] = \hat{z}_x, \quad \Pr(X=x) = \frac{1}{4}$$

$$E[\hat{Z}|X=x] = (1-\varepsilon)\hat{z}_x + \varepsilon\hat{z}_{x+1 \bmod 4}, \quad E[\hat{Z}^2] = \frac{1}{4}\sum_y \hat{z}_y^2 = \frac{5}{16}$$

so

$$D = \frac{1}{48} + \frac{3}{8}\varepsilon = \frac{\Delta^2}{12} + \frac{3}{8}\varepsilon$$

(b) The optimal $\hat{Z}$'s are given by $\hat{z}_y = E[X|Y=y]$, hence

$$\hat{z}_0 = \hat{z}_2 = 0, \quad \hat{z}_1 = -0.5, \quad \hat{z}_3 = 0.5$$

1–26 (a) Since $\hat{X}_n = \hat{Y}_n + \hat{X}_{n-1}$ and $\hat{X}_0 = 0$, we get

n	X_n	$\hat{X}_{n-1}$	$Y_n = X_n - \hat{X}_{n-1}$	$\hat{Y}_n = Q[Y_n]$	$\hat{X}_n = \hat{X}_{n-1} + \hat{Y}_n$
1	0.9	0	0.9	1	1
2	0.3	1	−0.7	−1	0
3	1.2	0	1.2	1	1
4	−0.2	1	−1.2	−1	0
5	−0.8	0	−0.8	−1	−1

(b) Since $\hat{X}_0 = 0$ we get $Y_1 = X_1$ and hence

$$D_1 = E[(Y_1 - \hat{Y}_1)^2] = E[(X_1 - Q[X_1])^2] = E[X_1^2] - 2E\big[X_1 Q[X_1]\big] + 1$$
$$= 2 - 2\left(\int_0^\infty x\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx - \int_{-\infty}^0 x\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx\right) = 2\left(1 - \sqrt{\frac{2}{\pi}}\right) \approx 0.40$$

(c) Now $Y_2 = X_2 - Q[X_1]$, so we get

$$D_2 = E\big\{\big(X_2 - Q[X_1] - Q\big[X_2 - Q[X_1]\big]\big)^2\big\}$$
$$= E[(X_2 - Q[X_1])^2] + 1 - 2E\big\{(X_2 - Q[X_1])Q\big[X_2 - Q[X_1]\big]\big\}$$
$$= 3 - 2\left(\frac{1}{2}E\big\{(X_2 - Q[X_1])Q\big[X_2 - Q[X_1]\big]\big|X_1 < 0\big\} + \frac{1}{2}E\big\{(X_2 - Q[X_1])Q\big[X_2 - Q[X_1]\big]\big|X_1 > 0\big\}\right)$$
$$= 3 - 2\left(\frac{1}{2}E\big\{(X_2+1)Q\big[X_2+1\big]\big\} + \frac{1}{2}E\big\{(X_2-1)Q\big[X_2-1\big]\big\}\right)$$

where

$$E\big\{(X_2+1)Q\big[X_2+1\big]\big\} = E\big\{(X_2-1)Q\big[X_2-1\big]\big\}$$
$$= \int_{-1}^{\infty}(x+1)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx - \int_{-\infty}^{-1}(x+1)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$$
$$= \sqrt{\frac{2}{\pi}}e^{-1/2} + 1 - 2Q(1)$$

where

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt$$

is the Q-function. Thus

$$D_2 = 1 - 2\left(\sqrt{\frac{2}{\pi}}e^{-1/2} - 2Q(1)\right) \approx 0.67$$

(d) DPCM (delta modulation) works by utilizing *correlation* between neighboring values of X_n, since such correlation will imply $E[Y_n^2] < E[X_n^2]$. However, in this problem $\{X_n\}$ is memoryless, and therefore $E[Y_2^2] = 2 > E[X_2^2] = 1$. This means that $D_2 > D_1$ and the feedback loop in the encoder increases the distortion instead of decreasing it.

1–27 To maximize $H(Y)$ the variable a that determines the encoder regions must be set so that all values of Y are equally probable. This is the case when

$$\int_0^a f_X(x)dx = \int_a^\infty f_X(x)dx \iff \int_0^a e^{-x}dx = \int_a^\infty e^{-x}dx$$

Solving gives $a = \ln 2$. The optimal recunstruction points, that minimize D, are then given as the centroids

$$y_4 = -y_1 = E[X|X \geq \ln 2] = 2\int_{\ln 2}^\infty xe^{-x}dx = 1 + \ln 2$$
$$y_3 = -y_2 = E[X|0 \leq X < \ln 2] = 2\int_0^{\ln 2} xe^{-x}dx = 1 - \ln 2$$

The resulting distortion is then

$$D = E[(X-Y)^2] = \int_0^a (x-y_3)^2 e^{-x}dx + \int_a^\infty (x-y_4)^2 e^{-x}dx = 1 + (\ln 2)^2 - \ln 2 \ln 4 \approx 0.52$$

1–28 (a) Given $X = W_1 + W_2$, the probability density function for X is

$$f_X = \begin{cases} \frac{1}{8}x + \frac{3}{8}, & X \in [-3,-1] \\ \frac{1}{4}, & |x| \leq 1 \\ -\frac{1}{8}x + \frac{3}{8}, & X \in [1,3] \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} h(X) &= -\int_{-3}^{-1}(\frac{1}{8}x + \frac{3}{8})\ln(\frac{1}{8}x + \frac{3}{8})dx - \int_{-1}^{1}\frac{1}{4}\ln(\frac{1}{4})dx \\ &\quad -\int_1^3(-\frac{1}{8}x + \frac{3}{8})\ln(-\frac{1}{8}x + \frac{3}{8})dx \\ &= 2\ln(2) + \frac{1}{4} \end{aligned}$$

(b) The optimum coefficients are obtained as

$$\begin{aligned} a_1 &= -a_4 \\ a_2 &= -a_3 \\ a_3 &= E\{X|X \in [0,1)\} \\ &= \frac{\int_0^1 xf_x dx}{\int_0^1 f_x dx} = \frac{1}{2} \\ a_4 &= E\{X|X \in [1,3)\} \\ &= \frac{\int_1^3 xf_x dx}{\int_1^3 f_x dx} = \frac{5}{3} \end{aligned}$$

The minimum distortion is

$$\begin{aligned} D &= E\{(X - Q(X))^2\} \\ &= 2\int_{-3}^{-1}(x + \frac{5}{3})^2(\frac{x+3}{8})dx + 2\int_{-1}^{0}\frac{1}{4}(x + \frac{1}{2})^2 dx \\ &= \frac{11}{72} \end{aligned}$$

1–29 (a) The pdf of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

The clipped signal has pdf

$$f_Z(z) = f_Z\big(z \,\big|\, |Z| < A\big)\Pr(|Z| < A) + f_z\big(z \,\big|\, |Z| \geq A\big)\Pr(|Z| \geq A) = g(z) + \delta(z-A)Q(A) + \delta(z+A)Q(A)$$

where

$$g(z) = \begin{cases} f_X(z), & |z| < A \\ 0, & |z| \geq A \end{cases}$$

(b) To fully utilize the available bits, it should hold that $c < A$. The mean square error is

$$E\{(Y-Z)^2\} = 2\int_0^c (x-b)^2 f_X(x)dx + 2\int_c^A (x-a)^2 f_X(x)dx + 2\int_A^\infty (A-a)^2 f_X(x)dx$$

The optimal a and b are:

$$\begin{aligned} b &= \frac{\int_0^c x f_X(x)dx}{\int_0^c f_X(x)dx} \\ &= \frac{\frac{1-exp(-\frac{1}{2}c^2)}{\sqrt{2\pi}}}{1-0.5-Q(c)} \\ a &= \frac{\int_c^A x f_X(x)dx + \int_A^\infty A f_X(x)dx}{\int_c^\infty f_X(x)dx} \\ &= \frac{\frac{e^{-\frac{1}{2}c^2} - e^{-\frac{1}{2}A^2}}{\sqrt{2\pi}} + 0.9Q(A)}{Q(c)} \end{aligned}$$

The optimal c satisfies $c = \frac{1}{2}(b+a)$.

1–30 Let

$$p = \Pr(X \leq -a) = \Pr(X > a) = \frac{(b-a)^2}{2b^2}$$

(noting that $0 \leq a \leq b$). Then $\Pr(-a < X \leq 0) = \Pr(0 < X \leq a) = 1/2 - p$ and the entropy $H(\hat{X})$ of the variable $\hat{X}$ is hence obtained as

$$H(\hat{X}) = -2p\log_2(2p) - 2(1/2-p)\log_2(1/2-p) = 1 + H_b(2p)$$

where

$$H_b(x) = -x\log_2 x - (1-x)\log_2(1-x)$$

is the binary entropy function. The requirement that it must be possible to compress $\hat{X}$ without errors at rates above (but not below) 1.5 bits per symbol is equivalent to requiring $H(\hat{X}) = 1.5$. Solving this equation gives the two possible solutions

$$p = p_1 \approx 0.0550 \quad \text{or} \quad p = p_2 \approx 0.4450$$

corresponding to

$$a = a_1 \approx 0.6683\, b \quad \text{or} \quad a = a_1 \approx 0.0566\, b$$

Now the parameter a is determined in terms of the given constant b, with two possible choices that will both satisfy the constraint on the entropy. Hence the quantization regions are completely specified in terms of b (for any of the two cases). Then we know that the quantization levels $y_1, \ldots, y_4$ that minimize the mean-square distortion D correspond to the centroids of the quantization regions. We can hence compute the quantization levels as

$$\begin{aligned} y_4 = -y_1 = E[X|X>a] &= \frac{1}{p}\int_a^b x f_X(x)dx = \frac{1}{p}\int_a^b x\,\frac{b-x}{b^2}dx \\ &= \frac{1}{p}\left[\frac{b^2-a^2}{2b} - \frac{b^3-a^3}{3b^2}\right] \approx \begin{cases} 0.7789\, b, & p = p_1, a = a_1 \\ 0.3711\, b, & p = p_2, a = a_2 \end{cases} \end{aligned}$$

and

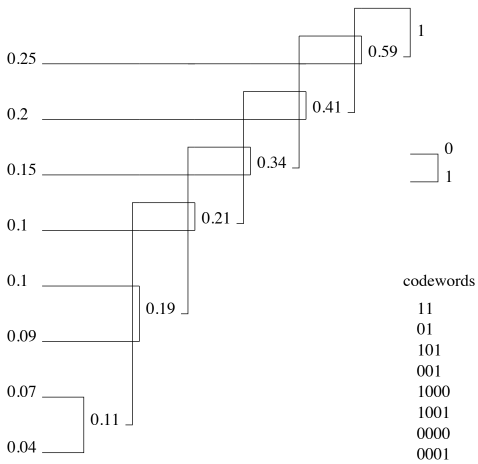
$$y_3 = -y_2 = E[X|0 < X \le a] = \frac{1}{1/2 - p} \int_0^a x f_X(x) dx$$
$$= \frac{1}{1/2-p}\left[\frac{a^2}{2b} - \frac{a^3}{3b^2}\right] \approx \begin{cases} 0.2782\,b, & p = p_1, a = a_1 \\ 0.0280\,b, & p = p_2, a = a_2 \end{cases}$$

The corresponding distortions, for the two possible solutions, are

$$\begin{aligned} D &= E[X^2] - E[\hat{X}^2] \\ &= E[X^2] - py_1^2 - (1/2-p)y_2^2 - py_3^2 - (1/2-p)y_4^2 = E[X^2] - 2py_1^2 - (1-2p)y_2^2 \\ &\approx \begin{cases} E[X^2] - 0.136\,b^2, & p = p_1, a = a_1 \\ E[X^2] - 0.123\,b^2, & p = p_2, a = a_2 \end{cases} \end{aligned}$$

We see that the difference in distortion is not large, but choosing $a = a_1$ clearly gives a lower distortion. Hence we should choose $a \approx 0.6683$, and the corresponding values for $y_1, \ldots, y_4$, as given above.

1–31 (a) The resulting Huffman tree is shown below, with the codewords listed to the down right.



The average codeword length is

$$L = 0.04 \cdot 4 + 0.07 \cdot 4 + 0.09 \cdot 4 + 0.1 \cdot 4 + 0.1 \cdot 3 + 0.15 \cdot 3 + 0.2 \cdot 2 + 0.25 \cdot 2 = 2.85$$

(bits per source symbol). The entropy (rate) of the source is

$$H = 0.04 \log 1/0.04 + \cdots + 0.25 \log 1/0.25 \approx 2.806$$

Hence the derived Huffman code performs about 0.044 bits worse than the theoretical optimum.

(b) It is readily seen that the given quantizer is uniform with stepsize Δ. The resulting distortion is

$$D = \int_0^\Delta \left(x - \frac{1}{2}\Delta\right)^2 e^{-x} dx + \int_\Delta^\infty \left(x - \frac{3}{2}\Delta\right)^2 e^{-x} dx = 2 - (1 + 2e^{-\Delta})\Delta + \frac{\Delta^2}{4}$$

The derivative of D wrt Δ is obtained as

$$\frac{\partial D}{\partial \Delta} = 2(\Delta - 1)e^{-\Delta} - 1 + \frac{\Delta}{2} = g(\Delta)$$

Solving $g(\Delta) = 0$ numerically or graphically then gives $\Delta \approx 1.5378$ (there is only one root and it corresponds to a minimum). Hence $1.5 < \Delta^* 1.55$ clearly holds true.

1–32 Uniform encoding with $\Delta = 1/4$ of the given pdf gives a discrete variable $\hat{X}$ with outcomes and corresponding probabilities according to

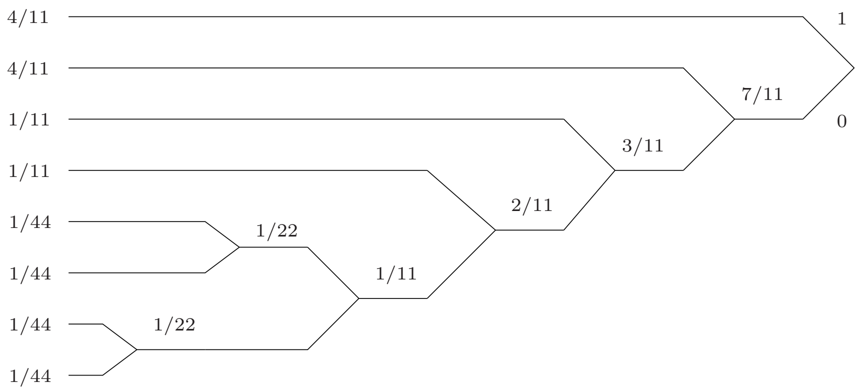
$p(\hat{x}_i)$	$\hat{X}$
1/44	$\hat{x}_0$
1/44	$\hat{x}_1$
1/11	$\hat{x}_2$
4/11	$\hat{x}_3$
4/11	$\hat{x}_4$
1/11	$\hat{x}_5$
1/44	$\hat{x}_6$
1/44	$\hat{x}_7$

where $p(\hat{x}_i) = \Pr(\hat{X} = \hat{x}_i)$.

(a) The entropy is obtained as

$$H(\hat{X}) = -\sum_{i=0}^{7} p(\hat{x}_i) \log p(\hat{x}_i) \approx 2.19 \ \text{[bits per symbol]}$$

(b) The Huffman code is constructed as



and it has codewords $\{1, 01, 001, 0001, 000011, 000010, 000001, 000000\}$ and average codeword length $L = 25/11 \approx 2.27 > H(\hat{X})$. Getting $L < H(\hat{X})$ must correspond to an error since $L \geq H(\hat{X})$ for all uniquely decodable codes.

(c) With uniform output levels $\hat{x}_i = (i - 7/2)\Delta$ the distortion is

$$\sum_{i=0}^{7} \int_{(i-4)\Delta}^{(i-3)\Delta} (x - \hat{x}_i)^2 f_X(x) dx = \frac{1}{\Delta} \sum_{i=0}^{7} p(\hat{x}_i) \int_{(i-4)\Delta}^{(i-3)\Delta} (x - \hat{x}_i)^2 dx = \frac{1}{\Delta} \sum_{i=0}^{7} p(\hat{x}_i) \int_{-\Delta/2}^{\Delta/2} \varepsilon^2 d\varepsilon = \frac{\Delta^2}{12} = \frac{1}{192}$$

(d) The optimal reconstruction levels are

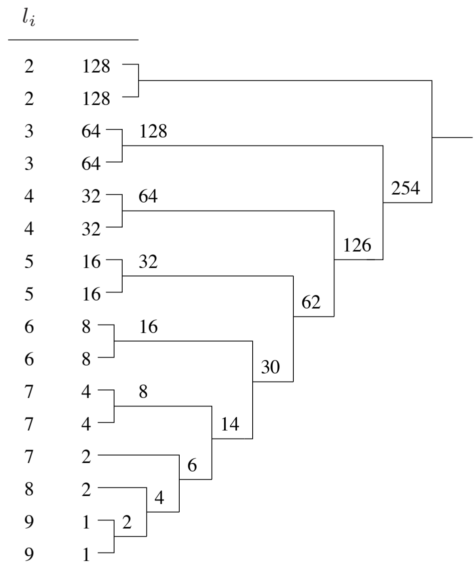
$$\hat{x}_i = E[X | (i-4)\Delta \leq X < (i-3)\Delta] = \frac{1}{\Delta} \int_{(i-4)\Delta}^{(i-3)\Delta} x \, dx = (i - \frac{7}{2})\Delta$$

That is, *the uniform output-levels are optimal* (since $f_X(x)$ is uniform over each encoder region) $\Rightarrow$ the distortion computed in (c) is the minimum distortion!

1–33 *Quantization and Huffman*: Since the quantizer is uniform with $\Delta = 1$, the 4-bit output-index I has the pmf

$$p(i) = \frac{128}{255} 2^{-(8-i)} \ , \ i = 0, \ldots, 7; \quad p(i) = \frac{128}{255} 2^{-(i-7)} \ , \ i = 8, \ldots, 15$$

A Huffman code for I, with codeword-lengths l_i (bits), is illustrated below.



The expected length of the output from the Huffman code is

$$L = \frac{128}{255}\left(2\,\frac{9}{256} + \frac{8}{128} + \frac{7}{128} + 2\,\frac{7}{64} + 2\,\frac{6}{32} + 2\,\frac{5}{16} + 2\,\frac{4}{8} + 2\,\frac{3}{4} + 2\,\frac{2}{2}\right) = \frac{756}{255} \approx 2.965 \text{ [bits/symbol]}$$

That is $\bar{L} = 3$.

Also, since the pdf $f(x)$ is uniform over the encoder regions, the quantization distortion with $k = 4$ bits is $\Delta^2/12 = 1/12$.

Quantization without Huffman: With $k = \bar{L} = 3$ we get $\Delta = 2$, and the resulting quantization distortion is

$$\begin{aligned} E[(X - \hat{X})^2] &= 2\int_0^2 (x-1)^2 f(x)dx + 2\int_2^4 (x-3)^2 f(x)dx + 2\int_4^6 (x-5)^2 f(x)dx + 2\int_6^8 (x-7)^2 f(x)dx \\ &= 2\int_{-1}^{1} x^2 \left[f(x+1) + f(x+3) + f(x+5) + f(x+7)\right]dx = \int_{-1}^{1} x^2 dx = \frac{1}{3} \end{aligned}$$

Consequently, when Huffman coding is employed to allow for spending one additional bit on quantization the distortion can be lowered a factor four!

1–34 (a) It is straightforward to calculate the exact quantization noise, however slightly tiring.

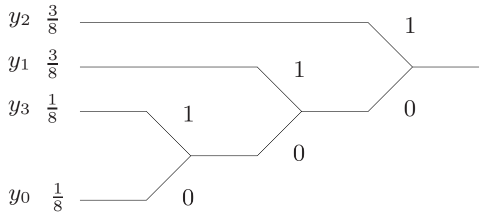
$$\begin{aligned} D_g &= \int_{-V}^{V} (x - Q(x))^2 f_X(x)dx = \{\text{symmetry}\} = 2\int_0^V (x - Q(x))^2 f_X(x)dx \\ &= \left\{f_X(x) = \frac{1}{V} - \frac{1}{V^2}x\right\} = 2\int_0^{V/2} (x - \frac{V}{4})^2(\frac{1}{V} - \frac{1}{V^2}x)dx + \\ &\quad 2\int_{V/2}^{V} (x - \frac{3V}{4})^2(\frac{1}{V} - \frac{1}{V^2}x)dx = \ldots = \frac{V^2}{48} + \frac{V^2}{192} = \frac{V^2}{48} \end{aligned}$$

The approximation of the granular quantization noise yields

$$D_g \approx \frac{\Delta^2}{12} = \frac{V^2}{48}$$

The approximation gives the exact value in this case. This is due to the fact that the quantization error $\tilde{X} = (X - Q(X))$ is uniformly distributed in $[-\frac{V}{4}, \frac{V}{4}]$. An alternative solution could use this approach.

(b) The probability mass function of Y_n is $\Pr(Y_n = y_0) = \Pr(Y_n = y_3) = \frac{1}{8}$ and $\Pr(Y_n = y_1) = \Pr(Y_n = y_2) = \frac{3}{8}$. A Huffman code for this random variable is shown below.



That is,

Symbol	Codeword
y_0	000
y_1	01
y_2	1
y_3	001

(c) A typical sequence of length 8 is $y_1^8 = y_0 y_3 y_1 y_1 y_1 y_2 y_2 y_2$. The probability of the sequence $\Pr(y_1^8) = 2^{-8H(Y_n)}$, where $H(Y_n)$ is the entropy of Y_n.

1–35 (a) The solution is straightforward.

$$
\begin{aligned}
E[X_n^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} x^2 e^{-|x|}dx = \{\text{even}\} = \\
&\int_0^{\infty} x^2 e^{-x}dx = \{\text{tables or integration by parts}\} = 2 \\
E[(X_n - \hat{X}_n)^2] &= \int_{-\infty}^{\infty} (x - \hat{x}(x))^2 f_X(x)dx = \{\text{even}\} = \int_0^{\infty} (x - \hat{x}(x))^2 e^{-x}dx = \\
&\int_0^1 (x - 1/2)^2 e^{-x}dx + \int_1^{\infty} (x - 3/2)^2 e^{-x}dx = \ldots = \\
&\frac{1}{4}(5 - \frac{13}{e}) + \frac{5}{4e} = \frac{5}{4} - \frac{2}{e} \approx 0.5142
\end{aligned}
$$

This gives the SQNR$= \frac{E[X_n^2]}{E[(X_n - \hat{X}_n)^2]} \approx \frac{2}{0.5142} \approx 3.9 \approx 5.9$ dB.

(b) To find the entropy of I_n, the probability mass function of I_n must be found.

$$
\begin{aligned}
\Pr(I_n = 0) = \Pr(I_n = 3) &= \int_{-\infty}^{-1} f_X(x)dx = \int_1^{\infty} f_X(x)dx = \\
&\frac{1}{2}\int_0^{\infty} e^{-x}dx = \frac{1}{2e} = p_0 \approx 0.184 \\
\Pr(I_n = 1) = \Pr(I_n = 2) &= \int_{-1}^{0} f_X(x)dx = \int_0^1 f_X(x)dx = \\
&\frac{1}{2}\int_0^1 e^{-x}dx = \frac{1}{2} - \frac{1}{2e} = p_1 \approx 0.316 \\
\Rightarrow H(I_n) &= -2p_0 \log p_0 - 2p_1 \log p_1 \approx 1.94 \text{ bits}
\end{aligned}
$$

Since there is no loss in information between I_n and $\hat{I}_n$, the other entropies follow easily.

$$
\begin{aligned}
H(\hat{X}_n) &= H(I_n) \\
H(\hat{X}_n | I_n) &= 0 \\
H(I_n | \hat{X}_n) &= 0 \\
H(\hat{X}_n | X_n) &= 0
\end{aligned}
$$

The differential entropy of X_n is by definition

$$\begin{aligned} h(X_n) &= -\int_{-\infty}^{\infty} f_X(x)\log f_X(x)dx = \{f_X(x) \text{ even}\} = -\int_0^{\infty} e^{-x}\log(\frac{1}{2}e^{-x})dx = \\ & -\int_0^{\infty} e^{-x}(\log e^{-x} - \log 2)dx = \int_0^{\infty} xe^{-x}\log e dx + \int_0^{\infty} e^{-x}dx = \ldots = \\ & \frac{\ln 2 + 1}{\ln 2} \approx 2.44 \text{ bits} \end{aligned}$$

(c) A Huffman code for I_n can be designed with standard techniques now that the pmf of I_n is known. One is given in the table below.

I_n	Codeword
0	11
1	10
2	01
3	00

The rate of the code is 2 bits per quantizer output, which means no compression. If the block-length is increased the rate can be improved.

(d) A code for blocks of two quantizer outputs is to be designed. First, the probability distribution of all possible combinations of (I_{n-1}, I_n) has to be computed. Then, the Huffman code can be designed using standard techniques. The probabilities and one Huffman code is given in the table below.

(I_{n-1}, I_n)	Probability	Codeword
00	p_0p_0	111
01	p_0p_1	110
02	p_0p_1	0111
03	p_0p_0	0110
10	p_1p_0	1011
11	p_1p_1	1010
12	p_1p_1	1001
13	p_1p_0	1000
20	p_1p_0	0011
21	p_1p_1	0010
22	p_1p_1	0001
23	p_1p_0	0000
30	p_0p_0	01011
31	p_0p_1	01010
32	p_0p_1	01001
33	p_0p_0	01000

The rate of the code is $\frac{1}{2}(6p_1p_1 + 8p_1p_1 + 32p_0p1 + 20p_0p_0) \approx 1.97$ bits per quantization output, which is a small improvement.

(e) The rate of the code would approach $H(I_n) = 1.95$ bits as the block-length would increase.

1–36 We have that $\{X_n\}$ and $\{W_n\}$ are stationary, $E[W_n] = 0$ and $E[W_n^2] = 1$. Since $\{W_n\}$ is Gaussian $\{X_n\}$ will be Gaussian (produced by linear operations on a Gaussian process). Let $\mu = E[X_n]$ and $\sigma^2 = E[(X_n - \mu)^2]$ (equal for all n), then

$$E[X_n] = a\,E[X_{n-1}] + E[W_n] \implies \mu(1-a) = 0 \implies \mu = 0$$
$$E[X_n^2] = a^2\,E[X_{n-1}^2] + E[W_n^2] \implies \sigma^2(1-a^2) = 1 \implies \sigma^2 = \frac{1}{1-a^2}$$

Also, since $\{X_n\}$ is stationary, $r(m) = E[X_n X_{n+m}]$ depends only on m, and $r(m)$ is symmetric in m. For $m > 0$, X_{m+n} and W_n are independent, and hence

$$r(m) = E[X_n X_{n+m}] = aE[X_{n-1}X_{n+m}] = a\,r(m+1) \implies r(m) = a^{-m}\sigma^2$$

Similarly $r(m) = a^m\sigma^2$ for $m < 0$, so $r(m) = a^{-|m|}\sigma^2$ for all m.

(a) Since X_n is zero-mean Gaussian with variance σ^2, the pdf of $X = X_n$ (for any n) is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{x^2}{2\sigma^2}\right)$$

and we hence get

$$\begin{aligned} h(X_n) &= -\int_{-\infty}^{\infty} f_X(x)\log f_X(x)dx = -\log e\int_{-\infty}^{\infty} f_X(x)\ln f_X(x)dx \\ &= -\log e\int_{-\infty}^{\infty} f_X(x)\left(\ln\frac{1}{\sqrt{2\pi\sigma^2}} - \frac{x^2}{2\sigma^2}\right)dx \\ &= -\log e\left(\ln\frac{1}{\sqrt{2\pi\sigma^2}} - \frac{1}{2}\right) = \frac{1}{2}\log(2\pi e\sigma^2) \end{aligned}$$

where "log" is the binary logarithm and "ln" is the natural, and with $\sigma^2 = (1-a^2)^{-1}$. Now we note that

$$h(X_n, X_{n-1}) = h(X_n|X_{n-1}) + h(X_{n-1}) = h(W_n) + h(X_{n-1})$$

by the chain rule, and since $h(X_n|X_{n-1}) = h(W_n)$ (the only "randomness" remaining in X_n when X_{n-1} is known is due to W_n). We thus get

$$h(X_n, X_{n-1}) = \frac{1}{2}\log(2\pi e) + \frac{1}{2}\log(2\pi e\sigma^2) = \log(2\pi e\sigma) = \log\frac{2\pi e}{\sqrt{1-a^2}}$$

(b) $X = X_n$ (any n) is zero-mean Gaussian with variance $\sigma^2 = (1-a^2)^{-1}$. The quantizer encoder is fixed, as in the problem description. The optimal value of b is hence given by the conditional mean

$$b = E[X|X>1] = \frac{1}{Q(1/\sigma)}\int_1^{\infty}\frac{x}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{x^2}{2\sigma^2}\right)dx = \frac{\sigma}{Q(1/\sigma)\sqrt{2\pi}}\exp\left(-\frac{1}{2\sigma^2}\right)$$

where

$$Q(x) = \int_x^{\infty}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt$$

is the Q-function.

(c) With $a = 0$, Y_n and Y_{n-1} are independent and X_n is zero-mean Gaussian with variance $\sigma^2 = 1$. The pmf for $Y = Y_m$, $m = n$ or $m = n-1$, is then

$$p(y) = \Pr(Y = y) = \begin{cases} Q(1), & y = -b \\ 1 - 2Q(1), & y = 0 \\ Q(1), & y = b \end{cases} \approx \begin{cases} 0.16, & y = -b \\ 0.68, & y = 0 \\ 0.16, & y = b \end{cases}$$

and since Y_n and Y_{n-1} are independent the joint probabilities are

y_n	y_{n-1}	$p(y_n, y_{n-1}) = p(y_n)p(y_{n-1})$
0	0	0.466
0	b	0.108
0	$-b$	0.108
b	0	0.108
$-b$	0	0.108
b	b	0.025
$-b$	b	0.025
b	$-b$	0.025
$-b$	$-b$	0.025

A Huffman code for these 9 different values, with codewords listed in decreasing order of probability, is (for example)

$$1, 001, 010, 011, 0000, 000100, 000101, 000110, 000111$$

with average length

$$L \approx 0.466 + 3 \cdot 3 \cdot 0.108 + 4 \cdot 0.108 + 4 \cdot 6 \cdot 0.025 \approx 2.47$$

(bits per source symbol). The joint entropy of Y_n and Y_{n-1} is

$$H(Y_n, Y_{n-1}) \approx -0.466 \log 0.466 - 4 \cdot 0.108 \log 0.108 - 4 \cdot 0.025 \log 0.025 \approx 2.43$$

so the average length is about 0.04 bits away from its minimum possible value.

---

2 Modulation and Detection

2–1 We need two orthonormal basis functions, and note that no two signals out of the four alternatives are orthogonal. All four signals have equal energy

$$E = \int_0^T s_i^2(t)dt = \frac{1}{3}C^2T$$

To find basis functions we follow the Gram-Schmidt procedure:

- Chose $\psi_1(t) = s_0(t)/\sqrt{E}$.
- Find $\psi_2(t) \perp \psi_1(t)$ so that $s_1(t)$ can be written as

$$s_1(t) = s_{11}\,\psi_1(t) + s_{12}\,\psi_2(t)$$

where

$$s_{11} = \int_0^T s_1(t)\psi_1(t)dt = \ldots = \frac{1}{2}\sqrt{E}.$$

Now $\phi_2(t) = s_1(t) - s_{11}\psi_1(t)$ is orthogonal to $\psi_1(t)$ but does not have unit energy (i.e., ϕ_2 and ψ_1 are orthogonal but not orthonormal). To normalize $\phi_2(t)$ we need its energy

$$E_{\phi_2} = \int_0^T \phi_2^2(t)dt = \ldots = \frac{1}{4}C^2T.$$

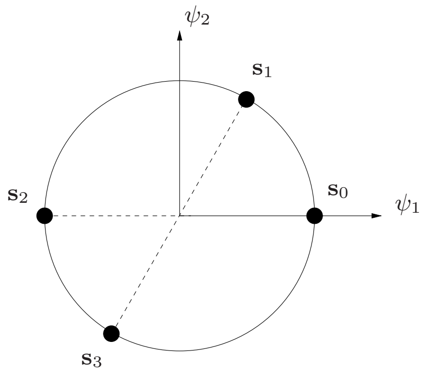
That is

$$\psi_2(t) = \sqrt{\frac{4}{C^2T}}\phi_2(t)$$

is orthonormal to $\psi_1(t)$. Also

$$s_{12} = \int_0^T s_1(t)\psi_2(t)dt = \frac{\sqrt{3}}{2}\sqrt{E}$$

Noting that $s_0(t) = -s_2(t)$ and $s_1(t) = -s_3(t)$ we finally get a signal space representation according to



with

$$\mathbf{s}_0 = -\mathbf{s}_2 = \sqrt{E}\,(1,0)$$
$$\mathbf{s}_1 = -\mathbf{s}_3 = \sqrt{E}\,(\frac{1}{2},\frac{\sqrt{3}}{2})$$

2–2 Since the signals are equiprobable, the optimal decision regions are given by the ML criterion. Denoting the received vector $\mathbf{r} = (r_1, r_2)$ we get

(a) $(r_1-2)^2 + (r_2-1)^2 \underset{2}{\overset{1}{\lessgtr}} (r_1+2)^2 + (r_2+1)^2$

(b) $|r_1-2| + |r_2-1| \underset{2}{\overset{1}{\lessgtr}} |r_1+2| + |r_2+1|$

2–3 (a) We get

$$f(r|s_0) = \frac{1}{\sqrt{2\pi}}\,\exp\big(-(r+1)^2/2\big)$$
$$f(r|s_1) = \frac{1}{\sqrt{2\pi}}\,\exp\big(-(r-1)^2/2\big)$$

The optimal decision rule is the MAP rule. That is

$$f(r|s_1)p_1 \underset{0}{\overset{1}{\gtrless}} f(r|s_0)p_0$$

which gives

$$r \underset{0}{\overset{1}{\gtrless}} \Delta$$

where $\Delta = 2^{-1}\,\ln 3$.

(b) In this case we get

$$f(r|s_0) = \frac{1}{2}\,\exp(-|r+1|)$$
$$f(r|s_1) = \frac{1}{2}\,\exp(-|r-1|)$$

The optimal decision rule is the same as in (a).

2–4 Optimum decision rule for S based on r. Since S is equiprobable the optimum decision rule is the ML criterion. That is $\hat{S} = \arg\min_{s\in\{\pm 5\}} f(r|s)$

(a) With $a = 1$ the conditional pdf is

$$f(r|s) = \frac{1}{\sqrt{2\pi}} \exp\big(-\frac{1}{2}(r-s)^2\big)$$

and the ML decision hence is

$$r \underset{-5}{\overset{+5}{\gtrless}} 0 \text{ or } \hat{S} = 5\,\mathrm{sgn}(r)$$

where $\mathrm{sgn}(x) = +1$ if $x > 0$ and $\mathrm{sgn}(x) = -1$ otherwise. The probability of error is

$$P_e = \frac{1}{2}\Pr(r > 0|S = +5) + \frac{1}{2}\Pr(r > 0|S = -5) = Q\left(\frac{5}{\sqrt{E[w^2]}}\right) = Q(5) \approx 2.9 \cdot 10^{-7}$$

(b) Conditioned that $S = s$, the received signal r is Gaussian with mean

$$E[a\,s + w] = s$$

and variance

$$\mathrm{Var}[a\,s + w] = 0.2\,s^2 + 1\ .$$

That is

$$f(r|S = s) = \frac{1}{\sqrt{2\pi(0.2\,s^2+1)}} \exp\left(-\frac{1}{2(0.2\,s^2+1)}(r-s)^2\right),$$

but since $s^2 = 25$ for both $s = +5$ and $s = -5$, the optimum decision is

$$f(r|S = +5) \underset{-5}{\overset{+5}{\gtrless}} f(r|S = -5) \iff r \underset{-5}{\overset{+5}{\gtrless}} 0\ .$$

That is, the same as in (a). The probability of error is

$$P_e = \Pr(r < 0|S = +5) = Q(5/\sqrt{2}) \approx 2.1 \cdot 10^{-7}$$

2–5 The optimal detector is the MAP detector, which chooses s_m such that $f(r|s_m)p_m$ is maximized. This results in the decision rule

$$p_0 \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{r^2}{2\sigma_0^2}\right) \underset{s_0}{\overset{s_1}{\lessgtr}} p_1 \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(r-\sqrt{E})^2}{2\sigma_1^2}\right)$$

Note that $\sigma_1^2 = 2\sigma_0^2$ and $p_0 = p_1 = 1/2$ (since $p_0 = p_1$, the MAP and ML detectors are identical). Taking the logarithm on both sides and reformulating the expression, the decision rule becomes

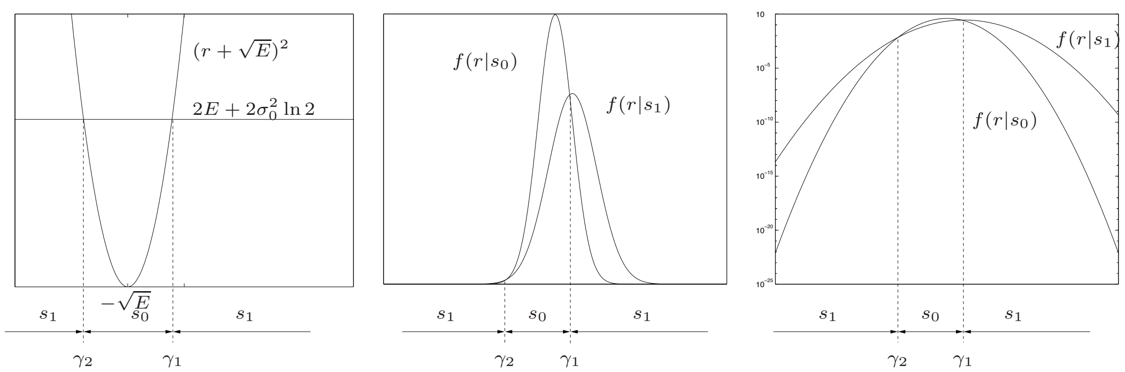
$$r^2 + 2\sqrt{E}\,r - E - 2\sigma_0^2 \ln 2 \underset{s_1}{\overset{s_0}{\lessgtr}} 0$$

$$(r + \sqrt{E})^2 \underset{s_1}{\overset{s_0}{\lessgtr}} 2E + 2\sigma_0^2 \ln 2$$

The two roots are

$$\gamma_{1,2} = \left[-1 \pm \sqrt{2 + 2(\sigma_0^2/E)\ln 2}\right]\sqrt{E}\ ,$$

and, hence, the receiver decides s_0 if $\gamma_2 < r < \gamma_1$ and s_1 otherwise. Below, $(r+\sqrt{E})^2$ and the probability density functions ($f(r|s_0)$ and $f(r|s_1)$) for the two cases are plotted together with the decision regions (middle: linear scale, right: logarithmic scale).



2–6 (a) We have

$$p_1 = 1-p,\ \ p_2 = \frac{1}{8}+\frac{p}{8},\ \ p_3 = \frac{7}{8}-\frac{p}{8},\ \ p_4 = p$$

(b) The optimal strategy is

– Strategy 4 for $0 \le p \le 1/7$
– Strategy 2 for $1/7 < p \le 7/9$
– Strategy 1 for $7/9 < p \le 1$

(c) Strategy 2

(d) Strategy 2

2–7 (a) Use the MAP rule (since the three alternatives, 900, 1000, and 1100, have different a priori probabilities), i.e.,

$$\max_{i\in\{900,1000,1100\}} p(i)f(h|i)$$

where h is the estimated height. The pdf for the estimation error ε is given by

$$f(\varepsilon) = \begin{cases} \frac{\varepsilon+100}{100\cdot 150} & -100 \le \varepsilon < 0 \\ \frac{200-\varepsilon}{200\cdot 150} & 0 \le \varepsilon \le 200 \\ 0 & \text{otherwise} \end{cases}$$

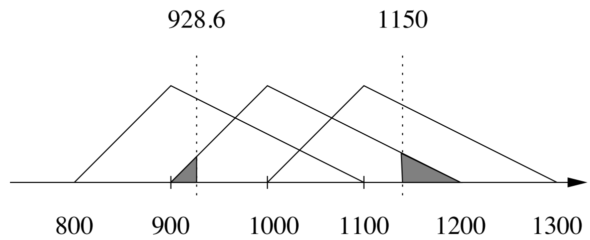
Solving for the decision boundaries yields

$$0.2\,\frac{200-(h_1-900)}{200\cdot 150} = 0.6\,\frac{(h_1-1000)+100}{100\cdot 150} \Rightarrow h_1 = 928.57$$
$$0.6\,\frac{200-(h_2-1000)}{200\cdot 150} = 0.2\,\frac{200-(h_2-1100)}{200\cdot 150} \Rightarrow h_2 = 1150$$

(b) The error probabilities conditioned on the three different types of peaks are given by

$$p_{\mathrm{e}|900} = \int_{h_1}^{1100} \frac{200-(h-900)}{200\cdot 150}\,dh = \int_{28.57}^{200} \frac{200-\varepsilon}{200\cdot 150}\,d\varepsilon = 0.4898$$
$$p_{\mathrm{e}|1000} = \int_{900}^{h_1} \frac{h-1000+100}{100\cdot 150}\,dh + \int_{h_2}^{1200} \frac{200-(h-1000)}{200\cdot 150}\,dh = 0.0689$$
$$p_{\mathrm{e}|1100} = \int_{1000}^{1100} \frac{h-1100+100}{100\cdot 150}\,dh + \int_{1100}^{h_2} \frac{200-(h-1100)}{200\cdot 150}\,dh = 0.625$$

This is illustrated below, where the shadowed area denotes $p_{\mathrm{e}|1000}$.



The total error probability is obtained as $p_\mathrm{e} = p_{\mathrm{e}|900}p_{900} + p_{\mathrm{e}|1000}p_{1000} + p_{\mathrm{e}|1100}p_{1100} = 0.26$. Thus, B had better bring some extra candy to keep A in a good mood!

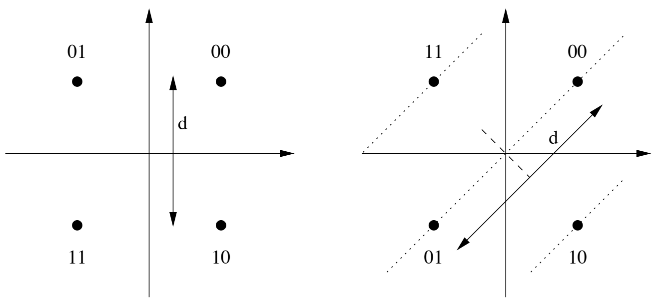
2–8 (a) The first subproblem is a standard formulation for QPSK systems. Gray coding, illustrated in the leftmost figure below, should be used for the symbols in order to minimize the error probability. The QPSK system can be viewed as two independent BPSK systems, one operating on the I channel (the second bit) and one on the Q channel (the first bit). The received vector can be anywhere in the two-dimensional plane. For the first bit, the two decision boundaries are separated by the horizontal axis. Similarly, for the second bit, the vertical axis separates the two regions. The bit error probability is given by

$$P_\mathrm{b} = P_{\mathrm{b},1} = P_{\mathrm{b},2} = Q\left(\frac{d/2}{\sigma}\right) = \left\{d = \sqrt{4E_\mathrm{b}}\right\} = Q\left(\sqrt{\frac{E_\mathrm{b}}{\sigma^2}}\right)$$

(b) In the second case, the noise components are fully correlated. Hence, the received vector is restricted to lie along one of the three dotted lines in the rightmost figure below. Note that for two of the symbols, errors can *never* occur! The other two symbol alternatives can be mixed up due to noise, and in order to minimize the bit error probability only one bit should differ between these two alternatives. For the first bit, it suffices to decide whether the received signal lies along the middle diagonal or one of the outer ones. For the second bit, the decision boundary (the dashed line in the figure) lies between the two symbols along the middle diagonal. Equivalently, the system can be viewed as a BPSK system operating in the same direction as the noise (the second bit) and one noise free tri-level PAM system for the first bit. Be careful when calculating the noise variance along the diagonal line. The noise contributions along the I and Q axes are identical, n, with variance σ^2. Hence, the noise contribution along the diagonal line is $n\sqrt{2}$ with variance $2\sigma^2$. This arrangement results in the bit error probabilities

$$\begin{aligned}
P_{\mathrm{b},1} &= 0 \\
P_{\mathrm{b},2} &= \frac{1}{2} Q\left(\frac{d/2}{\sqrt{4\sigma^2}}\right) = \left\{d = \sqrt{8E_\mathrm{b}}\right\} = \frac{1}{2} Q\left(\sqrt{\frac{E_\mathrm{b}}{\sigma^2}}\right) \\
P_\mathrm{b} &= \frac{1}{2}\left(P_{\mathrm{b},1} + P_{\mathrm{b},2}\right) \quad ,
\end{aligned}$$

where the factor $1/2$ in the expression for $P_{\mathrm{b},2}$ is due to the fact that the middle diagonal only occurs half the time (on average) and there cannot be an error in the second bit if any of the two outer diagonals occur. With this (strange) channel, a better idea would be to use a multilevel PAM scheme in the "noise-free" direction, which results in an error-free system!



Uncorrelated   Correlated

2–9 The ML decision rule, $\max f(y_1, y_2|x)$ is optimal since the two transmitted symbols are equally likely. Since the two noise components are independent,

$$\begin{aligned}
f(y_1, y_2|x = -1) &= f(y_1|x = -1) f(y_2|x = -1) = \frac{1}{4} e^{-(|y_1+1|+|y_1+1|)} \\
f(y_1, y_2|x = +1) &= f(y_1|x = +1) f(y_2|x = +1) = \frac{1}{4} e^{-(|y_1-1|+|y_1-1|)} \quad .
\end{aligned}$$

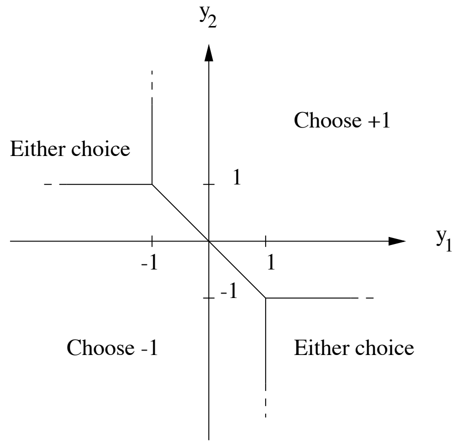
Assuming $x = -1$ is transmitted, the ML criterion results in

Choose -1	$f(y_1, y_2\|x = -1) > (y_1, y_2\|x = +1)$
Either choice	$f(y_1, y_2\|x = -1) = (y_1, y_2\|x = +1)$
Choose $+1$	$f(y_1, y_2\|x = -1) < (y_1, y_2\|x = +1)$

which reduces to

Choose -1	$\|y_1 + 1\| + \|y_2 + 1\| < \|y_1 - 1\| + \|y_2 - 1\|$
Either choice	$\|y_1 + 1\| + \|y_2 + 1\| = \|y_1 - 1\| + \|y_2 - 1\|$
Choose $+1$	$\|y_1 + 1\| + \|y_2 + 1\| > \|y_1 - 1\| + \|y_2 - 1\|$

After some simple manipulations, the above relations can be plotted as shown below. Similar calculations are also done for $x = +1$.



2–10 It is easily recognized that the two signals s_1 and s_2 are orthogonal, and that they can be represented in terms of the orthonormal basis functions $\phi_1(t) = 1/(A\sqrt{T})s_1(t)$ and $\phi_2(t) = 1/(A\sqrt{T})s_2(t)$.

(a) Letting $r_1 = \int_0^T R(t)\phi_1(t)dt$ and $r_2 = \int_0^T R(t)\phi_2(t)dt$, we know that the detector that minimizes the average error probability is defined by the MAP rule: "Choose s_1 if $f(r_1, r_2|s_1)P(s_1) > f(r_1, r_2|s_2)P(s_2)$ otherwise choose s_2." It is straightforward to check that this rule is equivalent to choosing s_1 if

$$r_1 - r_2 > -\Delta \quad \Delta = \frac{N_0/2}{A\sqrt{T}} \ln((1-p)/p) = \sqrt{N_0/2}10^{-0.4} \ln(0.85/0.15) \;\text{ since } A\sqrt{T} = \sqrt{N_0/2}10^{0.4}$$

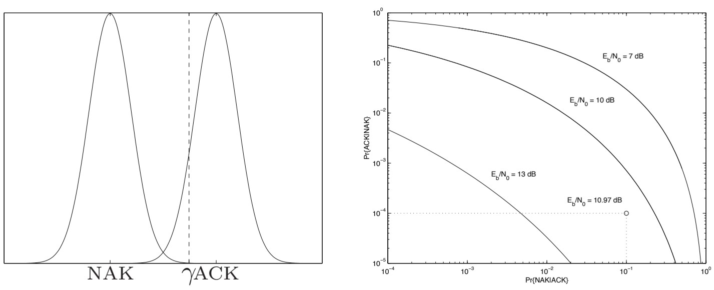
When s_1 is transmitted, r_1 is Gaussian$(A\sqrt{T}, N_0/2)$ and r_2 is Gaussian$(0, N_0/2)$, and vice versa. Thus

$$\begin{aligned}
P(\mathrm{e}) &= (1-p)\Pr(n > d_{min}/2 + \Delta/\sqrt{2}) + p\Pr(n > d_{min}/2 - \Delta/\sqrt{2}) \\
&= (1-p)Q\left(\frac{A\sqrt{T}+\Delta}{\sqrt{N_0}}\right) + pQ\left(\frac{A\sqrt{T}-\Delta}{\sqrt{N_0}}\right) \\
&= 0.85Q\left(\frac{1}{\sqrt{2}}(10^{0.4} + 10^{-0.4}\ln(\frac{0.85}{0.15})\right) + 0.15Q\left(\frac{1}{\sqrt{2}}(10^{0.4} - 10^{-0.4}\ln(\frac{0.85}{0.15})\right) \\
&\approx 0.85Q(2.26) + 0.15Q(1.29) \approx 0.024
\end{aligned}$$

(b) We get

$$P(\mathrm{e}) = (1-p)Q\left(\frac{A\sqrt{T}}{\sqrt{N_0}}\right) + pQ\left(\frac{A\sqrt{T}}{\sqrt{N_0}}\right) = Q\left(\frac{10^{0.4}}{\sqrt{2}}\right) \approx Q(1.78) \approx 0.036$$

2–11 According to the text, $\Pr\{\text{ACK}|\text{NAK}\} = 10^{-4}$ and $\Pr\{\text{NAK}|\text{ACK}\} = 10^{-1}$. The received signal has a Gaussian distribution around either 0 or $\sqrt{E_b}$, as illustrated to the left in the figure below.



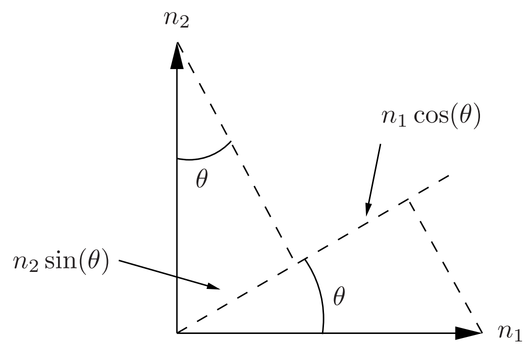
For the OOK system with threshold γ,

$$\Pr\{\text{NAK}|\text{ACK}\} = \Pr\{\sqrt{E_b} + n < \gamma\} = Q\left(\frac{\sqrt{E_b} - \gamma}{\sqrt{N_0/2}}\right) = 10^{-1}$$

$$\Pr\{\text{ACK}|\text{NAK}\} = \Pr\{n > \gamma\} = Q\left(\frac{\gamma}{\sqrt{N_0/2}}\right) = 10^{-4}$$

Solving the second equation yields $\gamma \geq 3.72\sqrt{N_0/2}$, which is the answer to the first question. Inserting this into the first equation gives $E_b/N_0 = 10.97$ dB as the minimum value. With this value, the threshold can also be expressed as $\gamma = 0.7437\,E_b$. The type of curve plotted to the right in the figure is usually denoted ROC (Receiver Operating Characteristics) and can be used to study the trade-off between missing an ACK and making an incorrect decision on a NAK.

2–12 (a) An ML detector that neglects the correlation of the noise decodes the received vector according to $\hat{\mathbf{s}} = \arg\min_{\mathbf{s}_i \in \{\mathbf{s}_1, \mathbf{s}_2\}} \|\mathbf{r} - \mathbf{s}_i\|$, i.e. the symbol closest to $\mathbf{r}$ is chosen. This means that the two decision regions are separated by a line through origon that is perpendicular to the line connecting $\mathbf{s}_1$ with $\mathbf{s}_2$. To obtain $P_{\text{e}1}$ we need to find the noise component along the line connecting the two constellation points. From the figure below it is seen that this component is $n' = n_1\cos(\theta) + n_2\sin(\theta)$.



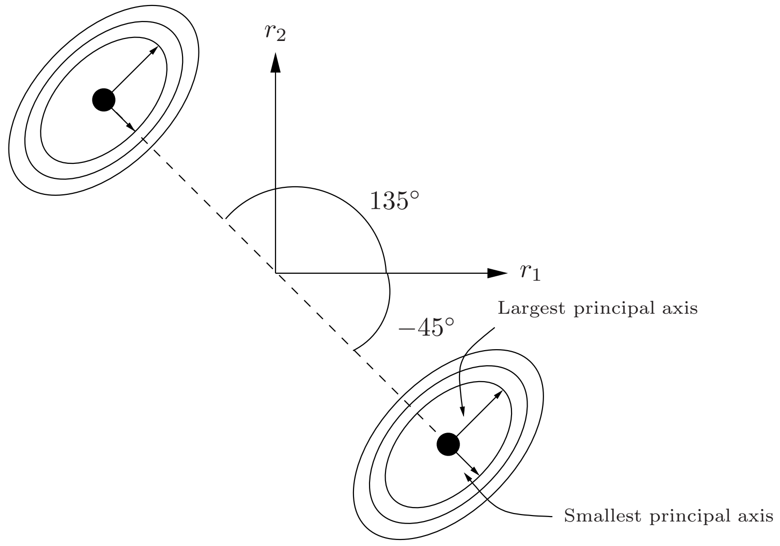
The mean and the variance of n' is then found as

$$\begin{aligned}
\text{E}[n'] &= \text{E}[n_1]\cos(\theta) + \text{E}[n_2]\sin(\theta) = 0 \\
\sigma_{n'}^2 = \text{E}[n'^2] &= \text{E}[n_1^2]\cos^2(\theta) + 2\text{E}[n_1 n_2]\cos(\theta)\sin(\theta) + \text{E}[n_2^2]\sin^2(\theta) \\
&= 0.1\cos^2(\theta) + 0.1\cos(\theta)\sin(\theta) + 0.1\sin^2(\theta) \\
&= 0.1 + 0.05\sin(2\theta).
\end{aligned}$$

Due to the symmetri of the problem and since n' is Gaussian we can find the symbol error probability as

$$\begin{aligned}
P_{\text{e}1} &= \Pr[e|\mathbf{s}_1]\Pr[\mathbf{s}_1] + \Pr[e|\mathbf{s}_2]\Pr[\mathbf{s}_2] = \Pr[e|\mathbf{s}_1] = \Pr[n' > 1] \\
&= \Pr[n'/\sigma_{n'} > 1/\sigma_{n'}] = Q\left(1/\sigma_{n'}\right) = Q\left(1/\sqrt{0.1 + 0.05\sin(2\theta)}\right).
\end{aligned}$$

Since $Q(x)$ is a decreasing function, we see that the probability of a symbol error is minimized when $\sin(2\theta) = -1$. Hence, the optimum values of θ are $-45°$ and $135°$, respectively. By studying the contour curves of $p(\mathbf{r}|\mathbf{s})$, i.e. the probability density function (pdf) of the received vector conditioned on the constellation point (which is the same as the noise pdf centered around the respective constellation point), one can easily explain this result. In the figure below it is seen that at the optimum angles, the line connecting the two constellation points is aligned with the shortest axes of the contour ellipses, i.e the noise component affecting the detector is minimized.



(b) The optimal detector in this case is an ML detector where the correlation is taken into account. Let

$$p(\mathbf{r}|\mathbf{s}_i) = \frac{\exp(-0.5(\mathbf{r}-\mathbf{s}_i)^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{r}-\mathbf{s}_i))}{\pi \det(\mathbf{R})}$$

denote the PDF of $\mathbf{r}$ conditioned on the transmitted symbol. Maximum likelihood detection amounts to maximizing this function, i.e.

$$\hat{\mathbf{s}} = \arg \max_{\mathbf{s}_i \in \{\mathbf{s}_1, \mathbf{s}_2\}} p(\mathbf{r}|\mathbf{s}_i) = \arg \min_{\mathbf{s}_i \in \{\mathbf{s}_1, \mathbf{s}_2\}} (\mathbf{r}-\mathbf{s}_i)^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{r}-\mathbf{s}_i) = \arg \min_{\mathbf{s}_i \in \{\mathbf{s}_1, \mathbf{s}_2\}} \|\mathbf{r}-\mathbf{s}_i\|^2_{\mathbf{R}^{-1}} ,$$

where the distance norm, $\|\mathbf{x}\|_{\mathbf{R}^{-1}} \triangleq \sqrt{\mathbf{x}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{x}}$, is now weighted using the covariance matrix

$$\mathbf{R} = \begin{bmatrix} \mathrm{E}[n_1^2] & \mathrm{E}[n_1 n_2] \\ \mathrm{E}[n_2 n_1] & \mathrm{E}[n_2^2] \end{bmatrix} = \begin{bmatrix} 0.1 & 0.05 \\ 0.05 & 0.1 \end{bmatrix} .$$

Using the above expression for the distance metric, the conditional probability of error is

$$\begin{aligned} \Pr[e|\mathbf{s}_1] &= \Pr[\|\mathbf{r}-\mathbf{s}_1\|^2_{\mathbf{R}^{-1}} > \|\mathbf{r}-\mathbf{s}_2\|^2_{\mathbf{R}^{-1}}] = \Pr[\|\mathbf{n}\|^2_{\mathbf{R}^{-1}} > \|\mathbf{n}-(\mathbf{s}_2-\mathbf{s}_1)\|^2_{\mathbf{R}^{-1}}] \\ &= \Pr[\mathbf{n}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{n} > \mathbf{n}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{n} - \mathbf{n}^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{s}_2-\mathbf{s}_1) - (\mathbf{s}_2-\mathbf{s}_1)^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{n} + (\mathbf{s}_2-\mathbf{s}_1)^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{s}_2-\mathbf{s}_1)] \\ &= \Pr[2(\mathbf{s}_2-\mathbf{s}_1)^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{n} > (\mathbf{s}_2-\mathbf{s}_1)^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{s}_2-\mathbf{s}_1)] . \end{aligned}$$

The mean and variance of $2(\mathbf{s}_2-\mathbf{s}_1)^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{n}$ are given by

$$\begin{aligned} \mathrm{E}[2(\mathbf{s}_2-\mathbf{s}_1)^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{n}] &= 0, \\ \mathrm{E}[(2(\mathbf{s}_2-\mathbf{s}_1)^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{n})^2] &= \mathrm{E}[4(\mathbf{s}_2-\mathbf{s}_1)^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{n}\mathbf{n}^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{s}_2-\mathbf{s}_1)] \\ &= 4(\mathbf{s}_2-\mathbf{s}_1)^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{s}_2-\mathbf{s}_1) = 4\|\mathbf{s}_2-\mathbf{s}_1\|^2_{\mathbf{R}^{-1}} , \end{aligned}$$

which means that

$$\Pr[e|\mathbf{s}_1] = Q\left(\frac{\|\mathbf{s}_2-\mathbf{s}_1\|_{\mathbf{R}^{-1}}}{2}\right) .$$

Because the problem is symmetric, the conditional probabilities are equal and hence after inserting the numerical values

$$P_{\mathrm{e}2} = Q\left(\frac{\|\mathbf{s}_2-\mathbf{s}_1\|_{\mathbf{R}^{-1}}}{2}\right) = Q\left(20\sqrt{\frac{0.1-0.05\sin(2\theta)}{3}}\right) .$$

To see when the performance of the suboptimal detector equals the performance of the optimal detector, we equate the corresponding arguments of the Q-function, giving the equation

$$1/\sqrt{0.1+0.05\sin(2\theta)} = 20\sqrt{\frac{0.1-0.05\sin(2\theta)}{3}} .$$

After some straightforward manipulations, this equation can be written as

$$\sin^2(2\theta) = 1\,.$$

Solving for θ finally yields $\theta = -135°, -45°, 45°, 135°$. The result is again explained in the figure, which illustrates the situation in the case of $\theta = 45°, 135°$. We see that the line connecting the two constellation point is parallel with one of the principal axes of the contour ellipse. This means that the projection of the received signal onto the other principal axis can be discarded since it only contains noise (and hence no signal part) which is statistically independent of the noise along the connecting line. As a result, we get a one-dimensional problem involving only one noise component. Thus, there is no correlation to take into account and both detectors are thus optimal.

2–13 (a) The optimal bit-detector for the 0th bit is found using the MAP decision rule

$$\begin{aligned}
\hat{b}_0 &= \arg\max_{b_0} p(b_0|\boldsymbol{r}) \\
&= \{\text{Bayes theorem}\} \\
&= \arg\max_{b_0} \frac{p(\boldsymbol{r}|b_0)p(b_0)}{p(\boldsymbol{r})} \\
&= \{p(b_0) = 1/2 \text{ and } p(\boldsymbol{r}) \text{ are constant with respect to (w.r.t.) } b_0\} \\
&= \arg\max_{b_0} p(\boldsymbol{r}|b_0) \\
&= \arg\max_{b_0} p(\boldsymbol{r}|b_0, b_1 = 0)p(b_1 = 0|b_0) + p(\boldsymbol{r}|b_0, b_1 = 1)p(b_1 = 1|b_0) \\
&= \{b_0 \text{ and } b_1 \text{ are independent so } p(b_1|b_0) = p(b_1) = 1/2, \text{ which is constant w.r.t. } b_0\} \\
&= \arg\max_{b_0} p(\boldsymbol{r}|b_0, b_1 = 0) + p(\boldsymbol{r}|b_0, b_1 = 1) \\
&= \arg\max_{b_0} p(\boldsymbol{r}|\boldsymbol{s}(b_0, 0)) + p(\boldsymbol{r}|\boldsymbol{s}(b_0, 1)) \\
&= \{\boldsymbol{r}|\boldsymbol{s}(b_0, b_1) \text{ is a Gaussian distributed vector with mean } \boldsymbol{s}(b_0, b_1) \text{ and covariance matrix } \sigma^2\boldsymbol{I}\} \\
&= \arg\max_{b_0} \frac{\exp(-\|\boldsymbol{r} - \boldsymbol{s}(b_0, 0)\|^2/(2\sigma^2))}{\sqrt{2\pi\sigma^2}^2} + \frac{\exp(-\|\boldsymbol{r} - \boldsymbol{s}(b_0, 1)\|^2/(2\sigma^2))}{\sqrt{2\pi\sigma^2}^2} \\
&= \{\sqrt{2\pi\sigma^2}^2 \text{ is constant w.r.t. } b_0\} \\
&= \arg\max_{b_0} \exp(-\|\boldsymbol{r} - \boldsymbol{s}(b_0, 0)\|^2/(2\sigma^2)) + \exp(-\|\boldsymbol{r} - \boldsymbol{s}(b_0, 1)\|^2/(2\sigma^2))\,.
\end{aligned}$$

The optimal bit detector for the 1st bit follows in a similar manner and is given by

$$\hat{b}_1 = \arg\max_{b_1} \exp(-\|\boldsymbol{r} - \boldsymbol{s}(0, b_1)\|^2/(2\sigma^2)) + \exp(-\|\boldsymbol{r} - \boldsymbol{s}(1, b_1)\|^2/(2\sigma^2))\,.$$

(b) Consider the detector derived in (a) for the 0th bit. According to the hint we may replace

$$\exp(-\|\boldsymbol{r} - \boldsymbol{s}(b_0, 0)\|^2/(2\sigma^2)) + \exp(-\|\boldsymbol{r} - \boldsymbol{s}(b_0, 1)\|^2/(2\sigma^2))$$

with

$$\max\{\exp(-\|\boldsymbol{r} - \boldsymbol{s}(b_0, 0)\|^2/(2\sigma^2)), \exp(-\|\boldsymbol{r} - \boldsymbol{s}(b_0, 1)\|^2/(2\sigma^2))\}$$

Hence, when $\sigma^2 \to 0$, we may study the equivalent detector

$$\begin{aligned}
\hat{b}_0 &= \arg\max_{b_0} \max\{\exp(-\|\boldsymbol{r} - \boldsymbol{s}(b_0, 0)\|^2/(2\sigma^2)), \exp(-\|\boldsymbol{r} - \boldsymbol{s}(b_0, 1)\|^2/(2\sigma^2))\} \\
&= \arg\max_{b_0} \max_{\substack{b_1 \\ b_0 \text{fixed}}} \exp(-\|\boldsymbol{r} - \boldsymbol{s}(b_0, b_1)\|^2/(2\sigma^2))
\end{aligned}$$

But the two max operators are equivalent to $\max_{(b_0,b_1)}$, which means that the optimal solution $(\tilde{b}_0, \tilde{b}_1)$ to

$$\max_{(b_0,b_1)} \exp(-\|\boldsymbol{r} - \boldsymbol{s}(b_0, b_1)\|^2/(2\sigma^2))$$

also maximizes the criterion function for $\hat{b}_0$. Hence, $\hat{b}_0 = \tilde{b}_0$. The above development can be repeated for the other bit-detector resulting in $\hat{b}_1 = \tilde{b}_1$. To conclude the proof it remains to be shown that $(\tilde{b}_0, \tilde{b}_1)$ is the output of the optimal symbol detection approach. Since (b_0, b_1) determines $\boldsymbol{s}(b_0, b_1)$, we note that we get the optimal symbol detector

$$\begin{aligned}\tilde{\boldsymbol{s}} &= \arg \max_{\substack{\boldsymbol{s} \\ \boldsymbol{s} \in \{\boldsymbol{s}(b_0, b_1)\}}} p(\boldsymbol{s}|\boldsymbol{r}) \\ &= \arg \max_{\substack{\boldsymbol{s} \\ \boldsymbol{s} \in \{\boldsymbol{s}(b_0, b_1)\}}} \exp(-\|\boldsymbol{r} - \boldsymbol{s}(b_0, b_1)\|^2/(2\sigma^2))\end{aligned}$$

with the optimal solution given by $\tilde{\boldsymbol{s}} = \boldsymbol{s}(\tilde{b}_0, \tilde{b}_1)$. It follows that $(\tilde{b}_0, \tilde{b}_1)$ is the output from the optimal symbol detection approach and hence the proof is complete.

2–14 (a) The Maximum A Posteriori (MAP) decision rule minimizes the symbol error probability. Since the input symbols are equiprobable, the MAP decision rule is identical to the Maximum Likelihood (ML) decision rule, which is

$$\hat{x}_n(y_n) = \arg \max_{i \in \{0,1\}} f_{y_n}(y|x_n = i)$$

where $f_{y_n}(y|x_n)$ is the conditional pdf of the decision variable y_n:

$$f_{y_n}(y|x_n) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma_0^2}}\, e^{-\frac{y^2}{2\sigma_0^2}} & x_n = 0 \\ \frac{1}{\sqrt{2\pi\sigma_1^2}}\, e^{-\frac{(y-1)^2}{2\sigma_1^2}} & x_n = 1 \end{cases}$$

Hence, the decision rule can be written as

$$\frac{f_{y_n}(y|x_n = 1)}{f_{y_n}(y|x_n = 0)} = \frac{\frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(y-1)^2}{2\sigma_1^2}}}{\frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{y^2}{2\sigma_0^2}}} \underset{0}{\overset{1}{\gtrless}} 1$$

Simplifying and taking the natural logarithm of the inequalities gives

$$-\frac{(y-1)^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_0^2} \underset{0}{\overset{1}{\gtrless}} \ln \frac{\sigma_1}{\sigma_0}$$

which can be written as

$$ay^2 + by - c \underset{0}{\overset{1}{\gtrless}} 0$$

where $a = \frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} > 0$, $b = \frac{2}{\sigma_1^2} > 0$ and $c = \frac{1}{\sigma_1^2} + 2 \ln \frac{\sigma_1}{\sigma_0} > 0$. The equation $ay^2 + by - c = 0$ has the two solutions

$$y = -\frac{b}{2a} \pm \frac{\sqrt{b^2 + 4ac}}{2a}$$

The decision boundary in this case can not be negative, so the optimal decision boundary is

$$\tilde{y} = \frac{1}{2a}(\sqrt{b^2 + 4ac} - b)$$

To summarize, the ML decision rule is

$$y_n \underset{0}{\overset{1}{\gtrless}} \tilde{y}$$

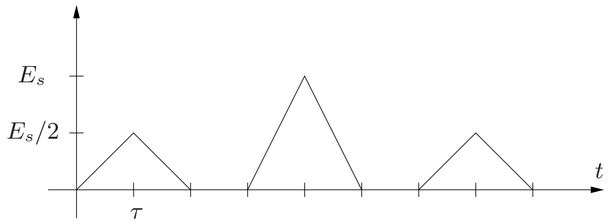
(b) The bit error probability is

$$\begin{aligned} P_b &= \Pr(\hat{x}_n = 1|x_n = 0) \Pr(x_n = 0) + \Pr(\hat{x}_n = 0|x_n = 1) \Pr(x_n = 1) \\ &= 0.5 \int_{-\infty}^{\tilde{y}} f_{y_n}(y|x_n = 1)\mathrm{d}y + 0.5 \int_{\tilde{y}}^{\infty} f_{y_n}(y|x_n = 0)\mathrm{d}y \\ &= 0.5 \left(Q\left(\frac{1 - \tilde{y}}{\sigma_1}\right) + Q\left(\frac{\tilde{y}}{\sigma_0}\right)\right) \end{aligned}$$

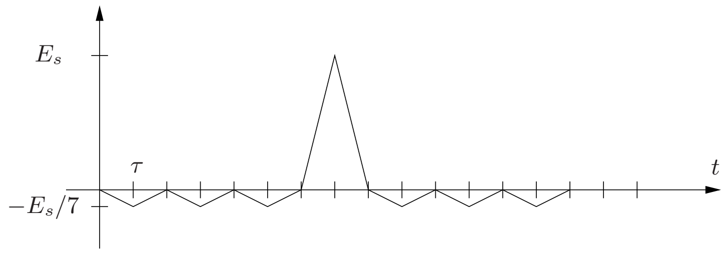
(c) The decision boundary $\tilde{y} \to 0$ as $\sigma_0^2 \to 0$, since the critical term $\frac{\sqrt{ac}}{a} = \sigma_0 \sqrt{2\ln(1/\sigma_0)} \to 0$ as $\sigma_0^2 \to 0$.
Furthermore, note that $\frac{\tilde{y}}{\sigma_0} \to \sqrt{2\ln 1/\sigma_0} \to \infty$ as $\sigma_0^2 \to 0$. Since $\lim_{x\to\infty} Q(x) = 0$,

$$\lim_{\sigma_0^2 \to 0} P_b = 0.5Q\left(\frac{1}{\sigma_1}\right). \tag{3.7}$$

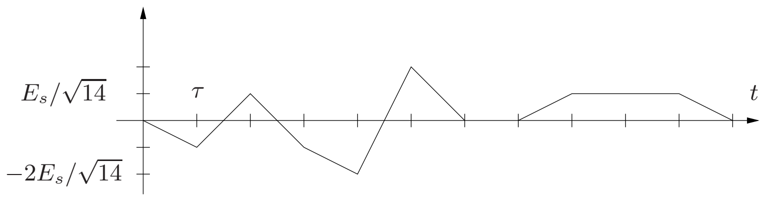
2–15 (a) The filter is $h_0(t) = s_0(4\tau - t)$, and the output of the filter when the input is the signal $s_0(t)$ is shown below.



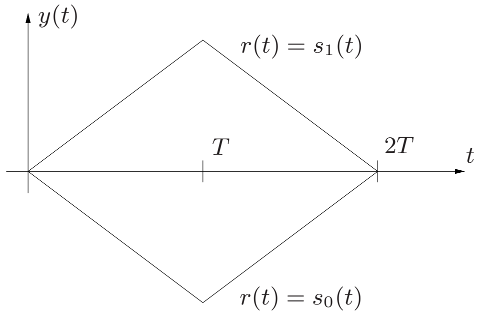
(b) The filter is $h_1(t) = s_1(7\tau - t)$, and the output of the filter when the input is the signal $s_1(t)$ is illustrated below.



(c) The output of $h_1(t)$ when the input is the signal $s_0(t)$ is shown below.



2–16 (a) The signal $y(t)$ is shown below.



(b) The error probability does not depend on which signal was transmitted. Assume $s_1(t)$ was transmitted. Then, for $|\Delta| \le T$, we see that

$$y = \sqrt{E}(1 - \frac{|\Delta|}{T})$$

in the absence of noise. Hence, with noise the symbol error probability is

$$P_e = Q\left((1 - \frac{|\Delta|}{T})\sqrt{\frac{2E}{N_0}}\right)$$

2–17 The received signal is $r(t) = s_i(t) + n(t)$, where $n(t)$ is AWGN with noise spectral density $N_0/2$. The signal fed to the detector is

$$z(T) = \left\{ \begin{array}{ll} -KABT & s_1(t) \text{ was sent} \\ 0 & s_2(t) \text{ was sent} \\ KABT & s_3(t) \text{ was sent} \end{array} \right\} + n \qquad n = KB\int_0^T n(t)dt$$

where n is Gaussian with mean value 0 and variance $K^2B^2TN_0/2$. The entity γ is chosen under the assumption $K = 1$. Nearest neighbor detection is optimal. This gives (with $K = 1$) the optimal γ according to $\gamma = ABT/2$. We get

$$\begin{aligned}
\Pr(\text{error}|s_1(t)) &= \Pr(-KABT + n > -\gamma) = \Pr\left(n > -\frac{ABT}{2} + KABT\right) \\
&= Q\left(\sqrt{\frac{2A^2T}{N_0}}\left(1 - \frac{1}{2K}\right)\right) \\
\Pr(\text{error}|s_2(t)) &= \Pr(|n| > \gamma) = 2\Pr\left(n > \frac{ABT}{2}\right) = 2Q\left(\sqrt{\frac{2A^2T}{N_0}}\frac{1}{2K}\right) \\
\Pr(\text{error}|s_3(t)) &= \Pr(\text{error}|s_1(t))
\end{aligned}$$

and, finally

$$\begin{aligned}
\Pr(\text{error}) &= \frac{1}{3}\Pr(\text{error}|s_1(t)) + \frac{1}{3}\Pr(\text{error}|s_3(t)) + \frac{1}{3}\Pr(\text{error}|s_3(t)) \\
&= \frac{2}{3}Q\left(\sqrt{\frac{2A^2T}{N_0}}\left(1 - \frac{1}{2K}\right)\right) + \frac{2}{3}Q\left(\sqrt{\frac{2A^2T}{N_0}}\frac{1}{2K}\right)
\end{aligned}$$

2–18 Let r be the sampled output from the matched filter. The decision variable r is Gaussian with variance $N_0/2$ and mean $\pm\sqrt{E}$. Decisions are based on comparing r with a threshold Δ. That is,

$$r \underset{\text{fire}}{\overset{\text{no fire}}{\gtrless}} \Delta$$

The miss probability is

$$\Pr(r > \Delta|\text{fire}) = Q\left(\frac{\sqrt{E} + \Delta}{\sqrt{N_0/2}}\right) = 10^{-7} \Rightarrow \Delta \approx 5.2\sqrt{N_0/2} - \sqrt{E}$$

The false alarm probability hence is

$$\Pr(r < \Delta|\text{no fire}) = Q\left(\frac{\sqrt{E} - \Delta}{\sqrt{N_0/2}}\right) \approx Q(2.8) \approx 3 \cdot 10^{-3}$$

2–19 We want to find the statistics of the decision variable after the integration.

First, let's find $\Psi(t)$, which is a unit-energy scaled $g_T(t)$. The energy of $g_T(t)$ is

$$\mathcal{E}_g = \int_0^T g_T^2(t)dt = \int_0^T A^2 dt = A^2T$$

which gives the basis function

$$\Psi(t) = \frac{1}{\sqrt{\mathcal{E}_g}} g_T(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

and the corrupted basis function is $\bar{\Psi}(t) = b\Psi(t)$. The decision variable out from the integrator is

$$\begin{aligned}\bar{r} &= \int_0^T (s_m(t)+n(t))\bar{\Psi}(t)dt = b\int_0^T (s_m(t)+n(t))\Psi(t)dt = \\ & \frac{b}{\sqrt{T}}\int_0^T s_m(t)g_T(t)dt + \frac{b}{\sqrt{T}}\int_0^T n(t)dt = \bar{s}_m + \bar{n}\end{aligned}$$

where

$$\bar{s}_m = \begin{cases} \frac{3}{2}bA\sqrt{T} & m=1 \\ \frac{1}{2}bA\sqrt{T} & m=2 \\ -\frac{1}{2}bA\sqrt{T} & m=3 \\ -\frac{3}{2}bA\sqrt{T} & m=4 \end{cases}$$

If the correct basis function would have been used, b would have been 1. $\bar{n}$ is the zero-mean Gaussian noise term, with variance

$$\sigma_{\bar{n}}^2 = E[\bar{n}^2] = E\left[\frac{b^2}{T}\int_0^T\int_0^T n(t)n(\tau)dtd\tau\right] = \frac{b^2}{T}\int_0^T\int_0^T \frac{N_0}{2}\delta(t-\tau)dtd\tau = \frac{b^2N_0}{2}$$

The detector uses minimum distance detection, designed for the correct $\Psi(t)$, so the decision regions for $\bar{r}$ are $[A\sqrt{T},\infty]$, $[0, A\sqrt{T})$, $[-A\sqrt{T}, 0)$ and $[-\infty, -A\sqrt{T})$. The symbol-error probability can now be computed as the probability that the noise brings the symbols into the wrong decision regions. Due to the complete symmetry of the problem, it's sufficient to analyze s_1 and s_2.

$$\begin{aligned}\Pr(\text{symbol-error}) &= 2\Big[\Pr(s_1 \text{ transmitted})\Pr(\bar{n} < -(\tfrac{3}{2}b-1)A\sqrt{T}) + \\ &\Pr(s_2 \text{ transmitted})\left(\Pr(\bar{n} \geq (1-\tfrac{b}{2})A\sqrt{T}) + \Pr(\bar{n} < -\tfrac{b}{2}A\sqrt{T})\right)\Big]\end{aligned}$$

Since $\bar{n}$ is Gaussian, this can be written in terms of Q-functions.

$$\Pr(\text{symbol-error}) = \frac{1}{2}\left[Q\left(\frac{(\frac{3}{2}b-1)A\sqrt{2T}}{b\sqrt{N_0}}\right) + Q\left(\frac{A\sqrt{2T}}{2\sqrt{N_0}}\right) + Q\left(\frac{(1-\frac{b}{2})A\sqrt{2T}}{b\sqrt{N_0}}\right)\right]$$

2–20

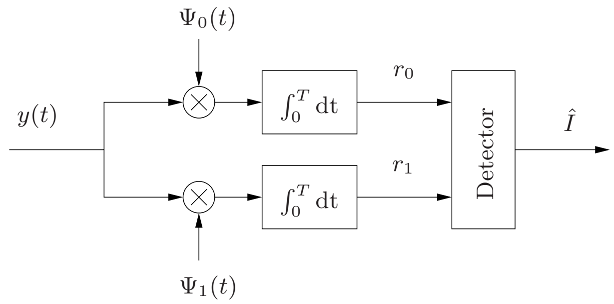
(a) First, find the basis waveforms. The signals $s_0(t)$ and $s_1(t)$ are already orthogonal, so the orthonormal basis waveforms $\Psi_0(t)$ and $\Psi_1(t)$ can be obtained by normalizing $s_0(t)$ and $s_1(t)$.

$$\begin{aligned}\Psi_0(t) &= \sqrt{\tfrac{2}{T}} \quad \text{for } 0 \leq t < \frac{T}{2} \\ \Psi_1(t) &= \sqrt{\tfrac{2}{T}} \quad \text{for } \frac{T}{2} \leq t < T\end{aligned}$$

Hence,

$$\begin{aligned}s_0(t) &= A\sqrt{\frac{T}{2}}\Psi_0(t) = B\Psi_0(t) \\ s_1(t) &= A\sqrt{\frac{T}{2}}\Psi_1(t) = B\Psi_1(t)\end{aligned}$$

The minimum-symbol-error receiver is shown below.



The signal alternatives can be written as vectors in the signal space,

$$\mathbf{s}_0 = \begin{bmatrix} B \\ 0 \end{bmatrix}$$
$$\mathbf{s}_1 = \begin{bmatrix} 0 \\ B \end{bmatrix}.$$

The decision variable vector, if $I = 0$, is

$$\mathbf{r} = \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} B + w_0 \\ w_1 \end{bmatrix}$$

If $I = 1$, the decision variable vector is

$$\mathbf{r} = \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} w_0 \\ B + w_1 \end{bmatrix}$$

where w_0 and w_1 are independent zero-mean Gaussian random variables with variance $N_0/2$. Since the symbol alternatives are not equally probable, the ML decision rule doesn't minimize the symbol error probability. Instead the general MAP criterion has to be used.

$$\hat{I} = \arg\max_I f_{\mathbf{r}}(\mathbf{r}|I)\Pr(I)$$

Since r_0 and r_1 are independent, the joint pdf can be split into the product of the marginal distributions:

$$f_{\mathbf{r}}(\mathbf{r}|I) = f_{r_0}(r_0|I) f_{r_1}(r_1|I)$$

The marginal distributions are given by

$$\begin{aligned}
f_{r_0}(r_0|0) &= \frac{1}{\sqrt{\pi N_0}} e^{-(r_0-B)^2/N_0} \\
f_{r_1}(r_1|0) &= \frac{1}{\sqrt{\pi N_0}} e^{-r_1^2/N_0} \\
f_{r_0}(r_0|1) &= \frac{1}{\sqrt{\pi N_0}} e^{-r_0^2/N_0} \\
f_{r_1}(r_1|1) &= \frac{1}{\sqrt{\pi N_0}} e^{-(r_1-B)^2/N_0}
\end{aligned}$$

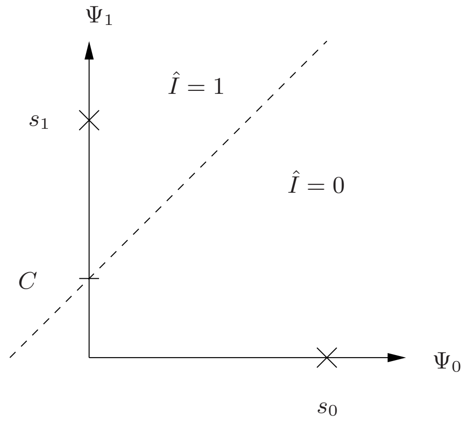
Hence, the joint distributions are given by

$$\begin{aligned}
f_{\mathbf{r}}(\mathbf{r}|0) &= \frac{1}{\pi N_0} e^{-\left((r_0-B)^2+r_1^2\right)/N_0} \\
f_{\mathbf{r}}(\mathbf{r}|1) &= \frac{1}{\pi N_0} e^{-\left((r_1-B)^2+r_0^2\right)/N_0}
\end{aligned}$$

The MAP decision rule can now be written as

$$\begin{aligned}
pe^{-\left((r_0-B)^2+r_1^2\right)/N_0} &\underset{1}{\overset{0}{\gtrless}} (1-p)e^{-\left((r_1-B)^2+r_0^2\right)/N_0} \\
\Rightarrow \ln p - \frac{(r_0-B)^2}{N_0} - \frac{r_1^2}{N_0} &\underset{1}{\overset{0}{\gtrless}} \ln(1-p) - \frac{(r_1-B)^2}{N_0} - \frac{r_0^2}{N_0} \\
\Rightarrow r_0 - r_1 &\underset{1}{\overset{0}{\gtrless}} \frac{N_0}{2B}\ln\frac{1-p}{p} = \frac{N_0}{A\sqrt{2T}}\ln\frac{1-p}{p} = -C
\end{aligned}$$

The decision regions are illustrated below.



(b)

$$y(t) = s_1(t) \Rightarrow \mathbf{r} = \begin{bmatrix} 0 \\ B \end{bmatrix}$$

The range of p's for which $y(t) = s_1(t) \Rightarrow \hat{I} = 0$ can be obtained from the decision rule derived in (a).

$$\begin{aligned} -B &> \frac{N_0}{2B} \ln \frac{1-p}{p} \\ \Rightarrow \frac{1-p}{p} &< e^{-2B^2/N_0} \\ \Rightarrow p &> \frac{1}{1+e^{-2B^2/N_0}} = \frac{1}{1+e^{-A^2T/N_0}} \end{aligned}$$

2–21 The received signal is

$$r(t) = s(t) + w(t)$$

where $s(t)$ is $s_1(t)$ or $s_2(t)$ and $w(t)$ is AWGN. The decision variable is

$$y_T = \int_0^\infty h(t) r(T-t) dt = s + w$$

where

$$s = \int_0^\infty e^{-t/T} s(T-t) dt = \begin{cases} 0 & \text{when } s(t) = s_1(t) \\ \sqrt{ET}(1-e^{-1}) & \text{when } s(t) = s_2(t) \end{cases}$$

and where

$$w = \int_0^\infty h(t) w(T-t) dt$$

is zero-mean Gaussian with variance

$$\frac{N_0}{2} \int_0^\infty e^{-2t/T} dt = \frac{TN_0}{4}$$

(a) We get

$$\begin{aligned} P_e &= \frac{1}{2}\Pr(y_T > b | s_1(t)) + \frac{1}{2}\Pr(y_T < b | s_2(t)) = \frac{1}{2}\Pr(w > b) + \frac{1}{2}\Pr\left(\sqrt{ET}(1-e^{-1}) + w < b\right) \\ &= \frac{1}{2} Q\left(\sqrt{\frac{4b^2}{TN_0}}\right) + \frac{1}{2} Q\left(\sqrt{\frac{4E}{N_0}}(1-e^{-1}) - \sqrt{\frac{4b^2}{TN_0}}\right) \end{aligned}$$

(b) P_e is minimized when $\Pr(y_T > b | s_1(t)) = \Pr(y_T < b | s_2(t))$, that is when

$$\sqrt{\frac{4b^2}{TN_0}} = \sqrt{\frac{4E}{N_0}}(1-e^{-1}) - \sqrt{\frac{4b^2}{TN_0}} \Rightarrow b = \sqrt{\frac{ET}{4}}\,(1-e^{-1})$$

2–22 The Walsh modulation in the problem is plain 64-ary orthogonal modulation with the 64 basis functions consisting of all normalized length 64 Walsh sequences. The bit error probability for orthogonal modulation is plotted in the textbook and for $E_\mathrm{b}/N_0 = 4$ dB, it is found that $P_{\mathrm{e,\ bit}} \approx 10^{-3}$. We get $P_{\mathrm{e,\ block}} \approx 2P_{\mathrm{e,\ bit}} \approx 2 \cdot 10^{-3}$. It is of course also possible to numerically compute the error probabilities, but it probably requires a computer.

2–23 Translation and rotation does not influence the probability of error, so the constellation in the figure is equivalent to QPSK with symbol energy $E = a^2/2$, and the exact probability of error is hence

$$P_e = 2Q\left(\sqrt{\frac{E}{N_0}}\right) - \left[Q\left(\sqrt{\frac{E}{N_0}}\right)\right]^2 = 2Q\left(\frac{a}{\sqrt{2N_0}}\right) - \left[Q\left(\frac{a}{\sqrt{2N_0}}\right)\right]^2$$

2–24 The decision rule "decide closest alternative" is optimal, since equiprobable symbols and an AWGN channel is assumed. After matched filtering the two noise components, n_1 and n_2, are independent and each have variance $\sigma^2 = N_0/2$. The error probabilities for bits b_1 and b_2, conditioning on $b_1 b_2 = 01$ transmitted, are denoted p_1 and p_2, respectively. Due to the symmetry in the problem, p_1 and p_2 are identical conditioning on any of the four symbols. The average bit error probability is equal to $p = (p_1 + p_2)/2$.

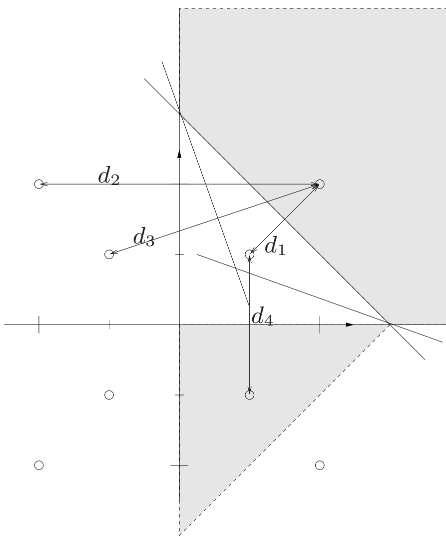
For the left constellation:

$$\begin{aligned}
p_1 &= \Pr\{n_1 < d/2, n_2 > d/2\} + \Pr\{n_1 > d/2, n_2 < d/2\} \\
&= \Pr\{n_1 < d/2\} \Pr\{n_2 > d/2\} + \Pr\{n_1 > d/2\} \Pr\{n_2 < d/2\} \\
p_2 &= \Pr\{n_1 > d/2\} \\
p &= \frac{1}{2}\left[2Q\left(\frac{d/2}{\sigma}\right)\left[1 - Q\left(\frac{d/2}{\sigma}\right)\right] + Q\left(\frac{d/2}{\sigma}\right)\right] = \frac{3}{2}Q\left(\frac{d/2}{\sigma}\right) - Q^2\left(\frac{d/2}{\sigma}\right)
\end{aligned}$$

For the right constellation:

$$\begin{aligned}
p_1 &= \Pr\{n_2 > d/2\} \\
p_2 &= \Pr\{n_1 > d/2\} \\
p &= \frac{1}{2}\left[Q\left(\frac{d/2}{\sigma}\right) + Q\left(\frac{d/2}{\sigma}\right)\right] = Q\left(\frac{d/2}{\sigma}\right)
\end{aligned}$$

2–25 It is natural to choose a such that the distance between the signals $\|(1,1)-(a,a)\| = d_1 = \sqrt{2}(1-a)$ is equal to the distance between $\|(a,a)-(a,-a)\| = d_4 = 2a$. Thus, $a = 1/(1+\sqrt{2})$. In the figure, the two types of decision regions are displayed. Note that the decision boundary between the signals $(1,1)$ and $(a,-a)$ (with distance d_3) does not effect the shape of the decision regions.



The union bound gives

$$(M-1)Q\left(\frac{d_1}{2\sigma_n}\right) = 7Q\left(\frac{1}{(1+\sqrt{2})\sigma_n}\right) \approx 0.0308 > 10^{-2}$$

Thus, we must make more accurate approximations. Condition on the signal $(1,1)$ being sent.

$$\Pr(\text{error}|(1,1)) \leq Q\left(\frac{d_1}{2\sigma_n}\right) + 2Q\left(\frac{d_2}{2\sigma_n}\right) = Q\left(\frac{1}{(1+\sqrt{2})\sigma_n}\right) + 2Q\left(\frac{1}{\sigma_n}\right)$$

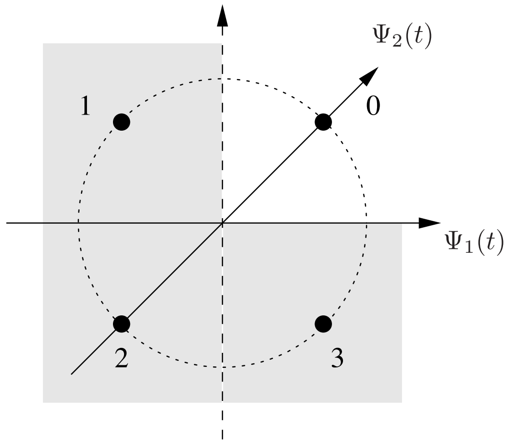
Condition on the signal (a,a) being sent.

$$\Pr(\text{error}|(a,a)) \leq Q\left(\frac{d_1}{2\sigma_n}\right) + 2Q\left(\frac{d_4}{2\sigma_n}\right) = 3Q\left(\frac{1}{(1+\sqrt{2})\sigma_n}\right)$$

An upper bound on the symbol error probability is then

$$P_e \leq \frac{1}{2}(\Pr(\text{error}|(1,1)) + \Pr(\text{error}|(a,a))) = 2Q\left(\frac{1}{(1+\sqrt{2})\sigma_n}\right) + Q\left(\frac{1}{\sigma_n}\right) \approx 0.0088 < 10^{-2}$$

2–26 The trick in the problem is to observe that the basis functions used in the receiver, illustrated below, are *non-orthogonal*.



Basis functions used in the receiver (solid) and in the transmitter (dashed). The grey area is the region where an incorrect decision is made, given that the upper right symbol is transmitted.

Hence, the corresponding noise components n_1 and n_2 in y_1 and y_2 are *not* independent, which must be accounted for when forming the decision rule. If the non-orthogonality is taken into account, there is no loss in performance compared to the case with orthogonal basis functions.

The statistical properties for n_1 and n_2 can be derived as

$$\begin{aligned}
E\{n_1 n_1\} =& E\left\{\int_0^T \int_0^T n(t)n(t')\Psi_1(t)\Psi_1(t')\right\} \\
=& E\left\{\int_0^T \int_0^T n(t)n(t')\sqrt{\frac{2}{T}}\cos(2\pi f t)\sqrt{\frac{2}{T}}\cos(2\pi f t)\right\} = \frac{N_0}{2} = \sigma_1^2 \\
E\{n_2 n_2\} =& E\left\{\int_0^T \int_0^T n(t)n(t')\Psi_2(t)\Psi_2(t')\right\} \\
=& E\left\{\int_0^T \int_0^T n(t)n(t')\sqrt{\frac{2}{T}}\cos(2\pi f t+\pi/4)\sqrt{\frac{2}{T}}\cos(2\pi f t+\pi/4)\right\} = \frac{N_0}{2} = \sigma_2^2 \\
E\{n_1 n_2\} =& E\left\{\int_0^T \int_0^T n(t)n(t')\Psi_1(t)\Psi_2(t')\right\} \\
=& E\left\{\int_0^T \int_0^T n(t)n(t')\sqrt{\frac{2}{T}}\cos(2\pi f t)\sqrt{\frac{2}{T}}\cos(2\pi f t+\pi/4)\right\} = \frac{N_0}{2}\cos(\pi/4) = \rho\sigma_1\sigma_2
\end{aligned}$$

Starting from the MAP criterion, the decision rule is easily obtained as

$$i = \arg\min_i (\mathbf{y} - \mathbf{s}_i)^T \mathbf{C}^{-1} (\mathbf{y} - \mathbf{s}_i) \ .$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \mathbf{s}_i = \begin{bmatrix} \int_0^T s_i(t)\Psi_1(t)\,dt \\ \int_0^T s_i(t)\Psi_2(t)\,dt \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \ .$$

Conditioning on the upper right signal alternative (s_0) being transmitted, an error occurs whenever the received signal is somewhere in the grey area in the figure. Note that this grey area is more conveniently expressed by using an orthonormal basis. This is possible since an ML receiver has the same performance regardless of the basis functions chosen as long as they span the same plane and the decision rule is designed accordingly. Thus, the error probability conditioned on s_0 can be expressed as

$$P_{\mathrm{e}|s_0} = 1 - \Pr\{\text{correct}|s_0\} = 1 - \Pr\{n_1' > -d/2 \text{ and } n_2' > -d/2\} =$$

$$1 - \int_{-d/2}^{\infty}\int_{-d/2}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\left[\frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2}\right]\right) dx_1 dx_2 = 1 - \left(1 - Q\left(\frac{d/2}{\sigma}\right)\right)^2 \ ,$$

where $d/2 = \sqrt{E_s/2}$ and $\sigma = \sigma_1 = \sigma_2$. A similar argument can be made for the remaining three signal alternatives, resulting in the final symbol error probability being $P_{\mathrm{e}} = P_{\mathrm{e}|s_0}$.

2–27 Let's start with the distortion due to quantization noise. Since the quantizer has many levels and $f_{X_n}(x)$ is "nice", the granular distortion can be approximated as $D_c = \frac{\Delta^2}{12}$. To calculate Δ, we first need to find the range of the quantizer, $(-V, V)$, which is equal to the range of X_n. Since the pdf of the input variable is

$$f_{X_n}(x) = \begin{cases} \frac{1}{2V} & \text{when } -V \le x \le V \\ 0 & \text{otherwise} \end{cases}$$

the variance of X_n is

$$E[X_n^2] = \int_{-\infty}^{\infty} x^2 f_{X_n}(x)dx = \int_{-V}^{V} x^2 \frac{1}{2V} dx = \frac{V^2}{3}$$

Since $E[X_n^2] = 1$, $V = \sqrt{3}$. Thus, the quantizer step-size is $\Delta = 2V/256 = \sqrt{3}/128$, which finally gives $D_c = 1.53 \cdot 10^{-5}$.

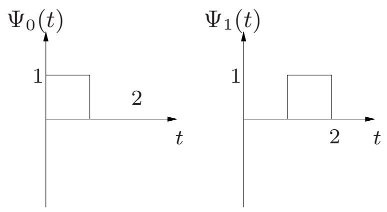
The distortion when a transmission error occurs is

$$D_e = E[(X_n - \hat{X}_n)^2] = E[X_n^2] + E[\hat{X}_n^2] = 1 + 1 = 2$$

since X_n and $\hat{X}_n$ are independent.

What is then the probability that all eight bits are detected correctly, P_a? Let's assume that the symbol error probability is P_s. The probability that one symbol is correctly received is $1 - P_s$. Then, the probability that all four symbols (that contain the eight bits) are correctly detected is $P_a = (1 - P_s)^4$.

The communication system uses the four signals $s_0(t), \ldots, s_3(t)$ to transmit symbols. It can easily be seen that two orthonormal basis functions are:



The four signals can then be expressed as:

$$\begin{aligned}
s_0(t) &= \Psi_0(t) - \Psi_1(t) \\
s_1(t) &= -\Psi_0(t) + \Psi_1(t) \\
s_2(t) &= \Psi_0(t) + \Psi_1(t) \\
s_3(t) &= -\Psi_0(t) - \Psi_1(t)
\end{aligned}$$

The constellation is QPSK! For QPSK, the symbol error probability is known as

$$P_s = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - \left(Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right)^2$$

Since $E_s/N_0 = 16$ and $E_s = 2E_b$,

$$\begin{aligned}
P_s &= 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - \left(Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right)^2 \\
&= 2Q(4) - (Q(4))^2 \approx 6.33 \cdot 10^{-5}
\end{aligned}$$

Using, $P_a = (1 - P_s)^4$, the expected total distortion is

$$D = D_c P_a + D_e(1 - P_a) \approx 0.15 \cdot 10^{-4} + 5.07 \cdot 10^{-4} \approx 5.22 \cdot 10^{-4}$$

2–28 Enumerate the signal points "from below" as $0, 1, \ldots 7$, let $P(i \to j) = \Pr(j$ received given i transmitted), and let

$$q(x) = Q\left(x\, A\sqrt{\frac{2}{N_0}}\right)$$

(a) *Assuming the NL*: Considering first b_1 we see that $\Pr(\hat{b}_1 \neq b_1 | b_1 = 0) = \Pr(\hat{b}_1 \neq b_1 | b_1 = 1)$, and

$$\begin{aligned}
&\Pr(\hat{b}_1 \neq 0 | b_1 = 0) \\
&= \frac{1}{4}\big(\Pr(\hat{b}_1 \neq 0|\text{point 0 sent}) + \Pr(\hat{b}_1 \neq 0|\text{point 1 sent}) + \Pr(\hat{b}_1 \neq 0|\text{point 2 sent}) + \Pr(\hat{b}_1 \neq 0|\text{point 3 sent})
\end{aligned}$$

where

$$\begin{aligned}
\Pr(\hat{b}_1 \neq 0|\text{point 0 sent}) &= P(0 \to 4) + P(0 \to 5) + P(0 \to 6) + P(0 \to 7) = q(7) \\
\Pr(\hat{b}_1 \neq 0|\text{point 1 sent}) &= P(1 \to 4) + P(1 \to 5) + P(1 \to 6) + P(1 \to 7) = q(5) \\
\Pr(\hat{b}_1 \neq 0|\text{point 2 sent}) &= P(2 \to 4) + P(2 \to 5) + P(2 \to 6) + P(2 \to 7) = q(3) \\
\Pr(\hat{b}_1 \neq 0|\text{point 3 sent}) &= P(3 \to 4) + P(3 \to 5) + P(3 \to 6) + P(3 \to 7) = q(1)
\end{aligned}$$

Hence

$$\Pr(\hat{b}_1 \neq b_1) = \frac{1}{4}\big(q(1) + q(3) + q(5) + q(7)\big)$$

Studying b_2 we see that $\Pr(\hat{b}_2 \neq b_2 | b_2 = 0) = \Pr(\hat{b}_2 \neq b_2 | b_2 = 1)$, and

$$\begin{aligned}
&\Pr(\hat{b}_2 \neq 0 | b_2 = 0) \\
&= \frac{1}{4}\big(\Pr(\hat{b}_2 \neq 0|\text{point 0 sent}) + \Pr(\hat{b}_2 \neq 0|\text{point 1 sent}) + \Pr(\hat{b}_2 \neq 0|\text{point 4 sent})\Pr(\hat{b}_2 \neq 0|\text{point 5 sent})\big)
\end{aligned}$$

where

$$\begin{aligned}
\Pr(\hat{b}_2 \neq 0|\text{point 0 sent}) &= P(0 \to 2) + P(0 \to 3) + P(0 \to 6) + P(0 \to 7) = q(3) - q(7) + q(11) \\
\Pr(\hat{b}_2 \neq 0|\text{point 1 sent}) &= P(1 \to 2) + P(1 \to 3) + P(1 \to 6) + P(1 \to 7) = q(1) - q(5) + q(9) \\
\Pr(\hat{b}_2 \neq 0|\text{point 4 sent}) &= P(4 \to 2) + P(4 \to 3) + P(4 \to 6) + P(4 \to 7) = q(1) - q(5) + q(3) \\
\Pr(\hat{b}_2 \neq 0|\text{point 5 sent}) &= P(5 \to 2) + P(5 \to 3) + P(5 \to 6) + P(5 \to 7) = q(3) - q(7) + q(1)
\end{aligned}$$

Hence
$$\Pr(\hat{b}_2 \neq b_2) = \frac{1}{4}\big(3q(1) + 3q(3) - 2q(5) - 2q(7) + q(9) + q(11)\big)$$
Now, for b_3 we again have the symmetry $\Pr(\hat{b}_3 \neq b_3|b_3 = 0) = \Pr(\hat{b}_3 \neq b_3|b_3 = 1)$, and we see that

$$\begin{aligned}
&\Pr(\hat{b}_3 \neq 0|b_3 = 0) \\
&\quad = \frac{1}{4}\big(\Pr(\hat{b}_3 \neq 0|\text{point 0 sent}) + \Pr(\hat{b}_3 \neq 0|\text{point 2 sent}) + \Pr(\hat{b}_3 \neq 0|\text{point 4 sent})\Pr(\hat{b}_3 \neq 0|\text{point 6 sent})\big)
\end{aligned}$$

where

$$\begin{aligned}
\Pr(\hat{b}_3 \neq 0|\text{point 0 sent}) &= P(0 \to 1) + P(0 \to 3) + P(0 \to 5) + P(0 \to 7) \\
&= q(1) - q(3) + q(5) - q(7) + q(9) - q(11) + q(13) \\
\Pr(\hat{b}_3 \neq 0|\text{point 2 sent}) &= P(2 \to 1) + P(2 \to 3) + P(2 \to 5) + P(2 \to 7) \\
&= q(1) - q(3) + q(1) - q(3) + q(5) - q(7) + q(9) \\
\Pr(\hat{b}_3 \neq 0|\text{point 4 sent}) &= P(4 \to 1) + P(4 \to 3) + P(4 \to 5) + P(4 \to 7) \\
&= q(3) - q(5) + q(1) - q(3) + q(1) - q(3) + q(5) \\
\Pr(\hat{b}_3 \neq 0|\text{point 6 sent}) &= P(6 \to 1) + P(6 \to 3) + P(6 \to 5) + P(6 \to 7) \\
&= q(9) - q(11) + q(5) - q(7) + q(1) - q(3) + q(1)
\end{aligned}$$

so

$$\Pr(\hat{b}_3 \neq b_3) = \frac{1}{4}\left(7q(1) - 5q(3) + 2q(5) - 2q(7) + 3q(9) - 2q(11) + q(13)\right)$$

Finally, for the NL, we get

$$P_b = \frac{1}{3}(P_{b,1} + P_{b,2} + P_{b,3}) = \frac{1}{12}\left(11q(1) - q(3) + q(5) - 3q(7) + 4q(9) - q(11) + q(13)\right)$$

As $A^2/N_0 \to \infty$ the $q(1)$ term dominates, so

$$\lim_{A^2/N_0 \to \infty} P_b = \lim_{A^2/N_0 \to \infty} \frac{11}{12} Q\left(x\sqrt{\frac{2A^2}{N_0}}\right)$$

(b) *Assuming the GL*: Obviously $P_b, 1$ is the same as for the NL. For $P_{b,2}$ we get

$$P_{b,2} = \frac{1}{2}\Big(\Pr(\hat{b}_2 \neq 0|b_2 = 0) + \Pr(\hat{b}_2 \neq 0|b_2 = 1)\Big)$$

where now $\Pr(\hat{b}_2 \neq 0|b_2 = 0) \neq \Pr(\hat{b}_2 \neq 0|b_2 = 1)$. In a very similar manner as above for the NL we get
$$\Pr(\hat{b}_2 \neq 0|b_2 = 0) = \frac{1}{2}(q(1) + q(3) - q(9) - q(11))$$
and
$$\Pr(\hat{b}_2 \neq 0|b_2 = 1) = \frac{1}{2}(q(1) + q(3) + q(5) + q(7))$$
that is
$$P_{b,2} = \frac{1}{4}(2q(1) + 2q(3) + q(5) + q(7) - q(9) - q(11))$$
For the GL and b_3 we get

$$\Pr(\hat{b}_3 \neq 0|b_3 = 0) = q(1) - q(5) + \frac{1}{2}(q(9) - q(13) + q(3) - q(7))$$

and

$$\Pr(\hat{b}_3 \neq 0 | b_3 = 1) = q(1) + q(3) + \frac{1}{2}(q(9) + q(11) - q(7) - q(5))$$

so

$$P_{b,3} = q(1) + \frac{3}{4}(q(3) - q(5)) + \frac{1}{2}(q(9) - q(7)) + \frac{1}{4}(q(11) - q(13))$$

and finally

$$P_b = \frac{1}{12}\big(7q(1) + 6q(3) - q(5) + q(9) - q(13)\big)$$

Also, as $A^2/N_0 \to \infty$ we get

$$\lim_{A^2/N_0 \to \infty} P_b = \lim_{A^2/N_0 \to \infty} \frac{7}{12} Q\left(x\sqrt{\frac{2A^2}{N_0}}\right)$$

(c) That is, the GL is asymptotically better than the NL.

2–29 The constellation is 4-PAM (baseband). The basis waveform is

$$\psi(t) = \sqrt{\frac{1}{T}}, \quad 0 \le t < T$$

with $\psi(t) = 0$ for $t < 0$ and $t \ge T$, and the signal space coordinates for the four different alternatives are

$$s_0 = 3A\sqrt{T}, \;\; s_1 = A\sqrt{T}, \;\; s_2 = -A\sqrt{T}, \;\; s_3 = -3A\sqrt{T}$$

The received signal is $r(t) = s_I\,\psi(t) + w(t)$ where $w(t)$ is AWGN with psd $N_0/2$, and s_I is the coordinate of the transmitted signal (I is the transmitted data variable corresponding to the two bits). The probabilities of the different signal alternatives are

$$p_0 = \Pr(s_I = s_0) = \Pr(b_1 = 0, b_0 = 0) = 4/9, \quad p_1 = \Pr(s_I = s_1) = \Pr(b_1 = 0, b_0 = 1) = 2/9$$
$$p_2 = \Pr(s_I = s_2) = \Pr(b_1 = 1, b_0 = 1) = 1/9, \quad p_3 = \Pr(s_I = s_3) = \Pr(b_1 = 1, b_0 = 0) = 2/9$$

(a) Optimal demodulation is based on

$$r = \int_0^T r(t)\psi(t)dt = s_I + w$$

where w is zero-mean Gaussian with variance $N_0/2$. Optimal detection is defined by the MAP rule,

$$\hat{I} = \arg \max_{i \in \{0,1,2,3\}} f(r|s_i)p_i = \arg \max_{i \in \{0,1,2,3\}} p_i \exp\left(-\frac{1}{N_0}(r - s_i)^2\right)$$

The rule indirectly specifies the decision regions

$$\Omega_i = \{r : \hat{I} = i\}$$

with

$$\Omega_3 = (-\infty, a], \;\; \Omega_2 = (a, b], \;\; \Omega_1 = (b, c], \;\; \Omega_0 = (c, \infty)$$

where the decision boundaries can be computed as

$$a = \frac{N_0 \,\ln(p_3/p_2) + (s_2^2 - s_3^2)}{2(s_2 - s_3)} = \frac{N_0}{4A\sqrt{T}} \ln 2 - 2A\sqrt{T}$$
$$b = \frac{N_0 \,\ln(p_2/p_1) + (s_1^2 - s_2^2)}{2(s_1 - s_2)} = \frac{N_0}{4A\sqrt{T}} \ln \frac{1}{2} = -a - 2A\sqrt{T}$$
$$c = \frac{N_0 \,\ln(p_1/p_0) + (s_0^2 - s_1^2)}{2(s_0 - s_1)} = \frac{N_0}{4A\sqrt{T}} \ln \frac{1}{2} + 2A\sqrt{T} = b + 2A\sqrt{T}$$

(b) The average error probability is

$$P_e = \sum_{i=0}^{3} \Pr(\hat{I} \neq i | I = i) p_i$$

where the conditional error probabilities are obtained as

$$\begin{aligned}
\Pr(\hat{I} \neq 3|s_3) &= \Pr(s_3 + w > a) = \Pr(w > A\sqrt{T} + d) = Q\left(\frac{A\sqrt{T}+d}{\sqrt{N_0/2}}\right) \\
\Pr(\hat{I} \neq 2|s_2) &= \Pr(s_2 + w < a) + \Pr(s_2 + w > b) = 2\Pr(w > A\sqrt{T} - d) = 2Q\left(\frac{A\sqrt{T}-d}{\sqrt{N_0/2}}\right) \\
\Pr(\hat{I} \neq 1|s_1) &= \Pr(s_1 + w < b) + \Pr(s_1 + w > c) = \Pr(w > A\sqrt{T} + d) + \Pr(w > A\sqrt{T} - d) \\
&= Q\left(\frac{A\sqrt{T}+d}{\sqrt{N_0/2}}\right) + Q\left(\frac{A\sqrt{T}-d}{\sqrt{N_0/2}}\right) \\
\Pr(\hat{I} \neq 0|s_0) &= \Pr(\hat{I} \neq 3|s_3)
\end{aligned}$$

with

$$d = \frac{N_0}{4A\sqrt{T}} \ln 2$$

Hence we get

$$P_e = \frac{4}{9}\left(2\,Q\left(\frac{A\sqrt{T}+d}{\sqrt{N_0/2}}\right) + Q\left(\frac{A\sqrt{T}-d}{\sqrt{N_0/2}}\right)\right)$$

(c) Since $N_0 \ll A^2T$, "large" errors can neglected, and we get

$$\begin{aligned}
\Pr(\hat{b}_1 \neq b_1) &= \Pr(\hat{b}_1 \neq b_1|s_0)p_0 + \Pr(\hat{b}_1 \neq b_1|s_1)p_1 + \Pr(\hat{b}_1 \neq b_1|s_2)p_2 + \Pr(\hat{b}_1 \neq b_1|s_3)p_3 \\
&\approx 0 + \frac{2}{9}Q\left(\frac{A\sqrt{T}+d}{\sqrt{N_0/2}}\right) + \frac{1}{9}Q\left(\frac{A\sqrt{T}-d}{\sqrt{N_0/2}}\right) + 0 \\
\Pr(\hat{b}_0 \neq b_0) &= \Pr(\hat{b}_0 \neq b_0|s_0)p_0 + \Pr(\hat{b}_0 \neq b_0|s_1)p_1 + \Pr(\hat{b}_0 \neq b_0|s_2)p_2 + \Pr(\hat{b}_0 \neq b_0|s_3)p_3 \\
&\approx \frac{4}{9}Q\left(\frac{A\sqrt{T}+d}{\sqrt{N_0/2}}\right) + \frac{2}{9}Q\left(\frac{A\sqrt{T}-d}{\sqrt{N_0/2}}\right) + \frac{1}{9}Q\left(\frac{A\sqrt{T}-d}{\sqrt{N_0/2}}\right) + \frac{2}{9}Q\left(\frac{A\sqrt{T}+d}{\sqrt{N_0/2}}\right)
\end{aligned}$$

so we get

$$P_b \approx \frac{2}{9}\left(2\,Q\left(\frac{A\sqrt{T}+d}{\sqrt{N_0/2}}\right) + Q\left(\frac{A\sqrt{T}-d}{\sqrt{N_0/2}}\right)\right) = \frac{P_e}{2}$$

again with

$$d = \frac{N_0}{4A\sqrt{T}} \ln 2$$

2–30 For the constellation in figure, the 4 signals on the outer square each contribute with the term

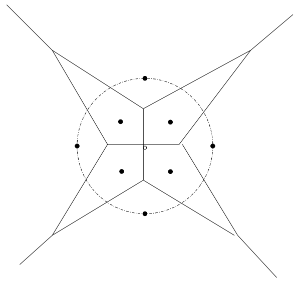
$$I_1 = 2Q(\frac{\sqrt{5-2\sqrt{2}}\sqrt{E_s}}{\sqrt{2N_0}}) + 2Q(\frac{2\sqrt{2}\sqrt{E_s}}{\sqrt{2N_0}})$$

and the other 4 signals, on the inner square, contribute with

$$I_2 = 2Q(\frac{\sqrt{5-2\sqrt{2}}\sqrt{E_s}}{\sqrt{2N_0}}) + 2Q(\frac{\sqrt{2}\sqrt{E_s}}{\sqrt{2N_0}})$$

The total bound is obtained as

$$\frac{1}{2}I_1 + \frac{1}{2}I_2 \tag{3.8}$$



2–31 We chose the orthonormal basis functions

$$\psi_j(t) = \sqrt{\frac{2}{T}} \sin \frac{2\pi j}{T} t, \;\; 0 \le t \le T$$

for $j = 1, 2, 3, 4, 5$. The signals can then be written as vectors in signal space according to

$$\mathbf{s}_i = A\sqrt{\frac{T}{2}}\,(a_i, b_i, c_i, d_i, e_i)$$

where

i	a_i	b_i	c_i	d_i	e_i
0	1	1	0	0	0
1	1	0	1	0	0
2	1	0	0	1	0
3	1	0	0	0	1
4	0	1	1	0	0
5	0	1	0	1	0
6	0	1	0	0	1
7	0	0	1	1	0
8	0	0	1	0	1
9	0	0	0	1	1

We see that there is full symmetry, i.e., the probability of error is the same for all transmitted signals. We can hence condition that $\mathbf{s}_0$ was transmitted and study

$$P_e = \Pr(\text{error}) = \Pr(\text{error}|\mathbf{s}_0 \text{ transmitted})$$

Now, we see that $\mathbf{s}_0$ differs from $\mathbf{s}_i,\; i = 1, \dots, 6$ in 2 coordinates, and from $\mathbf{s}_i,\; i = 7, 8, 9$ in 4 coordinates. Hence the squared distances in signal space from $\mathbf{s}_0$ to the other signals are

$$d_i^2 = A^2 T,\; i = 1, \dots, 6, \quad d_i^2 = 2A^2 T,\; i = 7, 8, 9$$

The Union Bound technique thus gives

$$P_e \le \sum_{i=1}^{9} Q\left(\frac{d_i/2}{\sqrt{N_0/2}}\right) = 6\,Q\left(\frac{A\sqrt{T}}{\sqrt{2N_0}}\right) + 3\,Q\left(\frac{A\sqrt{T}}{\sqrt{N_0}}\right)$$

2–32 We see that one signal point (the one in Origo) has 6 neighbors at distance d, and 6 points (the ones on the inner circle) have 5 neighbors at distance d, and finally 6 points (the ones on the outer circle) have two neighbors at distance d. Hence the Union Bound gives

$$P_e \le \frac{1}{13}(1\cdot 6 + 6 \cdot 5 + 6 \cdot 2)\,Q\left(\frac{d/2}{\sqrt{N_0/2}}\right) = \frac{48}{13}\,Q\left(\frac{d}{\sqrt{2N_0}}\right)$$

with approximate equality for high SNRs. Thus $K = 48/13$.

2–33 This is a PSK constellation. The signal points are equally distributed on a circle of radius $\sqrt{E}$ in signal space. The distance between two neighboring signals hence is

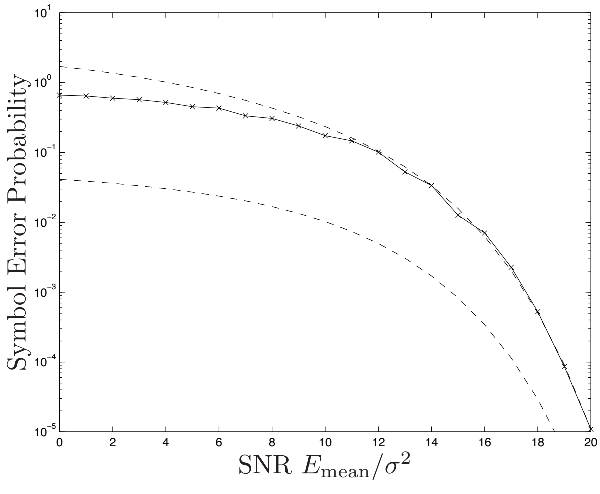
$$2\sqrt{E}\sin(\pi/L)$$

Each signal point has two neighbors at this distance. There is total symmetry. Conditioning that e.g. $s_0(t)$ is transmitted, the probability of error is the probability that any other signal is favored by the receiver. Hence $P_e < 2\,Q(\beta)$ since this counts the antipodal region twice. Clearly it also holds that $P_e > Q(\beta)$, since this does not count all regions.

2–34 The average energy is $E_{\text{mean}} = (E_1 + E_2)/2 = 2$ and, hence, the average SNR is $\text{SNR} = 2E_{\text{mean}}/N_0 = E_{\text{mean}}/\sigma^2$, where σ^2 is the noise variance in each of the two noise components $\mathbf{n} = (n_1, n_2)$. Let the eight signal alternatives be denoted by $\mathbf{s}_i$, where the four innermost alternatives are given by (in complex notation) $\mathbf{s}_i = \sqrt{E_1}e^{j(i\pi/2-\pi/4)}$, $i = 1, \ldots, 4$ and the four outermost alternatives by $\mathbf{s}_i = \sqrt{E_2}e^{j((i-5)\pi/2)}$, $i = 5, \ldots, 8$. Assuming that symbol i is transmitted, the received vector is $\mathbf{r} = \mathbf{s}_i + \mathbf{n}$. The optimal receiver is the MAP receiver, which, since the symbols are equiprobable, reduces to the ML receiver. Since the channel is an AWGN channel, the decision rule is to choose the closest (in the Euclidean sense) signal, and, as there is no memory in the channel, no sequence detection is necessary. As shown in class, the symbol error probability can, by using the union bound technique, be upper bounded as

$$\begin{aligned} P_\text{e} &= \sum_i \Pr(\mathbf{s}_i) \sum_{j\neq i} \Pr(\text{error}|\mathbf{s}_i) \\ &\leq \sum_{i=1}^{8} \frac{1}{8} \sum_{\substack{j=1\\ j\neq i}}^{8} Q\left(\frac{d_{ij}/2}{\sigma}\right) \quad , \end{aligned}$$

where $d_{ij} = \|\mathbf{s}_i - \mathbf{s}_j\|$ is the distance between $\mathbf{s}_i$ and $\mathbf{s}_j$. A lower bound on P_e is obtained by including only the dominating term in the double sum above. The upper and lower bounds are plotted below. The figure also shows a simulation estimating the true symbol error.



As can be seen from the plots, the upper bound is loose for low SNRs (it is even larger than one!), but improves as the SNR increases and is quite tight for high SNRs. The lower bound is quite loose in this problem (although this it not necessarily the case for other problems). The reason for this is that there are several distances in the constellation that are approximately equal and thus the dominating term in the summation is not "dominating enough". At high SNRs the error probability is dominated by the smallest d_{ij}. Since d_{15} is smaller than d_{12}, one way of improving the error probability is to decrease E_1 and increase E_2 such that $d_{15} = d_{12}$. By inspecting the signal constellation, $d_{12}^2 = 2E_1$ and $d_{15}^2 = E_1 + E_2 - \sqrt{2E_1E_2}$. Keeping $E_1 + E_2$ constant and solving for E_1 in $d_{12} = d_{15}$, $E_1 \approx 0.8453$ is obtained. Unfortunately, the improvement obtained is negligible as E_1 was close to this value from the beginning.

2–35 For $0 \leq t \leq T$ we get, using the relations $\cos(\alpha - \pi/2) = \sin\alpha$, $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ and $\cos\pi/4 = \sin\pi/4 = 1/\sqrt{2}$, that

$$s_1(t) = \sqrt{\frac{2E}{T}}\cos 4\pi\frac{t}{T} = \sqrt{E}\sqrt{\frac{2}{T}}\cos 4\pi\frac{t}{T} \qquad s_2(t) = \sqrt{\frac{2E}{T}}\cos\left(4\pi\frac{t}{T} - \frac{\pi}{2}\right) = \sqrt{E}\sqrt{\frac{2}{T}}\sin 4\pi\frac{t}{T}$$

$$\begin{aligned} s_3(t) &= \sqrt{\frac{E}{T}}\sin\left(4\pi\frac{t}{T} + \frac{\pi}{4}\right) = \sqrt{\frac{E}{T}}\sin\frac{\pi}{4}\cos 4\pi\frac{t}{T} + \sqrt{\frac{E}{T}}\cos\frac{\pi}{4}\sin 4\pi\frac{t}{T} \\ &= \frac{\sqrt{E}}{2}\sqrt{\frac{2}{T}}\cos 4\pi\frac{t}{T} + \frac{\sqrt{E}}{2}\sqrt{\frac{2}{T}}\sin 4\pi\frac{t}{T} \; \left(= \frac{1}{2}s_1(t) + \frac{1}{2}s_2(t)\right) \end{aligned}$$

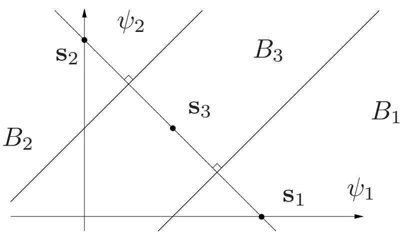
Now introduce the orthonormal basis functions

$$\psi_1(t) = \begin{cases} \sqrt{\frac{2}{T}}\cos\left(4\pi\frac{t}{T}\right) & 0 \leq t < T \\ 0 & \text{otherwise} \end{cases} \qquad \psi_2(t) = \begin{cases} \sqrt{\frac{2}{T}}\sin\left(4\pi\frac{t}{T}\right) & 0 \leq t < T \\ 0 & \text{otherwise} \end{cases}$$

The signal alternatives can then be expressed in signal space according to

$$\mathbf{s}_1 = \left(\sqrt{E}, 0\right) \qquad \mathbf{s}_2 = \left(0, \sqrt{E},\right) \qquad \mathbf{s}_3 = \left(\sqrt{E}/2, \sqrt{E}/2\right)$$

The optimal receiver is based on nearest-neighbor detection as illustrated below, where the solid lines mark the boundaries of the decision regions.



Since all signal alternatives lie on a straight line we can derive an exact expression for the error probability. We get

$$\Pr(\text{error}|\mathbf{s}_1) = \Pr(|\mathbf{r}-\mathbf{s}_3|^2 < |\mathbf{r}-\mathbf{s}_1|^2 | \mathbf{r} = \mathbf{s}_1 + \mathbf{n}) = \ldots = Q\left(\frac{|\mathbf{s}_1 - \mathbf{s}_3|}{\sqrt{N_0/2}}\right)$$

and $\Pr(\text{error}|\mathbf{s}_2) = \ldots = Q\left(\frac{|\mathbf{s}_2 - \mathbf{s}_3|}{\sqrt{N_0/2}}\right)$. We also get

$$\begin{aligned} \Pr(\text{error}|\mathbf{s}_3) &= \Pr(|\mathbf{r}-\mathbf{s}_1|^2 < |\mathbf{r}-\mathbf{s}_3|^2 | \mathbf{r} = \mathbf{s}_3 + \mathbf{n}) + \Pr(|\mathbf{r}-\mathbf{s}_2|^2 < |\mathbf{r}-\mathbf{s}_3|^2 | \mathbf{r} = \mathbf{s}_3 + \mathbf{n}) \\ &= Q\left(\frac{|\mathbf{s}_1 - \mathbf{s}_3|}{\sqrt{N_0/2}}\right) + Q\left(\frac{|\mathbf{s}_2 - \mathbf{s}_3|}{\sqrt{N_0/2}}\right) \end{aligned}$$

Since $|\mathbf{s}_1 - \mathbf{s}_3| = |\mathbf{s}_2 - \mathbf{s}_3| = \sqrt{E/2}$ we finally conclude that

$$\Pr(\text{error}) = \sum_{i=1}^{3}\Pr(\text{error}|\mathbf{s}_i)\Pr(\mathbf{s}_i) = \frac{4}{3}Q\left(\sqrt{\frac{E}{4N_0}}\right)$$

Also, the Union Bound is computed according to

$$\begin{aligned} \Pr(\text{error}|\mathbf{s}_1) &\leq Q\left(\frac{|\mathbf{s}_2 - \mathbf{s}_1|}{\sqrt{N_0/2}}\right) + Q\left(\frac{|\mathbf{s}_3 - \mathbf{s}_1|}{\sqrt{N_0/2}}\right) \\ \Pr(\text{error}|\mathbf{s}_2) &\leq Q\left(\frac{|\mathbf{s}_1 - \mathbf{s}_2|}{\sqrt{2N_0}}\right) + Q\left(\frac{|\mathbf{s}_3 - \mathbf{s}_2|}{\sqrt{2N_0}}\right) \\ \Pr(\text{error}|\mathbf{s}_3) &\leq Q\left(\frac{|\mathbf{s}_1 - \mathbf{s}_3|}{\sqrt{2N_0}}\right) + Q\left(\frac{|\mathbf{s}_2 - \mathbf{s}_3|}{\sqrt{2N_0}}\right) \end{aligned}$$

$$\Pr(\text{error}) = \frac{1}{3}\left(\Pr(\text{error}|\mathbf{s}_1) + \Pr(\text{error}|\mathbf{s}_2) + \Pr(\text{error}|\mathbf{s}_3)\right) \le \frac{4}{3}Q\left(\sqrt{\frac{E}{4N_0}}\right) + \frac{2}{3}Q\left(\sqrt{\frac{E}{N_0}}\right)$$

2–36 Note that the symbol period is so short so that the symbols disturb each other, i.e we have ISI. In the general case, the presence of ISI makes an analysis of the system difficult. However, the problem that is considered here is considerably simplified by the fact that only two symbols are transmitted and the receiver is supposed to minimize the probability of a sequence error (as opposed to a symbol error). This means that, from a performance point of view, the two symbols can be considered as one "super symbol" corresponding to four different messages (and hence message waveforms). We can therefore convert the problem into the standard form taught in class where one out of a set of waveforms (in this case four) is transmitted and the receiver is designed to minimize the probability of a message error. Thus, the solution strategy is as follows:

i. Find an orthonormal basis for the four message waveforms

ii. Draw the signal constellation with decision boundaries

iii. Use the union bound technique to get a tight upper bound for the error probability.

i. The four message waveforms are given by $s(t) = d_0p(t) + d_1p(t-1)$, where $d_0 = \pm 1$, $d_1 = \pm 1$. We'll use Gram-Schmidt for expressing $s(t)$ in terms of an orthonormal basis. The basis functions are given by

$$\varphi_1(t) = \frac{p(t)}{\|p(t)\|} = \frac{p(t)}{\sqrt{2}} = \begin{cases} 1/\sqrt{2}, & 0 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\tilde{\varphi}_2(t) = p(t-1) - < p(t-1), \varphi_1(t) > \varphi_1(t) = p(t-1) - p(t)/2$$

$$\varphi_2(t) = \frac{\tilde{\varphi}_2(t)}{\|\tilde{\varphi}_2(t)\|} = \frac{\tilde{\varphi}_2(t)}{\sqrt{3/2}} = \sqrt{2/3} \cdot \begin{cases} -1/2, & 0 \le t < 1 \\ 1/2, & 1 \le t < 2 \\ 1, & 2 \le t < 3 \end{cases}$$

Hence, we have

$$p(t) = \sqrt{2}\varphi_1(t)$$
$$p(t-1) = \varphi_1(t)/\sqrt{2} + \sqrt{3/2}\varphi_2(t)$$

and can therefore write the transmitted signal as

$$s(t) = d_0\sqrt{2}\varphi_1(t) + d_1\left(\varphi_1(t)/\sqrt{2} + \sqrt{3/2}\varphi_2(t)\right) = (d_0\sqrt{2} + d_1/\sqrt{2})\varphi_1(t) + d_1\sqrt{3/2}\varphi_2(t)$$

ii. From the above expression, we see that the coordinates of the signal constellation points are given by

$$(s_x, s_y) = (d_0\sqrt{2} + d_1/\sqrt{2}, d_1\sqrt{3/2}), \quad d_0 = \pm 1, d_1 = \pm 1$$

Writing out the coordinates of the four points we get

$$\mathbf{s}_1 = \left(-3\sqrt{2}/2, -\sqrt{3/2}\right), \quad \mathbf{s}_2 = \left(-1/\sqrt{2}, \sqrt{3/2}\right), \quad \mathbf{s}_3 = -\mathbf{s}_2, \quad \mathbf{s}_4 = -\mathbf{s}_1$$

If you make a picture of the signal constellation you will find that two points have three neighbors each and the two other points have two neighbors each (two points are neighbors if they share a decision boundary).

iii. All the neighbors are at the same distance $d = |\mathbf{s}_1 - \mathbf{s}_2| = 2\sqrt{2}$. Hence, by also noting that the symbols are equally likely (since d_0 and d_1 are independent and uniformly distributed), we finally get the desired answer by upperbounding the probability of a sequence error as

$$P_e = \sum_{i=1}^{4} \Pr[e|\mathbf{s}_i]\Pr[\mathbf{s}_i] < \frac{1}{4}\left(2 \cdot 3Q\left(\frac{d/2}{\sqrt{N_0/2}}\right) + 2 \cdot 2Q\left(\frac{d/2}{\sqrt{N_0/2}}\right)\right) = \frac{5}{2}Q\left(\frac{2}{\sqrt{N_0}}\right)$$

2–37 (a) We can chose

$$\psi_1(t) = \frac{s_0(t)}{\|s_0(t)\|} = \frac{1}{2\sqrt{3}}\, s_0(t)$$

as the first basis waveform. The component of $s_1(t)$ that is orthogonal to $s_0(t)$ and $\psi_1(t)$ then is

$$s_1(t) - \left(\int_0^T s_1(t)\psi_1(t)dt \right) \psi_1(t) = s_1(t) + \frac{1}{2}\, s_0(t)$$

so we have the second basis waveform as

$$\psi_2(t) = \frac{s_1(t) + \frac{1}{2}\, s_0(t)}{\|s_1(t) + \frac{1}{2}\, s_0(t)\|} = \frac{1}{\sqrt{8}}\big(s_1(t) + \frac{1}{2}\, s_0(t)\big) = \frac{1}{\sqrt{8}}\big(s_1(t) + \sqrt{3}\,\psi_1(t)\big)$$

(Check that $\|\psi_1(t)\| = \|\psi_2(t)\| = 1$ and that $\psi_1(t) \perp \psi_2(t)$.) We also get

$$s_1(t) = -\sqrt{3}\,\psi_1(t) + \sqrt{8}\,\psi_2(t)$$

Now, the component of $s_2(t)$ being orthogonal to both ψ_1 and ψ_2 is

$$s_2(t) - \left(\int_0^T s_2(t)\psi_1(t)dt \right) \psi_1(t) - \left(\int_0^T s_2(t)\psi_2(t)dt \right) \psi_2(t) = s_2(t) - \sqrt{3}\,\psi_1(t) + \sqrt{2}\,\psi_2(t) = 0$$

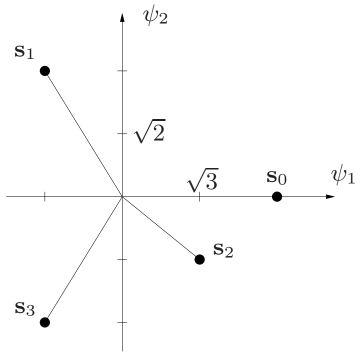
That is, $s_2(t)$ has no component orthogonal to $\psi_1(t)$ and $\psi_2(t)$ and is hence a two-dimensional signal that can be fully described in terms of $\psi_1(t)$ and $\psi_2(t)$ as

$$s_2(t) = \sqrt{3}\,\psi_1(t) - \sqrt{2}\,\psi_2(t)$$

A similar check on $s_3(t)$ gives that also $s_3(t)$ can be fully described in terms of $\psi_1(t)$ and $\psi_2(t)$ as

$$s_3(t) = -\sqrt{3}\,\psi_1(t) - \sqrt{8}\,\psi_2(t)$$

In signal space we hence get the representation illustrated below.



(b) An upper bound to P_e is given by the union bound. Letting $d_{ij} = \|\mathbf{s}_i - \mathbf{s}_j\|$ and

$$P_{ij} = Q\left(\frac{d_{ij}}{\sqrt{2N_0}}\right)$$

we get

$$\begin{aligned}
\Pr(\text{error}|s_0) &\le P_{01} + P_{02} \\
\Pr(\text{error}|s_1) &\le P_{10} + P_{12} + P_{13} \\
\Pr(\text{error}|s_2) &\le P_{20} + P_{21} + P_{23} \\
\Pr(\text{error}|s_3) &\le P_{31} + P_{32}
\end{aligned}$$

and hence

$$P_e \le \frac{1}{2}\big(P_{01} + P_{02} + P_{12} + P_{13} + P_{23}\big)$$

which, with $N_0 = 1$, gives

$$P_e \leq 0.0305492\ldots$$

A lower bound to P_e can be obtained by observing that

$$\begin{aligned}
\Pr(\text{error}|s_0) &\geq P_{02} \\
\Pr(\text{error}|s_1) &\geq P_{12} \\
\Pr(\text{error}|s_2) &\geq P_{20} \\
\Pr(\text{error}|s_3) &\geq P_{32}
\end{aligned}$$

since using only the "nearest neighbors" in the union bounds gives lower bounds on the conditional error probabilities. Consequently we get

$$P_e \geq \frac{1}{4}(P_{02} + P_{12} + P_{20} + P_{32}) = 0.0294939\ldots$$

We have hence shown that

$$0.029 < P_e < 0.031$$

2–38 We enumerate the signal alternatives from $\mathbf{s}_0$ to $\mathbf{s}_7$ counter-clockwise starting with the signal at coordinates $(\sqrt{2}d, 0)$. The symmetry of the constellation then gives

$$P_e = \Pr(\text{symbol error}) = \frac{1}{2}\Pr(\text{error}|\mathbf{s}_0) + \frac{1}{2}\Pr(\text{error}|\mathbf{s}_1)$$

Letting $P_e(i, j|a)$ denote the pairwise error probability between $\mathbf{s}_i$ and $\mathbf{s}_j$, conditioned on a known amplitude a, the union bound gives

$$\begin{aligned}
\Pr(\text{error}|\mathbf{s}_0, a) &\leq P_e(0,1|a) + P_e(0,7|a) = 2P_e(0,1|a) \\
\Pr(\text{error}|\mathbf{s}_1, a) &\leq P_e(1,0|a) + P_e(1,2|a) + P_e(1,3|a) + P_e(1,7|a) \\
&= 2P_e(1,0|a) + 2P_e(1,3|a)
\end{aligned}$$

(Note that the terms included above are enough to ensure an upper bound, since they "cover the whole plane" [c.f. the signal space diagram].) Let $P_e(i, j) = E[P_e(i, j|a)]$, where the expectation is with respect to the Rayleigh distribution of the amplitude, denote the average pairwise error probability. Then since $\mathbf{s}_0$ and $\mathbf{s}_1$ are at distance $a \cdot d$ and $\mathbf{s}_1$ and $\mathbf{s}_3$ are at distance $a \cdot \sqrt{2}d$ (for a known value of a), we get (using the hint)

$$\begin{aligned}
P_e(0,1) = P_e(1,0) &= \int_0^\infty Q\left(\frac{ad}{\sqrt{2N_0}}\right) a \exp\big(-\frac{a^2}{2\sigma^2}\big)da = \frac{1}{2}\left[1 - \frac{\sqrt{\gamma/2}}{\sqrt{1+\gamma/2}}\right] \\
P_e(1,3) &= \int_0^\infty Q\left(\frac{ad}{\sqrt{N_0}}\right) a \exp\big(-\frac{a^2}{2\sigma^2}\big) = \frac{1}{2}\left[1 - \frac{\sqrt{\gamma}}{\sqrt{1+\gamma}}\right]
\end{aligned}$$

where $\gamma \triangleq \sigma^2 d^2/N_0$. Hence we get the bound

$$P_e \leq \frac{3}{2} - \frac{\sqrt{\gamma/2}}{\sqrt{1+\gamma/2}} - \frac{1}{2}\frac{\sqrt{\gamma}}{\sqrt{1+\gamma}}$$

2–39 Using $\Psi_1(t) = 1,\ 0 \leq t < 1$ and $\Psi_2(t) = 1,\ 1 \leq t \leq 2$ as orthonormal basis, the signal constellation can be written as:

$$\bar{s}_i = (< s_i(t), \Psi_1(t) >, < s_i(t), \Psi_2(t) >)\ ,\ i = 1, 2, 3$$

and

$$\begin{aligned}
\bar{s}_1 &= A(2, 2) \\
\bar{s}_2 &= A(\sqrt{3} - 1, -(\sqrt{3} + 1)) \\
\bar{s}_3 &= A(-(\sqrt{3} + 1), \sqrt{3} - 1)
\end{aligned}$$

which yields a 3-PSK constellation, where $||\bar{s}_i|| = \sqrt{8}A\ \forall i$

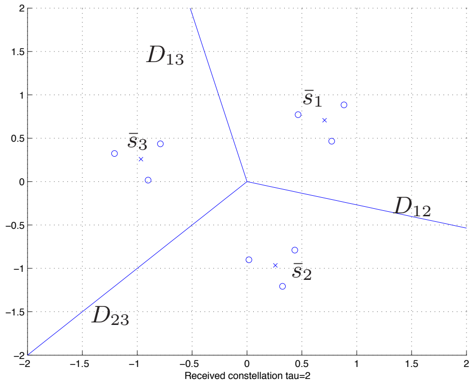
(a) The received signal can be written as

$$r(t) = s(t) + \frac{1}{4}s(t-\tau) + w(t) \ , \ \tau = T = 2$$

or in vector form, conditioned that $s_n(t)$ was transmitted and the previous symbol was $s_m(t)$:

$$\bar{r}_{nm} = \bar{s}_n + \frac{1}{4}\bar{s}_m + \bar{w}$$

which yields nine equiprobable received signals (disregarding the noise $\bar{w}$):



Due to symmetry, the error probability is equal for all three transmitted symbols ($= Pr$[error]). Two non-trivial upper-bounds can be expressed as:

- The tighter alternative:

$$\begin{aligned} Pr[\text{error}] &= Pr[\text{error } |\bar{s}_1 \text{ transmitted}] = \frac{1}{3}Pr[\text{error } |\bar{r}_{11} \text{ received}] + \frac{1}{3}Pr[\text{error } |\bar{r}_{12} \text{ received}] + \\ & \quad \frac{1}{3}Pr[\text{error } |\bar{r}_{13} \text{ received}] \end{aligned}$$

where

$$Pr[\text{error } |\bar{r}_{1j} \text{ received}] < Pr[|\bar{w}| > d_{j12}] + Pr[|\bar{w}| > d_{j13}] = Q\left(\frac{d_{j12}}{\sqrt{N_0/2}}\right) + Q\left(\frac{d_{j13}}{\sqrt{N_0/2}}\right)$$

and d_{j12} and d_{j13} denote the closest distance from $\bar{r}_{1j}$ to boundary D_{12} and D_{13} respectively.
Thus,

$$\begin{aligned} Pr[\text{error}] &< \frac{1}{3}\left[Q\left(\frac{d_{112}}{\sqrt{N_0/2}}\right) + Q\left(\frac{d_{113}}{\sqrt{N_0/2}}\right) + Q\left(\frac{d_{212}}{\sqrt{N_0/2}}\right) + \right. \\ & \left. Q\left(\frac{d_{213}}{\sqrt{N_0/2}}\right) + Q\left(\frac{d_{312}}{\sqrt{N_0/2}}\right) + Q\left(\frac{d_{313}}{\sqrt{N_0/2}}\right)\right] \end{aligned}$$

where

$$\begin{aligned} d_{j12} &= \tfrac{5}{8}\sqrt{3}, \tfrac{3}{8}\sqrt{3}, \tfrac{1}{2}\sqrt{3} \quad j = 1, 2, 3 \\ d_{j13} &= \tfrac{5}{8}\sqrt{3}, \tfrac{1}{2}\sqrt{3}, \tfrac{3}{8}\sqrt{3} \quad j = 1, 2, 3 \end{aligned}$$

which finally gives

$$Pr[\text{error}] < \frac{1}{3}\left(2Q\left(\sqrt{\frac{75}{32N_0}}\right) + 2Q\left(\sqrt{\frac{3}{2N_0}}\right) + 2Q\left(\sqrt{\frac{27}{32N_0}}\right)\right)$$

- A less tight upper-bound can be found by only considering the worst case ISI, i.e. either $\bar{r}_{13}$ or $\bar{r}_{12}$, i.e.

$$Pr[\text{error}] = Pr[\text{error } |\bar{r}_{13} \text{ received}] < Q\left(\frac{d_{312}}{\sqrt{N_0/2}}\right) + Q\left(\frac{d_{313}}{\sqrt{N_0/2}}\right)$$
$$= Q\left(\sqrt{\frac{3}{2N_0}}\right) + Q\left(\sqrt{\frac{27}{32N_0}}\right)$$

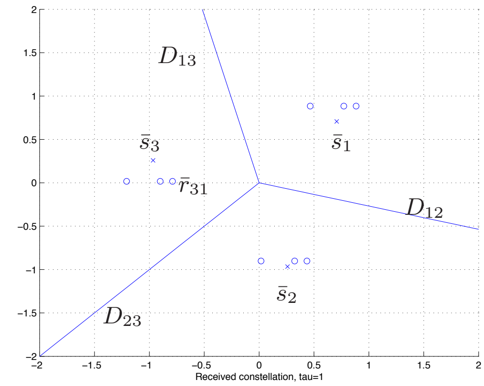
(b) Now the delay τ of the second tap in the channel is half a symbol period. Again, let $s_n(t)$ be the current transmitted symbol, and $s_m(t)$ be the previous transmitted symbol. Furthermore, let $s_{kl} =< s_k(t), \Psi_l(t) >$, e.g. s_{m1} is the Ψ_1-component of s_m. Also, let $\bar{r}_{nm}$ be the received signal vector (after demodulation), when the current transmitted signal is $s_n(t)$ and the previous transmitted signal is $s_m(t)$.

Due to the time-orthonormal basis we have chosen, the Ψ_1-component of $\bar{r}_{nm}$ is the result of integration of the received signal over the first half of the symbol period ($0 \le t \le 1$). Due to the ISI, the received signal in this time-interval consists of the first half of the desired symbol $s_n(t)$, but also the second half of the previous symbol $s_m(t)$ and noise.

Similarly, the Ψ_2-component of $\bar{r}_{nm}$ contains the desired contribution from the second half of $s_n(t)$, but also an ISI-component from the first half of $s_n(t)$. This can be expressed as:

$$\bar{r}_{nm} = (s_{n1} + \frac{1}{4}s_{m2}, s_{n2} + \frac{1}{4}s_{n1}) + \bar{w}$$

Disregarding the noise term, received constellation looks like below. Note that the constellation is not symmetric.



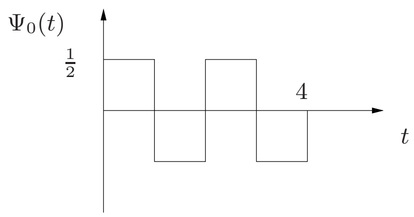
Several bounds of different tightness can be obtained. A fairly simple, but not very tight bound can be found by considering the overall worst case ISI. The bound takes only the constellation point closest to the decision boundaries into account. The point is $\bar{r}_{31}$ with distances $d_{13} = 0.679$ and $d_{23} = 0.787$ to the two boundaries D_{13} and D_{23} respectively.

The upper-bound is then

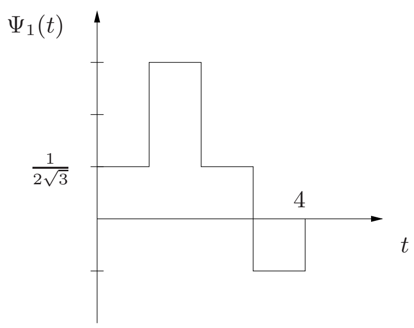
$$Pr[\text{error}] < Q\left(\frac{d_{13}}{\sqrt{N_0/2}}\right) + Q\left(\frac{d_{23}}{\sqrt{N_0/2}}\right)$$

Of course, more sophisticated and tighter bounds can be found.

2–40 First, orthonormal basis functions need to be found. To do this, the Gram-Schmidt orthonormalization procedure can be used. Starting with $s_0(t)$, the first basis function $\Psi_0(t)$ becomes



To find the second basis function, the projection of $s_1(t)$ onto $\Psi_0(t)$ is subtracted from $s_1(t)$, which after normalization to unit energy gives the second basis function $\Psi_1(t)$. This procedure is easily done graphically, since all of the functions are piece-wise constant.

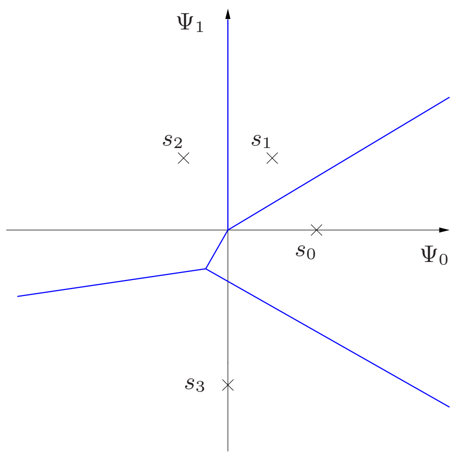


It can be seen that also $s_2(t)$ and $s_3(t)$ are linear combinations of $\Psi_0(t)$ and $\Psi_1(t)$. Alternatively, the continued Gram-Schmidt procedure gives no more basis functions.

To summarize:

$$\begin{aligned}
s_0(t) &= 2\Psi_0(t) \\
s_1(t) &= \Psi_0(t) + \sqrt{3}\Psi_1(t) \\
s_2(t) &= -\Psi_0(t) + \sqrt{3}\Psi_1(t) \\
s_3(t) &= -2\sqrt{3}\Psi_1(t)
\end{aligned}$$

(a) Since the four symbols are equiprobable, the optimal receiver uses ML detection, i.e. the nearest symbol is detected. The ML-decision regions look like:



The upper bound on the probability of error is given by

$$P_e \leq \frac{1}{4}\sum_{i=0}^{3}\sum_{\substack{k=0\\k\neq i}}^{3} Q(\frac{d_{ik}}{\sqrt{2N_0}})$$

Some terms in the sum can be skipped, since the error regions they correspond to are already covered by other regions. Calculating the distances between the symbols gives the following expression on the upper bound

$$P_e \leq \frac{1}{2}(2Q(\sqrt{\frac{2}{N_0}} + Q(\sqrt{\frac{6}{N_0}}) + Q(\sqrt{\frac{8}{N_0}}) + Q(\sqrt{\frac{14}{N_0}}))$$

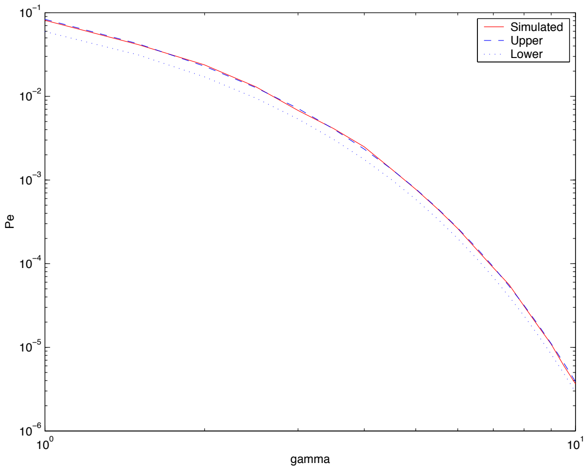
(b) The analytical lower bound on the probability of error is given by

$$P_e \geq \frac{1}{4}\sum_{i=0}^{3} Q(\frac{\min d_{ik}}{\sqrt{2N_0}})$$

where min d_{ik} is the distance from the i:th symbol to its nearest neighbor.
Calculating the distances gives min $d_{0k} = 2$, min $d_{1k} = 2$, min $d_{2k} = 2$, min $d_{3k} = 4$, which gives

$$P_e \geq \frac{3}{4}Q(\frac{2}{\sqrt{2N_0}}) + \frac{1}{4}Q(\frac{4}{\sqrt{2N_0}})$$

To estimate the true P_e the system needs to be simulated using a software tool like MATLAB. By using the equivalent vector model, it's easy to generate a symbol from the set defined above, add white gaussian noise, and detect the symbol. Doing this for very many symbols and counting the number of erroneosly detected symbols gives a good estimate of the true symbol error probability. Repeating the procedure for different noise levels gives a curve like below. As you can see, the "true" P_e from simulation is very close to the upper bound.



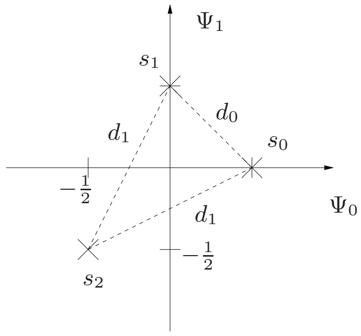
2–41 • First, observe that $s_2(t) = -s_0(t) - s_1(t)$, and that the energy of $s_0(t)$ equals the energy of $s_1(t)$. It can also directly be seen that $s_0(t)$ and $s_1(t)$ are orthogonal, so normalized versions of $s_0(t)$ and $s_1(t)$ can be used as basis waveforms.

$$\begin{aligned} \|s_0(t)\|^2 = \|s_1(t)\|^2 &= \int_0^{T/2} s_0(t)^2 dt = \left\{ \text{symmetry of } s_0(t)^2 \text{ around } t = \frac{T}{4} \right\} \\ &= 2\int_0^{T/4} s_0(t)^2 dt = \left\{ s_0(t) = \sqrt{\frac{24}{T^3}}t \right\} = \frac{48}{T^3}\int_0^{T/4} t^2 dt = \frac{1}{4} \end{aligned}$$

This gives the orthonormal basis waveforms

$$\Psi_0(t) = 2s_0(t) \quad \Psi_1(t) = 2s_1(t)$$

and the signal constellation



where $d_0 = \frac{1}{\sqrt{2}}$ and $d_1 = \frac{\sqrt{5}}{2}$.

- The union bound on the probability of symbol error (for ML detection) on one link is

$$P_e \leq \sum_{i=0}^{2} \Pr(s_i) \sum_{\substack{k=0\\k\neq i}}^{2} Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right)$$

where d_{ik} is the distance between s_i and s_k. Writing out the terms of the sums gives

$$P_e \leq \quad 2\tfrac{1}{4}\left(Q\left(\tfrac{d_0}{\sqrt{2N_0}}\right) + Q\left(\tfrac{d_1}{\sqrt{2N_0}}\right)\right) + \tfrac{1}{2}\left(2Q\left(\tfrac{d_1}{\sqrt{2N_0}}\right)\right) = \quad \frac{1}{2}Q\left(\frac{d_0}{\sqrt{2N_0}}\right) + \frac{3}{2}Q\left(\frac{d_1}{\sqrt{2N_0}}\right)$$

Now, a bound on the probability that the reception of one symbol on *one* link is correct is

$$1 - P_e \geq \quad 1 - \tfrac{1}{2}Q\left(\tfrac{d_0}{\sqrt{2N_0}}\right) - \tfrac{3}{2}Q\left(\tfrac{d_1}{\sqrt{2N_0}}\right)$$

- The probability that the communication of one symbol over all $n+1$ links is successful is then

$$\Pr(\hat{s} = s) = (1 - P_e)^{n+1} \geq \left(1 - \frac{1}{2}Q\left(\frac{d_0}{\sqrt{2N_0}}\right) - \frac{3}{2}Q\left(\frac{d_1}{\sqrt{2N_0}}\right)\right)^{n+1}$$

- This finally gives the requested bound

$$\begin{aligned}\Pr(\hat{s} \neq s) &= 1 - (1 - P_e)^{n+1} \leq 1 - \left(1 - \frac{1}{2}Q\left(\frac{d_0}{\sqrt{2N_0}}\right) - \frac{3}{2}Q\left(\frac{d_1}{\sqrt{2N_0}}\right)\right)^{n+1} \\ &= 1 - \left(1 - \frac{1}{2}Q\left(\frac{1}{\sqrt{4N_0}}\right) - \frac{3}{2}Q\left(\sqrt{\frac{5}{8N_0}}\right)\right)^{n+1}\end{aligned}$$

- Note that if we assume that $\frac{1}{2}Q\left(\frac{d_0}{\sqrt{2N_0}}\right) + \frac{3}{2}Q\left(\frac{d_1}{\sqrt{2N_0}}\right)$ is small, the approximation $(1+x)^\alpha \approx 1 + \alpha x$ (x small) can be applied with the result

$$1 - \left(1 - \frac{1}{2}Q\left(\frac{d_0}{\sqrt{2N_0}}\right) - \frac{3}{2}Q\left(\frac{d_1}{\sqrt{2N_0}}\right)\right)^{n+1} \approx \frac{n+1}{2}\left(Q\left(\frac{d_0}{\sqrt{2N_0}}\right) + 3Q\left(\frac{d_1}{\sqrt{2N_0}}\right)\right)$$

2–42 (a) It can easily be seen from the signal waveforms that they can be expressed as functions of only two basis waveforms, $\phi_1(t)$ and $\phi_2(t)$.

$$\begin{aligned}\phi_1(t) &= \psi_1(t) \\ \phi_2(t) &= K\left(\frac{3}{4}\psi_2(t) + \frac{\sqrt{3}}{4}\psi_3(t)\right)\end{aligned}$$

where K is a normalization factor to make $\phi_2(t)$ unit energy. Let's find K first.

$$\begin{aligned}
\|K\left(\frac{3}{4}\psi_2(t)+\frac{\sqrt(3)}{4}\psi_3(t)\right)\|^2 &= \int_{T/3}^{T} K^2\left(\frac{3}{4}\psi_2(t)+\frac{\sqrt(3)}{4}\psi_3(t)\right)^2 dt = \frac{9K^2}{16}\int_{T/3}^{2T/3}\psi_2^2(t)dt + \\
& \frac{3K^2}{16}\int_{2T/3}^{T}\psi_3^2(t)dt = \frac{3K^2}{4} = 1 \\
\Rightarrow & \quad K = \frac{2}{\sqrt{3}}
\end{aligned}$$

Hence, $\phi_2(t) = \frac{\sqrt{3}}{2}\psi_2(t) + \frac{1}{2}\psi_3(t)$. The signal waveforms can be written as

$$\begin{aligned}
s_1(t) &= A\phi_1(t) + \frac{\sqrt{3}}{2}A\phi_2(t) \\
s_2(t) &= -A\phi_1(t) + \frac{\sqrt{3}}{2}A\phi_2(t) \\
s_3(t) &= -\frac{\sqrt{3}}{2}A\phi_2(t)
\end{aligned}$$

The optimal decisions can be obtained via two correlators with the waveforms $\phi_1(t)$ and $\phi_2(t)$.

(b) The constellation is an equilateral triangle with side-length $2A$ and the cornerpoints in $s_1 = (A, \frac{\sqrt{3}}{2}A)$, $s_2 = (-A, \frac{\sqrt{3}}{2}A)$ and $s_3 = (0, -\frac{\sqrt{3}}{2}A)$. The union bound gives an upper bound by including, for each point, the other two. The lower bound is obtained by only counting one neighbor.

(c) The true error probability is somewhere between the lower and upper bounds. To guarantee an error probability below 10^{-4}, we must use the upper bound.

$$\begin{aligned}
P_e < 2Q\left(\sqrt{\frac{2A^2}{N_0}}\right) &= 10^{-4} \\
\text{Tables} \Rightarrow \sqrt{\frac{2A^2}{N_0}} \approx 3.9 \quad &\Rightarrow \quad A^2 \approx \frac{N_0}{2}3.9^2
\end{aligned}$$

To find the average transmitted power, we first need to find the average symbol energy. The three signals have energies:

$$\begin{aligned}
E_{s_1} = E_{s_2} &= \frac{7}{4}A^2 \\
E_{s_3} &= \frac{3}{4}A^2
\end{aligned}$$

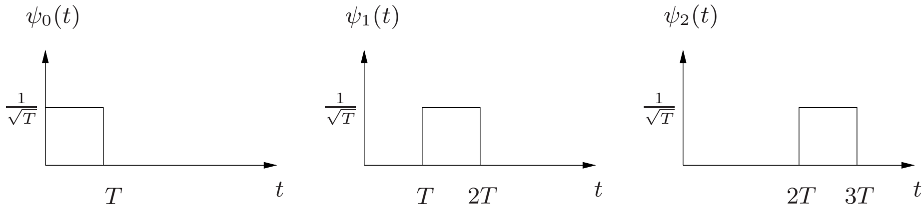
Since the signals are equally likely, we get an average energy per symbol $\bar{E}_s = \frac{17}{12}A^2$. The average transmitted power is then $\bar{P} = \frac{\bar{E}_s}{T} = \frac{17A^2}{12T}$. Due to the transmit power constraint $\bar{P} \leq P$, we get a constraint on the symbol-rate:

$$R = \frac{1}{T} \leq \frac{12P}{17A^2}$$

Plugging in the expression for A^2 from above gives the (approximate) bound

$$R \leq \frac{24P}{3.9^2 \cdot 17N_0}$$

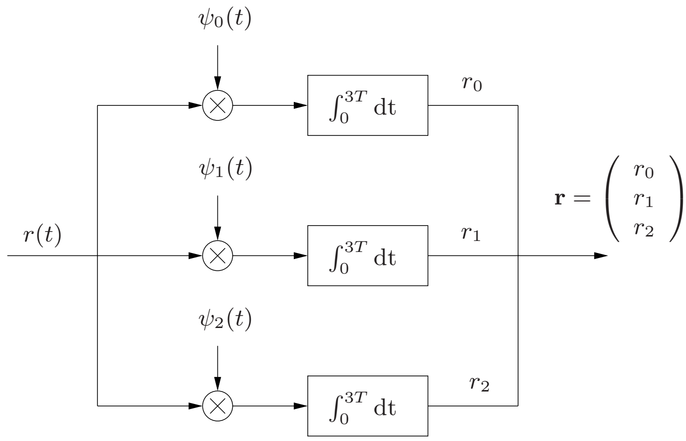
2–43 (a) The three waveforms can be represented by the three orthonormal basis waveforms $\psi_0(t)$, $\psi_1(t)$ and $\psi_2(t)$ below.



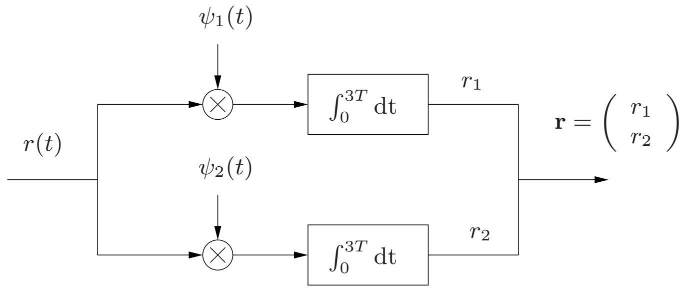
Orthonormal basis waveforms could also have been found with Gram-Schmidt orthonormalization. The waveforms can be written as linear combinations of the basis waveforms as

$$
\begin{aligned}
x_0(t) &= B\left(\psi_0(t)\right) \\
x_1(t) &= B\left(\psi_0(t)+\psi_1(t)\right) \\
x_2(t) &= B\left(\psi_0(t)+\psi_1(t)+\psi_2(t)\right)
\end{aligned}
$$

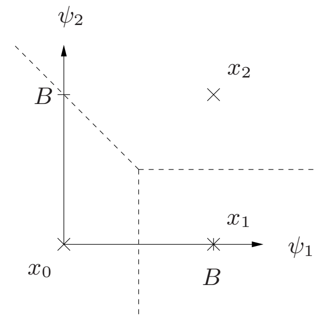
where $B = A\sqrt{T}$. The optimal demodulator for the AWGN channel is either the correlation demodulator or the matched filter. Using the basis waveforms above, the correlation demodulator is depicted in the figure below.



Note that the decision variable r_0 is equal for all transmitted waveforms. Consequently, it contains no information about which waveform was transmitted, so it can be removed.



(b) Since the transmitted waveforms are equiprobable and transmitted over an AWGN channel, the Maximum Likelihood (ML) decision rule minimizes the symbol error probability $\Pr(\hat{x} \neq x) = \Pr\left(\hat{I} \neq I\right)$, where x denotes the constellation point of the transmitted signal. The signal constellation with decision boundaries looks as follows.



From the figure, the detection rules are

$$\begin{aligned}
\psi_1 \le \frac{B}{2} \text{ and } \psi_1+\psi_2 < B \quad &\Rightarrow \quad x_0 \text{ was transmitted } (\hat{I}=0). \\
\psi_1 > \frac{B}{2} \text{ and } \psi_2 < \frac{B}{2} \quad &\Rightarrow \quad x_1 \text{ was transmitted } (\hat{I}=1). \\
\text{Otherwise} \quad &\Rightarrow \quad x_2 \text{ was transmitted } (\hat{I}=2).
\end{aligned}$$

(c) The distances between the constellation values $d_{i,k}$ can be obtained from the constellation figure above. A lower and upper bound on the symbol error probability is given by

$$\frac{1}{3}\sum_{i=0}^{2} Q\left(\frac{\min d_{i,k}}{\sqrt{2N_0}}\right) \le \Pr\left(\hat{I} \neq I\right) \le \sum_{i=0}^{2}\sum_{k=0,k\neq i}^{2} Q\left(\frac{d_{i,k}}{\sqrt{2N_0}}\right).$$

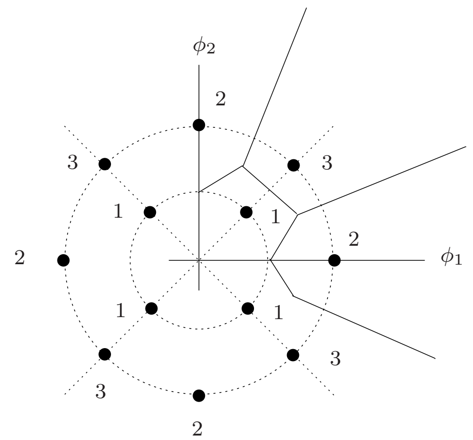
The distances are

$$\begin{aligned}
d_{0,1} = d_{1,0} = d_{1,2} = d_{2,1} &= B \\
d_{0,2} = d_{2,0} &= \sqrt{2}B
\end{aligned}$$

which gives the lower and upper bounds as ($B = A\sqrt{T}$)

$$Q\left(\frac{A\sqrt{T}}{\sqrt{2N_0}}\right) \le \Pr\left(\hat{I}\neq I\right) \le \frac{4}{3}Q\left(\frac{A\sqrt{T}}{\sqrt{2N_0}}\right) + \frac{2}{3}Q\left(\frac{A\sqrt{T}}{\sqrt{N_0}}\right)$$

2–44 All points enumerated “1” in the figure below have the same conditional error probability, and the same holds for the points labelled “2” and “3.” We can hence focus on the three points whose decision regions are scetched in the figure.



Point 1 has two nearest neighbors at distance $d_{11} = a\sqrt{2}$, two at distance $d_{12} = a\sqrt{5-2\sqrt{2}}$ and one at $d_{13} = a$. Point 2 has two neigbors at distance $d_{21} = d_{12}$ and two at $d_{23} = 4a\sin(\pi/8)$.

Finally, point 3 has two neighbors at distance $d_{32} = d_{23}$ and one at $d_{31} = d_{13}$. Hence, the union bound gives

$$
\begin{aligned}
P(\text{error}|1) &\le 2\,Q\left(\frac{d_{11}}{\sqrt{2N_0}}\right) + 2\,Q\left(\frac{d_{12}}{\sqrt{2N_0}}\right) + Q\left(\frac{d_{13}}{\sqrt{2N_0}}\right) \\
&= 2\,Q\left(\sqrt{\frac{a^2}{N_0}}\right) + 2\,Q\left(\sqrt{\frac{a^2(5-2\sqrt{2})}{2N_0}}\right) + Q\left(\sqrt{\frac{a^2}{2N_0}}\right) \\
P(\text{error}|2) &\le 2\,Q\left(\frac{d_{12}}{\sqrt{2N_0}}\right) + 2\,Q\left(\frac{d_{23}}{\sqrt{2N_0}}\right) = 2\,Q\left(\sqrt{\frac{a^2(5-2\sqrt{2})}{2N_0}}\right) + 2\,Q\left(\sin(\pi/8)\sqrt{\frac{8a^2}{N_0}}\right) \\
P(\text{error}|3) &\le 2\,Q\left(\frac{d_{23}}{\sqrt{2N_0}}\right) + Q\left(\frac{d_{13}}{\sqrt{2N_0}}\right) = 2\,Q\left(\sin(\pi/8)\sqrt{\frac{8a^2}{N_0}}\right) + Q\left(\sqrt{\frac{a^2}{2N_0}}\right)
\end{aligned}
$$

Using $a^2/N_0 = 10$ then gives

$$
\begin{aligned}
P(\text{error}|1) &\le 2\,Q\left(\sqrt{10}\right) + 2\,Q\left(\sqrt{5(5-2\sqrt{2})}\right) + Q\left(\sqrt{5}\right) \\
P(\text{error}|2) &\le 2\,Q\left(\sqrt{5(5-2\sqrt{2})}\right) + 2\,Q\left(\sin(\pi/8)\sqrt{80}\right) \\
P(\text{error}|3) &\le 2\,Q\left(\sin(\pi/8)\sqrt{80}\right) + Q\left(\sqrt{5}\right)
\end{aligned}
$$

and the overall average error probability can then be bounded as

$$
\begin{aligned}
P_e &= \frac{1}{3}(P(\text{error}|1) + P(\text{error}|2) + P(\text{error}|3)) \\
&\le \frac{1}{3}\left[2\,Q\left(\sqrt{10}\right) + 4\,Q\left(\sqrt{5(5-2\sqrt{2})}\right) + 4\,Q\left(\sin(\pi/8)\sqrt{80}\right) + 2\,Q\left(\sqrt{5}\right)\right] \approx 0.0100 < 0.011
\end{aligned}
$$

Similarly, lower bounds can be obtained as

$$
\begin{aligned}
P(\text{error}|1) &\ge Q\left(\frac{d_{13}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{5}\right) \\
P(\text{error}|2) &\ge Q\left(\frac{d_{21}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{5(5-2\sqrt{2})}\right) \\
P(\text{error}|3) &\ge Q\left(\frac{d_{31}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{5}\right)
\end{aligned}
$$

which gives

$$
P_e \ge \frac{1}{3}\left[2\,Q\left(\sqrt{5}\right) + Q\left(\sqrt{5(5-2\sqrt{2})}\right)\right] \approx 0.0086 > 0.0085
$$

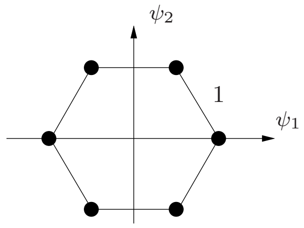
2–45 The signal set is two-dimensional, and can be described using the basis waveforms $\psi_1(t) = s_1(t)$ and

$$
\psi_2(t) = \begin{cases} +1, & 0 \le t < 1/2 \\ -1, & 1/2 \le t \le 1 \end{cases}
$$

It is straightforward to conclude that

$$
s_1(t) = \psi_1(t), \quad s_2(t) = \frac{1}{2}\,\psi_1(t) + \frac{\sqrt{3}}{2}\,\psi_2(t), \quad s_3(t) = -\frac{1}{2}\,\psi_1(t) + \frac{\sqrt{3}}{2}\,\psi_2(t)
$$

and, as given, $s_4(t) = -s_1(t)$, $s_5(t) = -s_2(t)$, $s_6(t) = -s_3(t)$, resulting in the signal space diagram shown below.



As we can see, the constellation is equivalent to 6-PSK with signal energy 1, and any two neighboring signals are at distance $d = 1$. The union bound technique thus gives

$$P_e < 2\,Q\left(\frac{d}{\sqrt{2N_0}}\right) = 2\,Q\left(\frac{1}{\sqrt{2N_0}}\right)$$

noting the symmetry of the constellation and the fact that, as in the case of general M-PSK, only the two nearest neighbors need to be accounted for. The lower bound is obtained by only counting one of the two terms in the union bound.

2–46 The signals are orthogonal if the inner product is zero.

$$\begin{aligned}\int_0^T \sin(2\pi ft)\cos(2\pi ft)dt &= \frac{1}{2}\int_0^T \underbrace{\sin(4\pi ft)}_{\text{Period } \frac{1}{2f}}dt = \\ &\underbrace{\frac{1}{2}\int_0^{\frac{K}{2f}} \sin(4\pi ft)dt}_{=0} + \underbrace{\frac{1}{2}\int_{\frac{K}{2f}}^T \sin(4\pi ft)dt}_{\approx 0} \approx 0\end{aligned}$$

where K is the number of whole periods of the sinusoid within the time T. The residual time $T - \frac{K}{2f}$ is very small if $f \gg 1/T$.

2–47 The received signal is $r(t) = \pm\sqrt{\frac{2E}{T}}\cos(2\pi f_c t) + n(t)$, where $n(t)$ is AWGN with p.s.d. $N_0/2$. The input signal to the detector is

$$r_0 = \int_0^T r(t)\cos(2\pi(f_c + \Delta f)t)dt = \ldots \approx \pm\sqrt{\frac{E}{2T}}\frac{\sin(2\pi\Delta fT)}{2\pi\Delta f} + n_0\ ,$$

where n_0 is independent Gaussian with variance $\mathrm{E}\left\{n_0^2\right\} = T/2 \cdot N_0/2$. We see that the SNR degrades as the frequency offset increases (initially). We have

$$P\,(\text{error}) = P\left(\sqrt{\frac{E}{2T}}\frac{\sin(2\pi\Delta fT)}{2\pi\Delta f} + n_0 < 0\right) = Q\left(\sqrt{\frac{2E}{N_0}}\frac{\sin(2\pi\Delta fT)}{2\pi\Delta fT}\right)$$

If $\Delta f = 0$, $P\,(\text{error}) = Q\left(\sqrt{\frac{2E}{N_0}}\right)$ and if $\Delta f = 1/2T$, $P\,(\text{error}) = 0.5$. Note that it is possible to obtain an error probability of more than 0.5 for $1/2T < \Delta f < 1/T$.

2–48 (a) In the analogue system it is a good approximation to say that each amplifier compensates for the loss of the attenuator, i.e. the gain of each amplifier is 130 dB. Consider the signal power S_f and the noise power N_f just before the final amplifier.

$$N_f = 3\left(kTB * 10^{5/10}\right)$$

$$S_f = P/10^{130/10}$$

The signal to noise ratio needs to be at least 10 dB so the minimum required power can be determined by solving the following equation.

$$\frac{S_f}{N_f} = 10 = \frac{P/10^{130/10}}{3\left(kTB * 10^{5/10}\right)}$$

$P = 11.4$ mW or -19.4 dBW.

(b) In the digital system, the maximum allowable error rate is 1×10^{-3} therefore each stage can only add approximately 3.3×10^{-4} to the error rate. Using a figure from the textbook it can be seen that an E_b/N_0 of 7.6 dB is required at each stage for BPSK.

$$P = E_b * 64000 * 10^{130/10}$$

The energy per bit E_b/N_0 is measured just before the demodulator.

$$N_0 = kT \times 10^{5/10}$$

Solving for P results in $P = 47$ mW or -13.3 dBW.

(c) Using the values given, which are reasonable, the digital system is less efficient than the analogue system. The digital system could be improved by using **speech compression coding**, code excited linear prediction (CELP) would allow the voice to be reproduced with a data rate of approximately only 5 kbps. **Error control coding** would also be able to improve the power efficiency of the system. Integrated circuits for 64 state Viterbi codecs are readily available.

2–49 (a) The transmitted signals:

$$\begin{aligned} s_1(t) &= g(t)\cos(2\pi f_c t) \\ s_2(t) &= -g(t)\cos(2\pi f_c t) \end{aligned}$$

The received signal is

$$r(t) = \pm g(t)\cos(2\pi f_c t) + n(t) \tag{3.9}$$

The demodulation follows

$$\begin{aligned} & \int_0^T r(t)\cos(2\pi f_c t + \Delta\phi)dt \\ = & \pm \int_0^T g(t)\frac{1}{2}(\cos(4\pi f_c t + \Delta\phi) + \cos(\Delta\phi))dt \\ & + \int_0^T n(t)\cos(2\pi f_c t + \Delta\phi)dt \\ \approx & \pm\frac{AT}{2}\cos(\Delta\phi) + \bar{n} \end{aligned}$$

where the integration of double carrier frequency term approximates to 0. For the zero-mean Gaussian noise term $\bar{n}$ we get

$$\begin{aligned} E[\bar{n}^2] &= E\int_0^T\int_0^T \bar{n}(t)\bar{n}(s)\cos(2\pi f_c t + \Delta\phi)\cos(2\pi f_c s + \Delta\phi)dtds \\ &\approx \frac{N_0 T}{4} \end{aligned}$$

Hence, the symbol error probability is obtained as

$$Q\left(\sqrt{\frac{A/2T\cos(\Delta\phi) - (-A/2T\cos(\Delta\phi))}{\sqrt{N_0T/4}}}\right) = Q\left(\sqrt{(\cos^2\Delta\phi)\frac{2E_b}{N_0}}\right)$$

(b)

$$E_b = E_s = \frac{1}{2}A^2T = 2 \times 10^{-9}$$

For the system without carrier phase error, the error probability is

$$Q(\sqrt{\frac{2E_b}{N_0}}) = 3.87 \times 10^{-6}$$

For the system with carrier phase error, the error probability is

$$Q(\sqrt{(\cos^2\Delta\phi)\frac{2E_b}{N_0}}) = 9.73\times 10^{-6}$$

2–50 The bit error probability for Gray encoded QPSK in AWGN is

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right).$$

A table for the Q-function, for instance in the textbook, gives that for a bit error probability of 0.1,

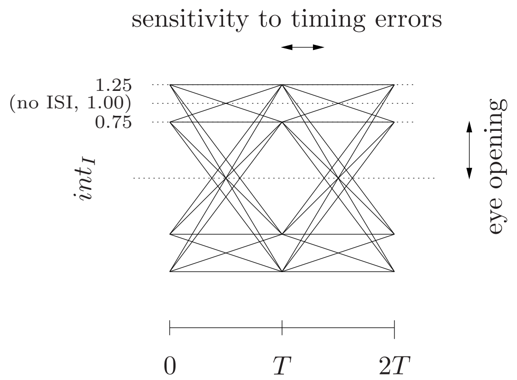
$$\sqrt{\frac{2E_b}{N_0}} \approx 1.3$$

which gives an SNR

$$10\log_{10}\frac{2E_b}{N_0} \approx 2.3 \text{ dB}.$$

2–51 (a) Due to the Doppler effect, the transmitter frequency seen by the receiver is higher than the receiver frequency. This frequency difference can be seen as a time-varying phase offset, i.e., the signal constellation in the receiver rotates with $2\pi f_{\mathrm{D}}T \approx 3.6°$ per symbol duration.

(b) This is an eye diagram with ISI.

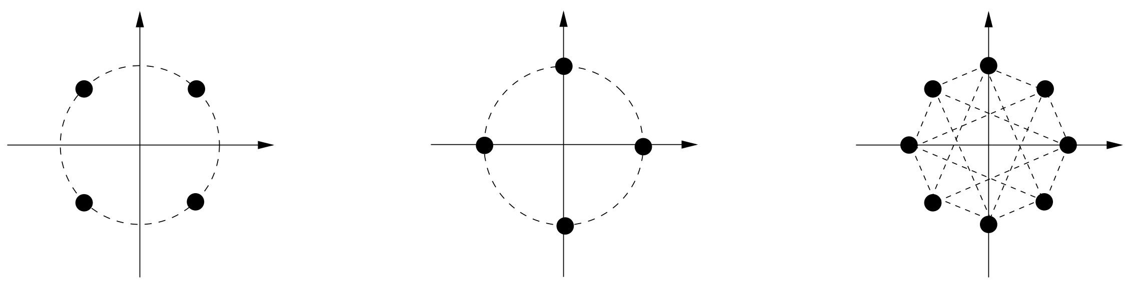


(c) An equalizer would help, e.g., an zero-forcing equalizer (if the noise level is not too high) or an MLSE equalizer. The equalizer would be included in the decision block, either as a filter preceding the two threshold devices (zero-forcing) or replacing the threshold devices completely (MLSE equalizer).

(d) The decision devices must be changed. In a 8-PSK system, the information about each of the three bits in a symbol is present on both the I and Q channels. Hence, the two threshold devices must be replaced by a decision making block taking two soft inputs (I and Q) and producing three hard bit outputs.

(e) Depends on the specific application and channel. PAM would probably be preferred over orthogonal modulation as the system is band-limited but (hopefully) not power-limited.

2–52 The increase in phase offset by $\pi/4$ between every symbol can be seen as a rotation of the signal constellation as illustrated below.



Since the receiver has knowledge of this rotation, it can simply derotate the constellation before detection. Hence, the $\pi/4$-QPSK modulation technique has the same bit and symbol error probabilities as ordinary QPSK.

One reason for using $\pi/4$-QPSK instead of ordinary QPSK is illustrated to the far right in the figure. The transmitted signal can have any of eight possible phase values, but at each symbol interval, only four of them are possible. The phase trajectories, shown with dashed lines, does not pass through the origin for $\pi/4$-QPSK, which is favorable for reducing the requirements on the power amplifiers in the transmitter.

Another reason for using $\pi/4$-QPSK is to make the symbol timing recovery circuit more reliable. An ordinary QPSK modulator can output a carrier wave if the input data is a long string of zeros thus making timing acquisition impossible. A $\pi/4$-QPSK modulator always has phase transitions in the output regardless of the input which is good for symbol timing recovery.

2–53 An important observation to make is that the basis functions used in the receiver are *non-orthogonal*. Consequently, the noise at the two outputs y_1 and y_2 from the front end, denoted n_1 and n_2, are *not* independent. The correlation matrix for the noise output $\mathbf{n} = [n_1\ n_2]^T$ can easily be show to be

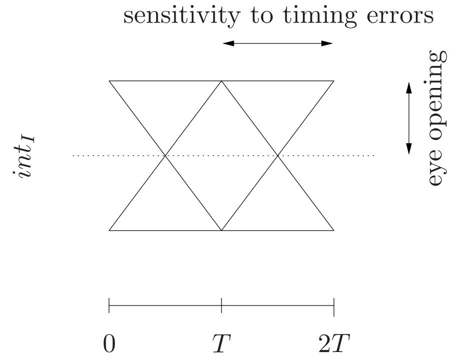
$$\mathbf{R} = E\{\mathbf{n}\mathbf{n}^*\} = \frac{N_0}{2}\begin{bmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{bmatrix}$$

Despite this, it is possible to derive an ML receiver. A simple way of approaching the problem is to transform it into a well-known form, e.g., a two-dimensional output with orthonormal basis functions and independent noise samples. Once this is done, the conventional decision rules as given in any text book can be used. The outputs $\mathbf{y} = [y_1\ y_2]^T$ can be transformed by

$$\mathbf{y}' = \mathbf{T}\mathbf{y} = \begin{bmatrix} 1 & 0 \\ 1 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

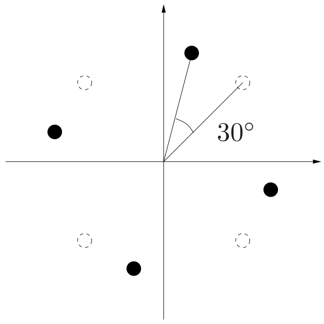
It is easy to verify that this will result in independent noise in y_1' and y_2', i.e., $E\{\mathbf{T}\mathbf{n}\mathbf{n}^*\mathbf{T}^*\} \propto \mathbf{I}$, and a square signal constellation after transformation. To summarize, the decision block should consist of the transformation matrix $\mathbf{T}$ followed by a conventional ML decision rule, i.e., choose nearest signal point.

2–54 (a) This is an eye diagram (the eye opening is sometimes called noise margin).



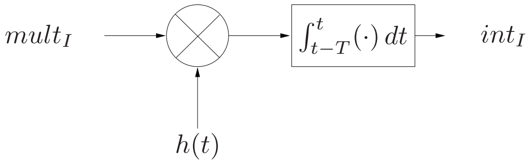
(b) A coherent receiver requires a phase reference, while a non-coherent receiver operates without a phase reference. Assuming a reliable phase reference, the coherent scheme outperforms the non-coherent counterpart (compare, for example, PSK and differentially demodulated DPSK in the textbook). Hence, the coherent receiver is preferable at high SNRs when a reliable phase estimate is present. At low SNRs, the non-coherent scheme is probably preferable, but a detailed analysis is necessary to investigate whether the advantage of not using an (unreliable) phase estimate outweighs the disadvantage of the higher error probability in the non-coherent scheme.

(c) This causes the rotation of the signal constellation as shown in the figure below (unfilled circles: without phase error, filled circles: 30° phase error). Depending on the sign of the phase error, the constellation rotates either clockwise or counterclockwise.



(d) Reducing the filter bandwidth without reducing the signaling rate will cause intersymbol interference in $d_I(n)$ and $d_Q(n)$.

(e) The integrators in the figure must be changed as (similar for the Q channel)



The BP filter might need some adjustment as well, depending on the bandwidth of the pulse, but this is probably less critical.

2–55 Denote the *transmitted* power in the Q channel with P_{Q}. The power is attenuated by 10 dB, which equals 10 times, in the channel. Hence, the received bit energy is

$$E_{\mathrm{b,Q}} = P_{\mathrm{Q}}T/10 \ ,$$

where $T = 1/4000$ (4 kbit/s bit rate). The bit error probability is given by

$$10^{-3} = Q\left(\sqrt{\frac{E_{\mathrm{b,Q}}}{R_0}}\right) = Q\left(\sqrt{\frac{P_{\mathrm{Q}}T/10}{R_0}}\right) \ ,$$

from which $P_{\mathrm{Q}} = 382\ \mu$W is solved. For the I channel, which is active 70% of the total time (30% of the time, nothing is transmitted and thus $P_{\mathrm{I}} = 0$), the data rate is 8 kbit/s (bit duration $T/2$) when active. Hence,

$$10^{-3} = 0.7Q\left(\sqrt{\frac{(P_{\mathrm{I}}/10)(T/2)/4}{R_0}}\right) \ .$$

From this, $P_{\mathrm{I}} = 712\ \mu$W is obtained. Hence, the average transmitted power in the I and Q channels are

$$\bar{P}_{\mathrm{I}} = 0.7P_{\mathrm{I}} + 0.3 \cdot 0 = 498\,\mu\mathrm{W}$$
$$\bar{P}_{\mathrm{Q}} = P_{\mathrm{Q}} = 382\,\mu\mathrm{W}$$

2–56 The received signal is $r(t) = s_m(t) + n(t)$, where $s_m(t) = \pm a\sqrt{\frac{2E}{T}}\cos(2\pi f_c t) \pm b\sqrt{\frac{2E}{T}}\sin(2\pi f_c t)$ is the transmitted signal and $n(t)$ is AWGN with p.s.d. $N_0/2$. The input signals to the detector are

$$r_0 = \int_0^T r(t)A\cos(2\pi f_c t)dt = \ldots \approx \pm aA\sqrt{\frac{ET}{2}} + n_0$$
$$r_1 = -\int_0^T r(t)B\sin(2\pi f_c t)dt = \ldots \approx \pm bB\sqrt{\frac{ET}{2}} + n_1 \,,$$

where n_0 and n_1 are independent Gaussian since

$$E\left\{n_0 n_1\right\} = \int_{t=0}^{T}\int_{s=0}^{T} AB\sin(2\pi f_c t)\cos(2\pi f_c s)\frac{N_0}{2}\delta(t-s)dtds \approx 0 \,,$$

$$
\begin{aligned}
E\left\{n_0^2\right\} &= \int_{t=0}^{T}\int_{s=0}^{T} A^2\cos(2\pi f_c t)\cos(2\pi f_c s)\frac{N_0}{2}\delta(t-s)dtds \approx A^2\frac{T}{2}\frac{N_0}{2}\,,\\
E\left\{n_1^2\right\} &= \int_{t=0}^{T}\int_{s=0}^{T} B^2\sin(2\pi f_c t)\sin(2\pi f_c s)\frac{N_0}{2}\delta(t-s)dtds \approx B^2\frac{T}{2}\frac{N_0}{2}\,.
\end{aligned}
$$

As all signals are equally probable, and the noise is Gaussian, the detector will select signal alternative closest to the received vector (r_0, r_1). (Draw picture !). Thus, if $r_0 > 0$ and $r_1 > 0$, the alternative $+(\dots)+(\dots)$ is selected, if $r_0 < 0$ and $r_1 > 0$, $-(\dots)+(\dots)$ is selected etc. Due to the symmetry, the error will not depend on the signal transmitted, therefore, assume that $s_1(t) = a\sqrt{\frac{2E}{T}}\cos(2\pi f_c t) + b\sqrt{\frac{2E}{T}}\sin(2\pi f_c t)$ is the transmitted signal. Then

$$
\begin{aligned}
\Pr(\text{error}) &= \Pr(\text{error}|s_1)\\
&= 1 - \Pr(\text{no error}|s_1) = 1 - \Pr\left(aA\sqrt{ET/2} + n_0 > 0,\ bB\sqrt{ET/2} + n_1 > 0\right)\\
&= 1 - \left(1 - Q\left(a\sqrt{\frac{2E}{N_0}}\right)\right)\left(1 - Q\left(b\sqrt{\frac{2E}{N_0}}\right)\right)
\end{aligned}
$$

2–57 (a) The mapping between the signals and the noise is: $r(t) \leftrightarrow$ Signal B, $r_{\mathrm{BP}}(t) \leftrightarrow$ Signal C, and $mult_Q(t) \leftrightarrow$ Signal A.

This can be seen by noting that

- the noise at the receiver is white and its spectrum representation should therefore be constant before the filter.
- the bandpass filter is assumed to filter out the received signal around the carrier frequency.
- after multiplication with the carrier the signal will contain two peaks, one at zero frequency and one at the double carrier frequency.

From the spectra in Signal C it is easily seen that the carrier frequency is approximately 500 Hz.

(b) First, note that a phase offset between the transmitter and the receiver rotates the signal constellation, except from this the constellation looks like it does when all parameters are assumed to be known. A frequency offset rotates the signal constellation during the complete transmission, making an initial QPSK constellation look like a circle. Ideally, when the SNR is high and the ISI is negligible, a received QPSK signal is four discrete points in the diagram. When ISI is present, several symbols interfere and several QPSK constellations can be seen simultaneously. From this we conclude that

Condition 1 $\leftrightarrow$ Constellation C	Condition 2 $\leftrightarrow$ Constellation A
Condition 3 $\leftrightarrow$ Constellation B	Condition 4 $\leftrightarrow$ Constellation D.

Thus, the signaling scheme is QPSK.

2–58 The received signal is $r(t) = s_m(t) + n(t)$, where $m \in \{1,2,3,4\}$ and $n(t)$ is AWGN with spectral density $N_0/2$. The detector receives the signals

$$
\begin{aligned}
r_0 &= \int_0^T r(t)\cos(2\pi f_c t + \hat{\phi})dt = \ldots \approx \sqrt{\frac{ET}{2}}\cos\left(\frac{2\pi}{4}m - \frac{\pi}{4} + \phi - \hat{\phi}\right) + n_0\\
r_1 &= -\int_0^T r(t)\sin(2\pi f_c t + \hat{\phi})dt = \ldots \approx \sqrt{\frac{ET}{2}}\sin\left(\frac{2\pi}{4}m - \frac{\pi}{4} + \phi - \hat{\phi}\right) + n_1
\end{aligned}
$$

where n_0 and n_1 are independent zero-mean Gaussian with variance

$$
E\left\{n_0^2\right\} = E\left\{n_1^2\right\} = \int_{t=0}^{T}\int_{s=0}^{T}\cos(2\pi f_c t + \hat{\phi})\cos(2\pi f_c s + \hat{\phi})\frac{N_0}{2}\delta(t-s)dtds \approx \frac{T}{2}\frac{N_0}{2}\,.
$$

Since all signal alternatives are equally probable the optimal detector is the defined by the nearest-neighbor rule. That is, the detector decides s_1 if $r_0 > 0$ and $r_1 > 0$, s_2 is $r_0 < 0$ and $r_1 > 0$, and so on. The impact of an estimation error is equivalent to rotating the signal constellation. Symmetry gives $P(\text{error}) = P(\text{error}|s_1 \text{ sent})$, and we get

$$\begin{aligned} P\left(\text{error}\right) &= P\left(\text{error}|s_1\right) = 1 - P\left(\text{correct}|s_1\right) \\ &= 1 - P\left(\sqrt{ET/2}\cos(\pi/4 + \phi - \hat{\phi}) + n_0 > 0, \sqrt{ET/2}\sin(\pi/4 + \phi - \hat{\phi}) + n_1 > 0\right) \\ &= 1 - \left(1 - Q\left(\sqrt{\frac{2E}{N_0}}\cos\left(\frac{\pi}{4} + \phi - \hat{\phi}\right)\right)\right)\left(1 - Q\left(\sqrt{\frac{2E}{N_0}}\sin\left(\frac{\pi}{4} + \phi - \hat{\phi}\right)\right)\right) \end{aligned}$$

2–59 In order to avoid interference, the four carriers in the multicarrier system should be chosen to be orthogonal and, in order to conserve bandwidth, as close as possible. Hence,

$$\Delta_f = f_i - f_j = \frac{1}{T} = \frac{R}{8} \ .$$

In the multicarrier system, each carrier carries symbols (of two bits each) with duration $T = 8/R$ and an average power of $P/4$. A symbol error occurs if one (or more) of the sub-channels is in error. Assuming no inter-carrier interference,

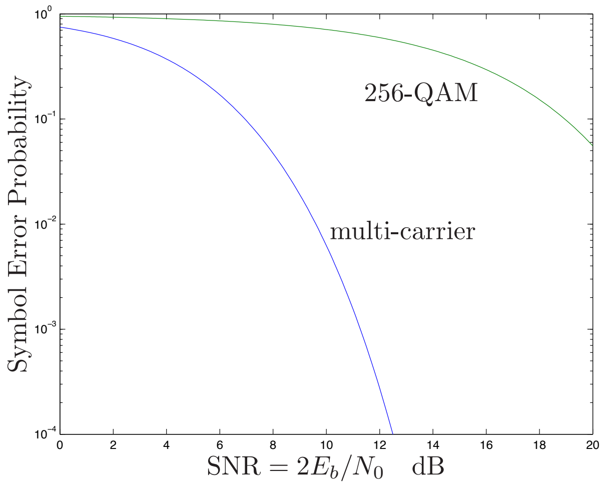
$$P_e = 1 - \left[1 - Q\left(\sqrt{\frac{2P}{RN_0}}\right)\right]^8$$

The 256-QAM system transmits 8 bits per symbol, which gives the symbol duration $T = 8/R$. The average power is P since there is only one carrier. The symbol error probability for rectangular 256-QAM is

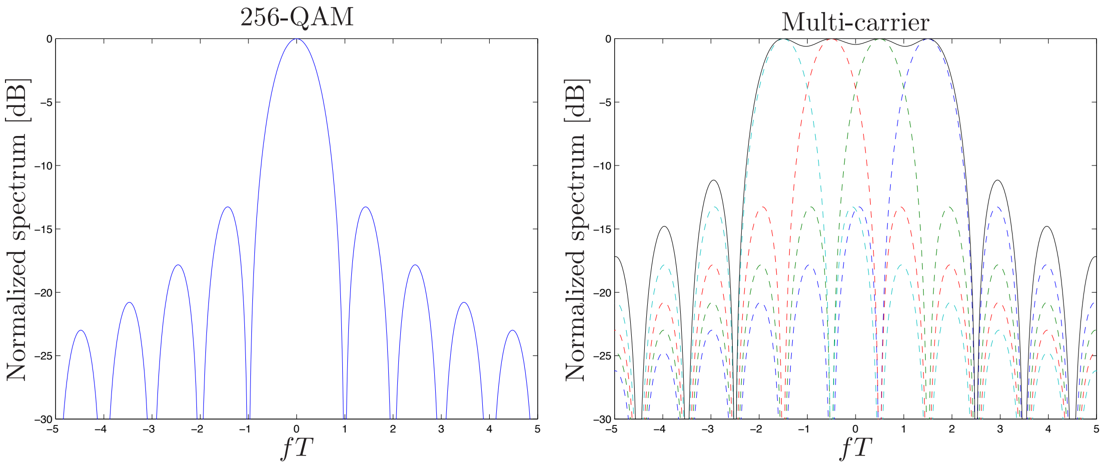
$$\begin{aligned} P_e &= 1 - (1 - p)^2 \\ p &= 2\left(1 - \frac{1}{\sqrt{256}}\right) Q\left(\sqrt{\frac{3}{256 - 1}\frac{8P}{RN_0}}\right) \end{aligned}$$

For a (time-varying) unknown channel gain, the 256-QAM system is less suitable since the amplitude is needed when forming the decision regions. This is one of the reasons why high-order QAM systems is mostly found in stationary environment, for example cable transmission systems.

Using $E_b = PT_b = P/R$ and $\text{SNR} = 2E_b/N_0$, the error probability as a function of the SNR is easily generated.



The power spectral density is proportional to $[\sin(\pi f T)/(\pi f T)]^2$ in both cases and the following plots are obtained (four QPSK spectra, separated in frequency by $1/T$).



Additional comments: multicarrier systems are used in several practical application. For example in the new Digital Audio Broadcasting (DAB) system, OFDM (Orthogonal Frequency Division Multiplex) is used. One of the main reasons for splitting a high-rate bitstream into several (in the order of hundreds) low-rate streams is equalization. A low-rate stream with relatively large symbol time (much larger than the *coherence time* (a concept discussed in 2E1435 Communication Theory, advanced course) and hence relatively little ISI) is easier to equalize than a single high-rate stream with severe ISI.

2–60 Beacuse of the symmetry of the problem, we can assume that the symbol $s_0(t)$ was transmitted and look at the probabilty of choosing another symbol at the receiver. Given that $i = 0$ (assuming the four QPSK phases are $\pm\pi/4, \pm 3\pi/4$), standard theory from the course gives

$$r_{cm} = a_m \sqrt{E/2} + w_{cm}, \quad r_{sm} = a_m \sqrt{E/2} + w_{sm}$$

where w_{cm} and w_{sm}, $m = 1, 2$, are independent zero-mean Gaussian variables with variance $N_0/2$. Letting $\mathbf{w}_m = (w_{cm}, w_{sm})$, the detector is hence fed the vector

$$\mathbf{u} = (u_1, u_2) = (b_1 a_1 + b_2 a_2)\sqrt{E/2}\,(1, 1) + b_1\mathbf{w}_1 + b_2\mathbf{w}_2$$

where we assume $b_1 > 0$ and $b_2 > 0$ (even if b_1 and b_2 are unknown, and to be determined, it is obvious that they should be chosen as positive). Given that $s_0(t)$ was transmitted, an ML decision based on $\mathbf{u}$ will be incorrect if at least one of the components u_1 and u_2 of $\mathbf{u}$ is negative (i.e., if $\mathbf{u}$ is not in the first quadrant). We note that u_1 and u_2 are independent and Gaussian with $E[u_1] = E[u_2] = (b_1 a_1 + b_2 a_2)\sqrt{E/2}$ and $\text{Var}[u_1] = \text{Var}[u_2] = (b_1^2 + b_2^2)N_0/2$. That is, we get

$$\begin{aligned}
P_e &= \Pr(\text{error}) = 1 - \Pr(\text{correct}) = 1 - \Pr(u_1 > 0, u_2 > 0) \\
&= 1 - \left[Q\left(\frac{-(b_1a_1 + b_2a_2)\sqrt{E}}{\sqrt{(b_1^2 + b_2^2)N_0}}\right)\right]^2 = 1 - \left[1 - Q\left(\frac{(b_1a_1 + b_2a_2)\sqrt{E}}{\sqrt{(b_1^2 + b_2^2)N_0}}\right)\right]^2 \\
&= 2Q\left(\frac{(b_1a_1 + b_2a_2)\sqrt{E}}{\sqrt{(b_1^2 + b_2^2)N_0}}\right) - \left[Q\left(\frac{(b_1a_1 + b_2a_2)\sqrt{E}}{\sqrt{(b_1^2 + b_2^2)N_0}}\right)\right]^2
\end{aligned}$$

We thus see that P_e is minimized by choosing b_1 and b_2 such that the ratio

$$\lambda \triangleq \frac{(b_1a_1 + b_2a_2)\sqrt{E}}{\sqrt{(b_1^2 + b_2^2)N_0}}$$

is maximized (for fixed E and N_0 and given values of a_1 and a_2). Since

$$\lambda = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|}\sqrt{\frac{E}{N_0}},$$

with $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$, we can apply Schwartz' inequality ($\mathbf{x}\cdot\mathbf{y} \le \|\mathbf{x}\|\,\|\mathbf{y}\|$ with equality only when $\mathbf{x} = (\text{positive constant}) \cdot \mathbf{y}$) to conclude that the optimal b_1 and b_2 are obtained when $\mathbf{b} = k\,\mathbf{a}$, where $k > 0$ is an arbitrary constant. This choice then gives

$$P_{e,\min} = 2Q\left(\sqrt{\frac{(a_1^2 + a_2^2)E}{N_0}}\right) - \left[Q\left(\sqrt{\frac{(a_1^2 + a_2^2)E}{N_0}}\right)\right]^2$$

The diversity combining method discribed in the problem, with weighting $b_1 = k\,a_1$ and $b_2 = k\,a_2$ for known or estimated values of a_1 and a_2, is called *maximal ratio combining* (since the ratio λ is maximized), and is frequently utilized in practice to obtain diversity gain in radio communications. In practice, the different signals $r_m(t)$ do not have to correspond to different receiver antennas, but can instead obtained e.g. by transmitted the same information at different carrier frequencies or different time-slots.

2–61 The correct pairing of the measurement signals and the system setup is

5A is the time-discrete signal at the oversampling rate at the output of the matched filter but before symbol rate sampling.

6B is an IQ-plot of the symbols after matched filtering and sampling at the symbol rate but before the phase correction.

7C is an IQ-plot of the received symbols after phase correction. However, the phase correction is not perfect as there is a residual phase error. This is the main contribution to the bad performance seen in the BER plot and an improvement of the phase estimation algorithm is suggested (on purpose, an error was added in the simulation setup to get this plot).

4D shows the received signal, including noise, before matched filtering.

The plot E is called an eye diagram and in this case we can guess that ISI is not present in the system considered. The plot was generated by using plotting the signal in point 5, but with the inclusion of the phase correction from the phase estimator (point 11). If the phase offset was not removed from the signal before plotting, the result would be rather meaningless as phase offsets in the channel would affect the result as well as any ISI present. This would only be meaningful in a system without any phase correction.

2–62

(a) Since the two signal sets have the same number of signal alternatives L and hence the same data rate, the peak signal power is proportional to the peak signal energy. The peak energy $\hat{E}_A$ of the PAM system obeys

$$\sqrt{\hat{E}_A} = \frac{L-1}{2} d_A$$

In system B we have

$$\frac{d_B}{2} = \sqrt{\hat{E}_B}\sin(\pi/L)$$

Setting $d_A = d_B$ and solving for the peak energies gives

$$\frac{\hat{P}_A}{\hat{P}_B} = \frac{\hat{E}_A}{\hat{E}_B} = \left[(L-1)\sin(\pi/L)\right]^2 \approx \pi^2$$

where the approximate equality is valid for large L since then $L-1 \approx L$ and $\sin(\pi/L) \approx \pi/L$. This means that system A needs about 9.9 dB higher peak power to achieve the same symbol error rate performance as system B.

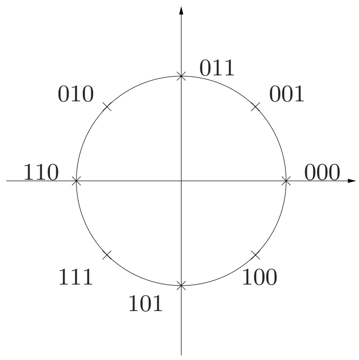
(b) The average signal power is in both cases proportional to the average energy. In the case of the PAM system (system A) the transmitted signal component is approximately uniformly distributed in the interval $[-\sqrt{\hat{E}_A}, \sqrt{\hat{E}_A}]$, since L is large and since transmitted signals are equally probable. Hence the mean energy of system A is $\bar{E}_A \approx \hat{E}_A/3$. In system B, on the other hand, all signals have the same energy so we have $\bar{E}_B = \hat{E}_B$. Hence using the results from (a) we have that in the case $d_A = d_B$ it holds

$$\frac{\bar{P}_A}{\bar{P}_B} = \frac{\bar{E}_A}{\bar{E}_B} \approx \frac{\pi^2}{3}$$

This means that the PAM system needs about 5.2 dB higher mean power to achieve the same symbol error rate performance as the PSK system.

2–63 Since the three independent bits are equally likely, the eight different symbols will be equally likely. The assumption that E/N_0 is large means that when the wrong symbol is detected in the receiver (symbol error), it is a neighboring symbol in the constellation.

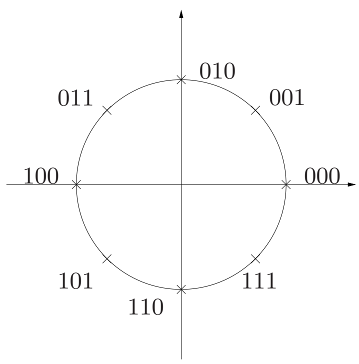
Let's name the three bits that are mapped to a symbol $b_3b_2b_1$ and let's assume that the *symbol* error probability is P_e.
The Gray-code that differs in only one bit between neighboring symbols looks like



The error probabilities for the three bits are

$$\begin{aligned}
P_{b1} = &\ \tfrac{1}{8}(\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2})P_e = \frac{P_e}{2}\\
P_{b2} = &\ \tfrac{1}{8}(0+\tfrac{1}{2}+\tfrac{1}{2}+0+0+\tfrac{1}{2}+\tfrac{1}{2}+0)P_e = \frac{P_e}{4}\\
P_{b3} = &\ \tfrac{1}{8}(\tfrac{1}{2}+0+0+\tfrac{1}{2}+\tfrac{1}{2}+0+0+\tfrac{1}{2})P_e = \frac{P_e}{4}
\end{aligned}$$

It can be seen that the mapping that yields the largest ratio between the maximum and minimum error probabilites for the three different bits is the regular binary representation of the numbers 0-7.



The right-most bit (b_1) changes between all neighboring symbols, whereas the left-most bit (b_3) changes only between four neighbors, which is the minimum in this case.

The error probabilities for b_1 and b_3 are

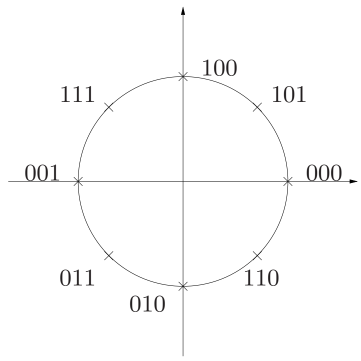
$$
\begin{aligned}
P_{b1} = &\ \tfrac{1}{8}(1+1+1+1+1+1+1+1)P_e = & P_e \\
P_{b3} = &\ \tfrac{1}{8}(\tfrac{1}{2}+0+0+\tfrac{1}{2}+\tfrac{1}{2}+0+0+\tfrac{1}{2})P_e = & \frac{P_e}{4}
\end{aligned}
$$

which gives the ratio

$$
\frac{P_{b1}}{P_{b3}} = 4
$$

Of course, various permutations of this mapping give the same result, like swapping the three bits, or rotating the mapping in the constellation.

A mapping the yields equal error probabilites for the bits can be found by clever trial and error. One example is



The error probabilities for the bits are

$$
\begin{aligned}
P_{b1} = &\ \tfrac{1}{8}(\tfrac{1}{2}+1+1+\tfrac{1}{2}+0+\tfrac{1}{2}+\tfrac{1}{2}+0)P_e = & \frac{P_e}{2} \\
P_{b2} = &\ \tfrac{1}{8}(\tfrac{1}{2}+0+\tfrac{1}{2}+1+1+\tfrac{1}{2}+0+\tfrac{1}{2})P_e = & \frac{P_e}{2} \\
P_{b2} = &\ \tfrac{1}{8}(1+\tfrac{1}{2}+0+\tfrac{1}{2}+\tfrac{1}{2}+0+\tfrac{1}{2}+1)P_e = & \frac{P_e}{2}
\end{aligned}
$$

Rotation of the mapping around the circle, and swapping of bits gives the same result.

2–64 Without loss of generality we can consider $n = 1$. We get

$$
\begin{aligned}
y_1 &= \int_T^{2T} g(t-T)\big(x_0 g(t-\tau) + x_1 g(t-\tau-T)\big)dt + \int_T^{2T} g(t)n(t)dt \\
&= x_0 \int_0^{\tau} g(t)g(T-\tau+t)dt + x_1 \int_{\tau}^{T} g(t)g(t-\tau)dt + w \\
&= x_0 \frac{4-\pi}{4\sqrt{2}\pi} + x_1 \frac{4+3\pi}{4\sqrt{2}\pi} + w = a\,x_0 + b\,x_1 + w
\end{aligned}
$$

with $a \approx 0.048$, $b \approx 0.755$, and where

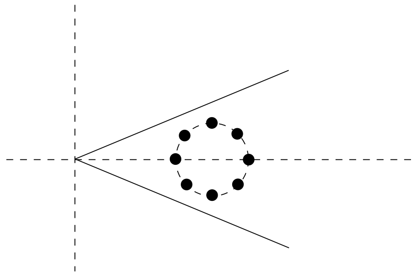
$$
g(t) = \sqrt{\frac{2}{T}}\sin(\pi t/T), \quad 0 \le t \le T
$$

and $\tau = T/4$ was used, and where w is complex Gaussian noise with independent zero-mean components of variance $N_0/2$.

Due to the symmetry of the problem, we can assume $x_1 = 1$ and compute the probability of error as

$$P_e = \Pr(\hat{x}_1 \neq x_1) = \Pr(b + a\,x_0 + w \notin \Omega_0)$$

where Ω_0 is the decision region of alternative zero. The eight equally likely different values for $b + a\,x_0$ are illustrated below (not to correct scale!). The figure also marks the boundaries of Ω_0.



When $N_0 \ll 1$, the error probability is dominated by the event that y_1 is outside Ω_0 for $x_0 = (-1+j)/\sqrt{2}$, $x_0 = j$, $x_0 = (-1+j)/\sqrt{2}$ or $x_0 = -j$, since these values for x_0 make $a + b\,x_0$ end up close to the decision boundaries. All these four alternatives are equally close to their closest decision boundary, and the distance is

$$d = b\,\sin\frac{\pi}{8} - a\,\cos\frac{\pi}{8} \approx 0.244$$

Hence, as $N_0 \to 0$ we get

$$P_e \approx \frac{1}{2} Q\left(\frac{d}{\sqrt{N_0/2}}\right) \approx \frac{1}{2} Q\left(\frac{0.244}{\sqrt{N_0/2}}\right)$$

2–65 First, substitute the expression for the received signal $r(t)$ into the expression for the decision variable $v_{0,m}$,

$$\begin{aligned} v_{0,m} &= \int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} r(t)e^{-j2\pi 500t}\,dt = \int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} (x(t) + n(t) + e^{j2\pi 2000t})e^{-j2\pi 500t}\,dt \\ &= \underbrace{\int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} x(t)e^{-j2\pi 500t}\,dt}_{s_{0,m}} + \underbrace{\int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} n(t)e^{-j2\pi 500t}\,dt}_{n_{0,m}} + \underbrace{\int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} e^{j2\pi 2000t}e^{-j2\pi 500t}\,dt}_{i_{0,m}}\ . \end{aligned}$$

The decision variable $v_{0,m}$ can be decomposed into 3 components, the signal $s_{0,m}$, the noise $n_{0,m}$ and the interference $i_{0,m}$. Examining the signal $s_{0,m}$ in branch zero results in

$$\begin{aligned} s_{0,m} &= \int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} x(t)e^{-j2\pi 500t}\,dt \\ &= \begin{cases} \int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} e^{j2\pi 500t}e^{-j2\pi 500t}\,dt = \int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} 1\,dt = \frac{1}{1000} & \text{if } d_m = 0 \\ \int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} e^{j2\pi 1500t}e^{-j2\pi 500t}\,dt = \int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} e^{j2\pi 1000t}\,dt = 0 & \text{if } d_m = 1. \end{cases} \end{aligned}$$

The noise $n_{0,m}$ is sampled Gaussian noise. Each sample with respect to m is independent as a different section of $n(t)$ is integrated to get each sample. Assuming the noise spectral density of $n(t)$ is N_0 then the energy of each sample $n_{0,m}$ can be calculated as

$$\begin{aligned} \sigma_n^2 = E\{|n_{0,m}|^2\} &= E\left\{\left|\int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} n(t)e^{-j2\pi 500t}\,dt\right|^2\right\} \\ &= N_0 \int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} \int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} \delta(t-t')e^{-j2\pi 500t}e^{j2\pi 500t'}\,dt\,dt' = \frac{N_0}{1000} \end{aligned}$$

Similarly, the interference in branch zero, $i_{0,m}$, can be calculated as

$$i_{0,m} = \int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} e^{j2\pi 2000t} e^{-j2\pi 500t}\, dt = \int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} e^{j2\pi 1500t}\, dt = -\frac{(-1)^m}{1500\pi} \ .$$

Repeating the above calculations for $v_{1,m}$ (i.e., the first branch) results in

$$s_{1,m} = \begin{cases} 0 & \text{if } d_m = 0 \\ \frac{1}{1000} & \text{if } d_m = 1. \end{cases}$$

$$i_{1,m} = \int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} e^{j2\pi 2000t} e^{-j2\pi 1500t}\, dt = \int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} e^{j2\pi 500t}\, dt = \frac{(-1)^m}{500\pi}$$

The variance of the noise $n_{1,m}$ is also $(N_0)/1000$. The correlation between $n_{1,m}$ and $n_{0,m}$ should be checked,

$$\begin{aligned} E\{n_{1,m} n_{0,m}^*\} &= E\left\{ \int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} n(t) e^{-j2\pi 1500t}\, dt \int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} n(t)^* e^{j2\pi 500t}\, dt \right\} \\ &= N_0 \int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} \int_{\frac{m-1/2}{1000}}^{\frac{m+1/2}{1000}} \delta(t-t') e^{-j2\pi 1500t} e^{j2\pi 500t'}\, dt\, dt' = 0 \ . \end{aligned}$$

Hence, $n_{1,m}$ and $n_{0,m}$ are uncorrelated.

Using the above, the decision variables $v_{0,m}$ and $v_{1,m}$ for $d_m = 0$ are

$$\begin{aligned} v_{0,m} &= \frac{1}{1000} + n_{0,m} - \frac{(-1)^m}{1500\pi} \\ v_{1,m} &= n_{1,m} + \frac{(-1)^m}{500\pi} \end{aligned}$$

and for $d_m = 1$

$$\begin{aligned} v_{0,m} &= n_{0,m} - \frac{(-1)^m}{1500\pi} \\ v_{1,m} &= \frac{1}{1000} + n_{1,m} + \frac{(-1)^m}{500\pi} \ . \end{aligned}$$

For uncorrelated Gaussian noise the optimum decision rule is

$$\hat{d_m} = \arg\min_{d_m} \left(|v_{0,m} - E\{v_{0,m}|d_m\}|^2 + |v_{1,m} - E\{v_{1,m}|d_m\}|^2 \right)$$

We calculate the distance from the received decision variables to their expected positions for each possible value of transmitted data ($d_m = 0$ or $d_m = 1$) and then chose the data corresponding to the smallest distance.

The signal components s_m and interference components i_m of the decision variables are real, so only the real parts of the decision variables need to be considered. Rewriting this formula for the given FSK system

$$\left[\begin{array}{c} (v_{0,m} - \frac{1}{1000} + \frac{(-1)^m}{1500\pi})^2 \\ +(v_{1,m} - \frac{(-1)^m}{500\pi})^2 \end{array} \right] \underset{d_m=1}{\overset{d_m=0}{\lessgtr}} \left[\begin{array}{c} (v_{0,m} + \frac{(-1)^m}{1500\pi})^2 \\ +(v_{1,m} - \frac{1}{1000} - \frac{(-1)^m}{500\pi})^2 \end{array} \right]$$

$$v_{1,m} - \frac{(-1)^m}{500\pi} \underset{d_m=1}{\overset{d_m=0}{\lessgtr}} v_{0,m} + \frac{(-1)^m}{1500\pi}$$

The decision rule can be found by simplifying the previous equation, resulting in

$$v_{1,m} - v_{0,m} \underset{d_m=1}{\overset{d_m=0}{\lessgtr}} \frac{(-1)^m}{375\pi}$$

2–66 (a) The receiver correlates with the functions $\varphi_1(t)$ and φ_2.

$$\varphi_1(t) = \begin{cases} \sqrt{\frac{2}{T}} \sin \frac{2\pi t}{T}, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases} \qquad \varphi_2(t) = \begin{cases} \sqrt{\frac{2}{T}} \sin \frac{4\pi t}{T}, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases}$$

Let $f_k(\rho) = \frac{1}{k\pi} \sin k\pi\rho$. Compute the inner products

$$(u_1(t), \varphi_1(t)) = \int_0^{\rho T} \sqrt{\frac{2E}{T}} \sin\left(\frac{2\pi t}{T}\right) dt \sqrt{\frac{2}{T}} \sin\left(\frac{2\pi t}{T}\right) = \sqrt{E}(\rho - f_4(\rho))$$

$$(u_1(t), \varphi_2(t)) = (u_2(t), \varphi_1(t)) = \int_0^{\rho T} \sqrt{\frac{2E}{T}} \sin\left(\frac{2\pi t}{T}\right) \sqrt{\frac{2}{T}} \sin\left(\frac{4\pi t}{T}\right) dt = \sqrt{E}(f_2(\rho) - f_6(\rho))$$

$$(u_2(t), \varphi_2(t)) = \int_0^{\rho T} \sqrt{\frac{2E}{T}} \sin\left(\frac{4\pi t}{T}\right) \sqrt{\frac{2}{T}} \sin\left(\frac{4\pi t}{T}\right) dt = \sqrt{E}(\rho - f_8(\rho))$$

$$\begin{aligned} d_E^2 &= ((u_1(t), \varphi_1(t)) - (u_2(t), \varphi_1(t)))^2 + ((u_1(t), \varphi_2(t)) - (u_2(t), \varphi_2(t)))^2 \\ &= E(\rho - f_2(\rho) - f_4(\rho) + f_6(\rho))^2 + E(\rho - f_2(\rho) + f_6(\rho) - f_8(\rho))^2 \\ &\Longrightarrow d_E = \sqrt{E}\sqrt{(\rho - f_2(\rho) - f_4(\rho) + f_6(\rho))^2 + (\rho - f_2(\rho) + f_6(\rho) - f_8(\rho))^2} \end{aligned}$$

(b) No, the receiver adds unnecessary noise during $\rho T \le t < T$.

2–67 When $s_0(t)$ is transmitted, the received signal can be modeled as

$$r(t) = \sqrt{\frac{2E_b}{T}} \cos(2\pi f_c t + \phi_0) + n(t)$$

Studying the coherent receiver, we thus have

$$\begin{aligned} r_0 &= \int_0^T r(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t + \hat{\phi}_0) dt \\ &= \frac{2\sqrt{E_b}}{T} \int_0^T \cos(2\pi f_c t + \phi_0) \cos(2\pi f_c t + \hat{\phi}_0) dt + \overbrace{\int_0^T n(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t + \hat{\phi}_0) dt}^{=n_0} \\ &= \frac{\sqrt{E_b}}{T} \int_0^T \cos(\phi_0 - \hat{\phi}_0) dt + \underbrace{\frac{\sqrt{E_b}}{T} \int_0^T \cos(4\pi f_c t + \phi_0 + \hat{\phi}_0) dt}_{f_c \gg 1/T \,\to\, \approx 0} + n_0 \approx \sqrt{E_b} \cos(\phi_0 - \hat{\phi}_0) + n_0 \end{aligned}$$

Similarly,

$$\begin{aligned} r_1 &= \int_0^T r(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t + 2\pi t/T + \hat{\phi}_1) dt \\ &= \frac{2\sqrt{E_b}}{T} \int_0^T \cos(2\pi f_c t + \phi_0) \cos(2\pi f_c t + 2\pi t/T + \hat{\phi}_1) dt + \overbrace{\int_0^T n(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t + 2\pi t/T + \hat{\phi}_1) dt}^{=n_1} \\ &= \frac{\sqrt{E_b}}{T} \int_0^T \cos(2\pi t/T + \hat{\phi}_1 - \phi + 0) dt + \underbrace{\frac{\sqrt{E_b}}{T} \int_0^T \cos(4\pi f_c t + 2\pi t/T + \phi_0 + \hat{\phi}_1) dt}_{f_c \gg 1/T \,\to\, \approx 0} + n_1 \approx n_1 \end{aligned}$$

The noise components are obtained as

$$n_0 = \int_0^T n(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t + \hat{\phi}_0) dt \qquad n_1 = \int_0^T n(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t + 2\pi t/T + \hat{\phi}_1) dt$$

These are independent zero-mean Gaussian with variance $N_0/2$. The error probability is hence obtained as

$$\begin{aligned}\Pr(\text{error}|s_0(t)) &= \Pr(r_0 < r_1|s_0(t)) = \Pr(\sqrt{E_b}\cos(\phi_0 - \hat{\phi}_0) + n_0 < n_1) \\ &= \Pr(\sqrt{E_b}\cos(\phi_0 - \hat{\phi}_0) < n_1 - n_0) \\ &= \Pr(\sqrt{E_b}\cos(\phi_0 - \hat{\phi}_0) < n') = Q\left(\sqrt{\frac{2E_b}{N_0}}\cos(\phi_0 - \hat{\phi}_0)\right)\end{aligned}$$

where $n' = n_1 - n_0$ is zero-mean Gaussian with variance $N_0/2 + N_0/2 = N_0$. According to the textbook, the error probability of the non-coherent receiver is obtained as

$$\Pr(\text{error}|s_0(t)) = \Pr(\text{error}) = \frac{1}{2}e^{-1/2\frac{E_b}{N_0}}$$

With $10\log_{10}(2E_b/N_0) = 10$ dB we get $E_b/N_0 = 5$, and hence

$$Q\left(\sqrt{5}\cos(\phi_0 - \hat{\phi}_0)\right) \le \frac{1}{2}e^{-5/2}$$

This gives

$$|\phi_0 - \hat{\phi}_0| \le \cos^{-1}\frac{1.74}{\sqrt{5}} \approx 39^\circ$$

2–68 (a) Signal set 1 is M-PSK and signal set 2 is M-FSK. Since the SNR is quite high, the upper bound on the symbol error probability is tight enough for the purpose of this problem. For the M-PSK and phase-coherent reception, the distance to the two nearest neighbors is $2\sqrt{E_s}\sin(\frac{\pi}{M})$. Hence, the upper bound is

$$P_e^1 = 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin(\frac{\pi}{M})\right).$$

For phase-coherent M-FSK, the distances to the $M-1$ neighbors are $\sqrt{2E_s}$. Hence, the upper bound is

$$P_e^2 = (M-1)Q\left(\sqrt{\frac{E_s}{N_0}}\right).$$

Table-lookup of the Q-function gives the following result for low values of M.

M	P_e^1 (M-PSK)	P_e^2 (M-FSK)
2	$7.7 \cdot 10^{-9}$	$3.2 \cdot 10^{-5}$
3	$4.8 \cdot 10^{-7}$	$6.3 \cdot 10^{-5}$
4	$3.2 \cdot 10^{-5}$	$9.5 \cdot 10^{-5}$
5	$4.4 \cdot 10^{-4}$	$1.3 \cdot 10^{-4}$

The error probability for M-PSK grows faster than for M-FSK, since the distance to the two nearest neighbors decreases with M. For M-FSK, the distance remains constant, but the number of nearest neighbors increases. For $M < 5$, M-PSK gives the lower symbol error probability, and for $M \ge 5$, M-FSK gives the lower symbol error probability.

(b) For high constellation orders ($M \ge 5$), M-FSK gives a lower symbol error probability. However, the required bandwidth grows linearly with M for M-FSK. If the target system is bandwidth limited, M-PSK is probably a more feasible choice, even for high M.

2–69 (a) At high SNR the symbol error probability is determined by the minimum signal space distance between pairs of signal points. Maximum possible distance for a fixed b is obtained when

$$\sqrt{2}a = b - a \qquad \Longrightarrow a = \frac{b}{1+\sqrt{2}}$$

(b) The given QAM constellation with $b = 3a$ has mean energy and minimum squared distance

$$\bar{E}_{QAM} = \frac{4}{8}a^2 + \frac{4}{8}(3a)^2 = 5a^2 \qquad d_{min}^{QAM} = \sqrt{2}a$$

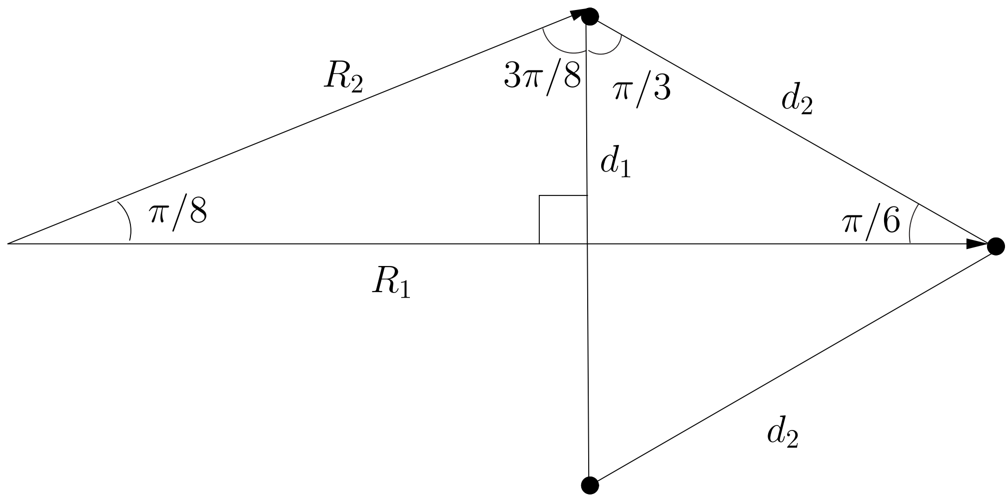
The 8PSK system has mean energy $\bar{E}_{PSK}$ and $d_{min}^{PSK} = 2\sqrt{\bar{E}_{PSK}}\sin\pi/8$. Setting the minimum distances equal gives

$$2\sqrt{\bar{E}_{PSK}}\sin\pi/8 = \sqrt{2}a = \sqrt{2}\sqrt{\frac{\bar{E}_{QAM}}{5}} \qquad \Longrightarrow$$

$$\frac{\bar{E}_{QAM}}{\bar{E}_{PSK}} = \frac{5}{2}4\sin^2\pi/8 = 1.4645, \qquad 10\log_{10}1.4645 = 1.65\text{dB}$$

That is, the QAM constellation needs about 1.65 dB higher energy.

2–70 (a) To minimize the symbol error probability of the 16-Star Constellation we need to maximize the distance between the 2 closest constellation points. This is done because errors due to the closest constellation points dominate when the E_s/N_0 is high. Referring to the figure below, as the ratio R_1/R_2 is increased then d_2 increases but d_1 decreases. The energy per symbol is kept constant, $E_s = (R_1^2 + R_2^2)/2$. In this situation the maximum of the minimum distance will occur when $d_1 = d_2$. This simplifies the geometry as an equilateral triangle is formed. The sin rule can be applied.



The ratio R_1/R_2 can be determined from the "Sine Rule".

$$\frac{R_1}{R_2} = \frac{\sin((1/3 + 3/8)\pi)}{\sin(\pi/6)} = 1.59$$

(b) Beginning with 16 PSK. The radius of the constellation is $\sqrt{E_s}$. The decision boundary is set at half way between 2 constellation points. The distance d from a constellation point to a decision boundary is $\sqrt{E_s}\sin(\pi/16)$. The distance can expressed in terms of number of standard deviations k.

$$\sigma = \sqrt{\frac{N_0}{2}}; \quad k\sigma = d; \quad k = \sqrt{\frac{2E_s}{N_0}}\sin(\pi/16)$$

The probability of the received signal crossing the boundary is then given by the Q(.) function.

$$P(X > \mu + k\sigma) = \text{Q}(k); \quad P_e = \text{Q}\left(\sqrt{\frac{2E_s}{N_0}}\sin(\pi/16)\right)$$

An error occurs if the decision boundary is crossed in either direction. Therefore the probability of symbol error P_s is:

$$P_s = 2\text{Q}\left(\sqrt{\frac{2E_s}{N_0}}\sin(\pi/16)\right)$$

Here lies the first assumption. Part of the error region has been counted twice by simply summing the probability of the 2 regions, however for 16 PSK and high E_s/N_0 this approximation is accurate. Four bits are transmitted for every symbol $E_s = 4E_b$, therefore

$$P_s = 2\mathrm{Q}\left(\sqrt{\frac{8E_b}{N_0}}\sin(\pi/16)\right)$$

When E_s/N_0 is high we can assume that the majority of the errors are produced by the neighboring symbols there $P_b \approx P_s/4$. The probability of bit error P_b in terms of E_b/N_0 can be approximated by:

$$P_b = \frac{1}{2}\mathrm{Q}\left(\sqrt{\frac{8E_b}{N_0}}\sin(\pi/16)\right)$$

Now consider the 16 Star constellation. $E_s = (R_1^2 + R_2^2)/2$ and $R_1/R_2 = 1.59$

$$2E_s = 1.59^2 R_2^2 + R_2^2; \quad R_2 = 0.75\sqrt{E_s}$$

The distance d from a constellation point to a decision boundary is $R_2 \sin(\pi/8)$.

$$k = 0.75\sqrt{\frac{2E_s}{N_0}}\sin(\pi/8)$$

The probability of the received signal crossing the boundary is then given by the Q(.) function.

$$P_e = \mathrm{Q}\left(0.75\sqrt{\frac{2E_s}{N_0}}\sin(\pi/8)\right)$$

An error occurs if the decision boundary is crossed in either direction. The outer constellations have 2 neighbors and the inner constellation points have 4 neighbors giving an average of 3. This is a loser approximation than for 16 PSK and will tend to give sightly high error probability results.

$$P_s = 3\mathrm{Q}\left(0.75\sqrt{\frac{2E_s}{N_0}}\sin(\pi/8)\right)$$

$E_s = 4E_b$

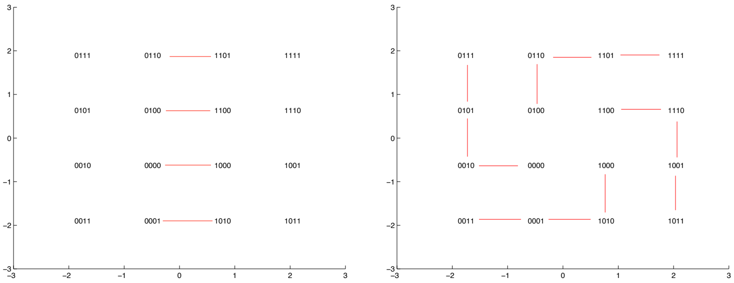
$$P_s = 3\mathrm{Q}\left(0.75\sqrt{\frac{8E_b}{N_0}}\sin(\pi/8)\right)$$

In the star constellation the outer constellation points have 2 neighbors each with one bit difference while the inner constellation points have 4 neighbors, 2 with 1 bit difference and 2 with 2 bits difference. When a symbol error occurs it is most likely that one of the neighbors is mistaken for the correct symbol. Here, it is assumed that each neighbor is equally likely to be the source of the error. Hence, the average number of bit errors per symbol error is $(2\times1+2\times1+2\times2)/(2+2+2) = 4/3$. However, there are 4 bits transmitted per symbol so the conversion factor from symbol error probability to bit error probability is $4/3/4 = 1/3$. Thus, the probability of bit error P_b in terms of E_b/N_0 can be approximated by:

$$P_b = \mathrm{Q}\left(0.75\sqrt{\frac{8E_b}{N_0}}\sin(\pi/8)\right)$$

It was assumed that only the neighbors influence the probability of bit error P_b.

2–71 At high E_b/N_0 the error probability is dominated by the nearest neighbors. Studying each bit separately and counting the number of nearest neighbors giving an error in the bit considered, the desired error probabilities can be found. Below, two figures illustrating the error events assuming a '0' is transmitted for the first and third bits are shown. Similar figures can be made for the second and fourth bit as well. Based upon the above, it is concluded that the bit error probability is three times higher for the two last bits compared to the two first ones.



Time-varying bit mappings could be employed to give each user the same time average bit error probability, e.g., swap the first/second bits with the third/fourth every second transmitted symbol. The receiver can do the reverse in the demapping process. This way, all bits will have the same error probability measured over a long sequence.

2–72 In the 64-QAM case, each symbol carries 6 bits and the symbol rate is $R/6$. The symbol error probability is given by

$$P_e^{(64)} = 1 - (1-p)^2$$
$$p = 2\left(1 - \frac{1}{\sqrt{64}}\right) Q\left(\sqrt{\frac{3}{64-1}\frac{6P}{RN_0}}\right) .$$

In the multicarrier case, each of the 3 carriers transfers 2 bits per symbol. The QPSK symbol duration is $6/R$ and each carrier is given a power of $P/3$. Hence, the QPSK symbol error probability for one carrier is

$$P_e^{(4)} = 1 - \left(1 - Q\left(\sqrt{\frac{P}{3}\frac{6}{R}\frac{1}{N_0}}\right)\right)^2$$

A symbol error occurs if any of the subcarriers are in error. Assuming no inter-carrier interference,

$$P_e^{(\mathrm{MC})} = 1 - \left(1 - P_e^{(4)}\right)^3$$

From the above, the required power at a symbol error rate of 10^{-2} can be derived. It amounts to roughly 8.2 dB in favor of the multicarrier solution. However, there are other issues affecting the final choice as well, for example the bandwidth required, which is not touched upon in this problem.

2–73 Considering first the bandwidth constraint, we know that the power spectral density of $u(t)$ is

$$S_u(f) = \frac{1}{4}\left[S_z(f+f_c) + S_z(f-f_c)\right]$$

where, in the case of rectangular QAM, uniform PSK or uniform ASK

$$S_z(f) = \frac{1}{T}E[|x_n|^2]\,|G(f)|^2$$

with

$$|G(f)|^2 = (AT)^2\mathrm{sinc}^2(fT)$$

Hence the fractional power within $|\pm f_c \pm B|$ is

$$2\int_0^{BT} \mathrm{sinc}^2(\tau)d\tau = 2f(BT)$$

where $f(x)$ is the function plotted in the problem formulation.

The different modulation formats considered in the problem can convey $L = 2, 3, 4$ or 6 bits per symbol, so since the bit rate is fixed to 1000 the possible values for the transmission rate

$1/T$ are $500, 1000/3, 250$ or $500/3$, giving the possible values $1/2$, $3/4$, 1 or $3/2$ for BT. To get $2f(BT) > 0.9$ we need $BT \geq 1$ (i.e., $1/2$ and $3/4$ will not work). Thus the possible values for L are 4 or 6. This means we can concentrate on investigating further the formats 16-ASK, 16-PSK, 16-QAM and 64-QAM.

The transmitted power is

$$P = \frac{E_g E[|x_n|]^2}{2T}$$

where $E_g = \|g(t)\|^2$. Assuming the levels $-15, -13, -11, \ldots, 11, 13, 15$ for the 16-ASK constellation, we get $E[|x_n|^2] = 85$. Thus, to satisfy the power constraint

$$E_g < \frac{200}{85 \cdot 250}$$

For 16-ASK, noting that $N_0 = 1/250$, we then get

$$P_e > Q\left(\sqrt{\frac{E_g}{N_0}}\right) = Q\left(\sqrt{\frac{200}{85}}\right) > 0.01$$

Thus 16-ASK will not work.

For 16-PSK, assuming $E[|x_n|^2] = 1$, we can use at most $E_g = 200/250$ to get

$$P_e < 2\,Q\left(\sqrt{200}\,\sin\frac{\pi}{16}\right) < 0.006$$

Hence 16-PSK will work. Problem solved!

For completeness, we list also the corresponding error probabilities for 16-QAM and 64-QAM: With 16-QAM we can achieve

$$P_e < 4\,Q(\sqrt{20}) \approx 0.000015$$

and 64-QAM will give

$$P_e = \frac{7}{4} Q\left(\sqrt{\frac{50}{7}}\right) \approx 0.0066$$

Hence, both 16- and 64-QAM will work as well.

2–74 (a) The spectral density of the complex baseband signal is given by Bennett's formula,

$$S_z(f) = \frac{1}{T} S_x(fT)|G(f)|^2.$$

The autocorrelation of the stationary information sequence is:

$$R_x(m) = E[x_n^* x_{n+m}] = \begin{cases} E[|x_n|^2] = 5a^2 & \text{when } m = 0 \\ E[x_n^*]E[x_{n+m}] = 0 & \text{when } m \neq 0 \end{cases} = 5a^2\delta(m)$$

The spectral density of the information sequence is the time-discrete Fourier-transform of the autocorrelation function:

$$S_x(fT) = \mathcal{F}_d\left\{R_x(m)\right\} = \mathcal{F}_d\left\{5a^2\delta(m)\right\} = 5a^2$$

Since the pulse $g(t)$ is limited to $[0, T]$, it can be written as

$$g(t) = \sqrt{\frac{2}{T}}\sin(\pi t/T)\mathrm{rect}_T(t - T/2)$$

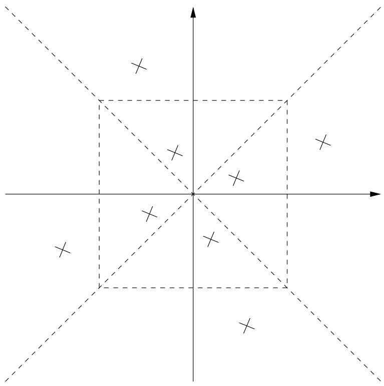
and the Fourier transform is obtained as

$$\begin{aligned}
G(f) &= \sqrt{\frac{2}{T}}\mathcal{F}\left\{\sin(\pi t/T)\right\} * \mathcal{F}\left\{\mathrm{rect}_T(t-T/2)\right\} \\
\mathcal{F}\left\{\sin(\pi t/T)\right\} &= \frac{1}{2j}\left(\delta(f-1/2T)-\delta(f+1/2T)\right) \\
\mathcal{F}\left\{\mathrm{rect}_T(t-T/2)\right\} &= e^{-j\pi fT}T\mathrm{sinc}(fT) \\
\Rightarrow G(f) &= \sqrt{2T}\frac{1}{2j}\left(\delta(f-1/2T)*\left(e^{-j\pi fT}\mathrm{sinc}(fT)\right)-\delta(f+1/2T)*\left(e^{-j\pi fT}\mathrm{sinc}(fT)\right)\right) \\
&= \sqrt{2T}\frac{1}{2j}\left(e^{-j\pi T(f-1/2T)}\mathrm{sinc}(fT-1/2)-e^{-j\pi T(f+1/2T)}\mathrm{sinc}(fT+1/2)\right) \\
&= \frac{j\sqrt{T}}{\sqrt{2}}e^{-j\pi fT}\left(e^{-j\pi/2}\mathrm{sinc}(fT+1/2)-e^{j\pi/2}\mathrm{sinc}(fT-1/2)\right) \\
&= \frac{j\sqrt{T}}{\sqrt{2}}e^{-j\pi fT}\left(-j\mathrm{sinc}(fT+1/2)-j\mathrm{sinc}(fT-1/2)\right) \\
&= \frac{\sqrt{T}}{\sqrt{2}}e^{-j\pi fT}\left(\mathrm{sinc}(fT+1/2)+\mathrm{sinc}(fT-1/2)\right) \\
\Rightarrow |G(f)|^2 &= \frac{T}{2}\left(\mathrm{sinc}(fT+1/2)+\mathrm{sinc}(fT-1/2)\right)^2 \\
\Rightarrow S_z(f) &= \frac{5a^2}{2}\left(\mathrm{sinc}(fT+1/2)+\mathrm{sinc}(fT-1/2)\right)^2
\end{aligned}$$

The spectral density of the carrier modulated signal is given by

$$S_u(f) = \frac{1}{4}\left[S_z(f-f_c)+S_z(f+f_c)\right]$$

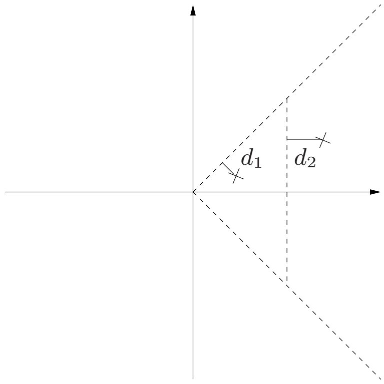
(b) The phase shift in the channel results in a constellation rotation as illustrated in the figure below. The dashed lines are the decision boundaries (ML detector).



The symbol error probability is

$$\Pr(\text{symbol error}) = \frac{1}{8}\sum_{i=0}^{7}\Pr(\text{symbol error} \mid \text{symbol } i \text{ transmitted})$$

Due to symmetry, only two of these probabilities have to be studied. Furthermore, since the signal to noise power ratio is very high, we approximate the error probability by the distance to the nearest decision boundary.



Hence,

$$\begin{aligned}
\Pr(\text{symbol error } | x_n = a) &\approx Q\left(\frac{d_1}{\sqrt{N_0/2}}\right) \\
\Pr(\text{symbol error } | x_n = 3a) &\approx Q\left(\frac{d_2}{\sqrt{N_0/2}}\right)
\end{aligned}$$

which gives the error probability

$$\Pr(\text{symbol error}) = \frac{1}{2}Q\left(\frac{d_1}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{d_2}{\sqrt{N_0/2}}\right)$$

The distances d_1 and d_2 can be computed as

$$\begin{aligned}
d_1 &= a\sin\phi \approx 0.38a \\
d_2 &= a(3\cos\phi - 2) \approx 0.77a
\end{aligned}$$

2–75 (a) Derive and plot the power spectral density (psd) of $v(t)$.

The modulation format is QPSK. Let's call the complex baseband signal after the transmit filter $z(t)$. The power spectral density of $z(t)$ is given by

$$S_z(f) = \frac{1}{T}S_x(fT)|G_T(f)|^2$$

where $S_x(fT)$ is the psd of x_n, and $G_T(f)$ is the frequency response of $g_T(t)$. The autocorrelation of x_n is

$$R_x(k) = E[x_n x_{n-k}^*] = \delta(k)$$

which gives the power spectral density as the time-discrete Fourier transform of $R_x(k)$.

$$S_x(fT) = 1$$

The frequency response of the transmit filter is

$$G_T(f) = \mathcal{F}\{g_T(t)\} = \text{rect}_{\frac{1}{T}}(f) = \begin{cases} 1 & |f| < \frac{1}{2T} \\ 0 & \text{otherwise} \end{cases}$$

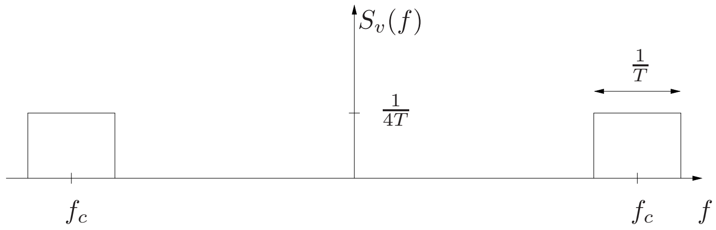
This gives the psd of $z(t)$

$$S_z(f) = \frac{1}{T}\text{rect}_{\frac{1}{T}}(f)$$

The psd of the carrier modulated bandpass signal $v(t)$ is given by

$$S_v(f) = \frac{1}{4}[S_z(f-f_c) + S_z(f+f_c)] = \frac{1}{4T}[\text{rect}_{\frac{1}{T}}(f-f_c) + \text{rect}_{\frac{1}{T}}(f+f_c)]$$

which looks like



(b) Derive an expression for $y(nT)$.
First, let's find an expression for $v(t)$ in the time-domain.

$$\begin{aligned}
v(t) &= \Re\left\{z(t)e^{j2\pi f_c}\right\} = \Re\left\{\sum_{n=-\infty}^{\infty} e^{j\phi_n} g_T(t-nT)e^{j2\pi f_c t}\right\} = \Re\left\{\sum_{n=-\infty}^{\infty} g_T(t-nT)e^{j(2\pi f_c t+\phi_n)}\right\} \\
&= \sum_{n=-\infty}^{\infty} g_T(t-nT)\cos(2\pi f_c t+\phi_n) = \frac{1}{2}\sum_{n=-\infty}^{\infty} g_T(t-nT)\left(e^{j(2\pi f_c t+\phi_n)} + e^{-j(2\pi f_c t+\phi_n)}\right)
\end{aligned}$$

Let's call the received baseband signal, before the receive filter, $u(t) = u_s(t) + u_n(t)$, where $u_s(t)$ is the signal part and $u_n(t)$ is the noise part.

$$u(t) = u_s(t) + u_n(t) = 2v(t)e^{-j(2\pi(f_c+f_e)+\phi_e)} + 2n(t)e^{-j(2\pi(f_c+f_e)+\phi_e)}$$

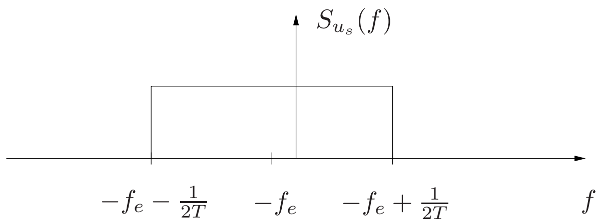
Let's analyze the signal part first.

$$\begin{aligned}
u_s(t) &= \sum_{n=-\infty}^{\infty} g_T(t-nT)\left(e^{j(2\pi f_c t+\phi_n)} + e^{-j(2\pi f_c t+\phi_n)}\right)e^{-j(2\pi(f_c+f_e)+\phi_e)} \\
&= \sum_{n=-\infty}^{\infty} g_T(t-nT)\left(e^{-j(2\pi f_e t+\phi_e-\phi_n)} + e^{-j(2\pi(2f_c+f_e)t+\phi_e+\phi_n)}\right)
\end{aligned}$$

Note that the signal contains low-frequency terms and high-frequency terms (around $2f_c$). The receiver filter $g_R(t) = \frac{\sin(2\pi t/T)}{\pi t}$ is a low-pass filter with frequency response

$$G_R(f) = \mathcal{F}\{g_R(t)\} = \mathrm{rect}_{\frac{2}{T}}(f) = \begin{cases} 1 & |f| < \frac{1}{T} \\ 0 & \text{otherwise} \end{cases}$$

Since $f_c \gg \frac{1}{T}$, the high-frequency components of $u_s(t)$ will not pass the filter. The psd of the low-frequency terms of $u_s(t)$ looks like



Since, $f_e < \frac{1}{2T}$, the low-frequency part of $u_s(t)$ will pass the filter undistorted. If we divide also $y(t)$ into its signal and noise parts, $y(t) = y_s(t) + y_n(t)$, we get

$$y_s(t) = \sum_{k=-\infty}^{\infty} g_T(t-kT)e^{-j(2\pi f_e t+\phi_e-\phi_k)}$$

Sample this process at $t = nT$,

$$\begin{aligned} y_s(nT) &= \sum_{k=-\infty}^{\infty} g_T(nT - kT)e^{-j(2\pi f_e nT + \phi_e - \phi_k)} = \left\{ g_T((n-k)T) = \frac{1}{T}\delta(n-k) \right\} \\ &= \frac{1}{T} e^{-j(2\pi f_e nT + \phi_e - \phi_n)} \end{aligned}$$

Now, let's analyze the noise.

$$\begin{aligned} u_n(t) &= 2n(t)e^{-j(2\pi(f_c+f_e)t+\phi_e)} \Rightarrow \\ y_n(t) &= u_n(t) \star g_R(t) = \int_{-\infty}^{\infty} 2n(v)e^{-j(2\pi(f_c+f_e)v+\phi_e)} \frac{\sin(2\pi(t-v)/T)}{\pi(t-v)} dv \Rightarrow \\ y_n(nT) &= \int_{-\infty}^{\infty} 2n(v)e^{-j(2\pi(f_c+f_e)v+\phi_e)} \frac{\sin(2\pi(nT-v)/T)}{\pi(nT-v)} dv \end{aligned}$$

This complicated expression for the noise in $y(nT)$ can not easily be simplified, so we leave it in this form. However, the statistical properties of the noise will be evaluated in (c).

(c) Find the autocorrelation function for the noise in $y(t)$.

The mean and autocorrelation function of $u_n(t)$ are

$$\begin{aligned} E[u_n(t)] &= 0 \\ R_{u_n}(t) &= E[u_n(t)u_n^*(t-\tau)] = 2N_0\delta(\tau) \end{aligned}$$

Hence, $u_n(t)$ is white. The psd of the noise is $S_{u_n}(f) = 2N_0$. The psd of the noise after the receive filter is

$$S_{y_n}(f) = S_{u_n}(f)|G_R(f)|^2 = 2N_0 \mathrm{rect}_{\frac{1}{T}}(f)$$

which gives the autocorrelation function

$$R_{y_n}(\tau) = 2N_0 \frac{\sin(2\pi\tau/T)}{\pi\tau}$$

The noise in $y(n)$ is colored, but still additive and Gaussian (Gaussian noise passing though LTI-systems keeps the Gaussian property).

(d) Find the symbol-error probability. Assuming that $f_e = 0$, the sampled received signal is

$$y(nT) = \frac{1}{T} e^{j(\phi_n - \phi_e)} + y_n(nT)$$

The real and imaginary parts of this signal are the decision variables of the detector. The detector was designed for the ideal case, i.e. the decision regions are equal to the four quadrants of the complex plane. The mean and variance of the complex noise is

$$\begin{aligned} E[y_n(nT)] &= 0 \\ \sigma_{y_n}^2 &= R_{y_n}(0) = 4N_0/T \end{aligned}$$

The complex noise is circular symmetric in the complex plane, so the variance of the real noise ($\sigma_{n_{Re}}^2$) is equal to the variance of the imaginary noise ($\sigma_{n_{Im}}^2$).

$$\begin{aligned} y_n(n) &= n_{Re}(n) + jn_{Im}(n) \\ \sigma_{n_{Re}}^2 &= \sigma_{y_n}^2/2 = 2N_0/T \\ \sigma_{n_{Im}}^2 &= \sigma_{y_n}^2/2 = 2N_0/T \end{aligned}$$

Due to the phase rotation ϕ_e, the distances to the decision boundaries are not equal, but

$$d_1 = \frac{1}{T}\cos(\frac{\pi}{4} - \phi_e)$$
$$d_2 = \frac{1}{T}\sin(\frac{\pi}{4} - \phi_e)$$

where The probability for a correct decision is then

$$\begin{aligned}
\Pr(\text{correct}) &= (1 - \Pr(n_{Re} > d_1))\,(1 - \Pr(n_{Im} > d_2)) \\
&= \left(1 - Q\left(\frac{\cos(\frac{\pi}{4} - \phi_e)}{\sqrt{2N_0T}}\right)\right)\left(1 - Q\left(\frac{\sin(\frac{\pi}{4} - \phi_e)}{\sqrt{2N_0T}}\right)\right)
\end{aligned}$$

The probability of symbol error is then

$$\begin{aligned}
\Pr(\text{error}) &= 1 - \Pr(\text{correct}) = Q\left(\frac{\cos(\frac{\pi}{4} - \phi_e)}{\sqrt{2N_0T}}\right) + Q\left(\frac{\sin(\frac{\pi}{4} - \phi_e)}{\sqrt{2N_0T}}\right) \\
&\quad - Q\left(\frac{\cos(\frac{\pi}{4} - \phi_e)}{\sqrt{2N_0T}}\right) Q\left(\frac{\sin(\frac{\pi}{4} - \phi_e)}{\sqrt{2N_0T}}\right)
\end{aligned}$$

---

3 Channel Capacity and Coding

3–1 The mutual information is

$$I(X;Y) = H(Y) - H(Y|X)$$

with

$$H(Y) = 1 - \frac{1}{2}(1-\varepsilon)\log(1-\varepsilon) - \frac{1}{2}(1+\varepsilon)\log(1+\varepsilon)$$

and

$$H(Y|X) = \frac{1}{2}H(Y|X=0) + \frac{1}{2}H(Y|X=1) = \frac{1}{2}h(\varepsilon) + 0$$

where $h(x)$ is the binary entropy function. Hence we get

$$\begin{aligned}
I(X;Y) &= 1 - \frac{1}{2}(1-\varepsilon)\log(1-\varepsilon) - \frac{1}{2}(1+\varepsilon)\log(1+\varepsilon) - \frac{1}{2}h(\varepsilon) \\
&= 1 - \frac{1}{2}(1+\varepsilon)\log(1+\varepsilon) + \frac{1}{2}\varepsilon\log\varepsilon \approx 0.85
\end{aligned}$$

3–2 (a) The average error probability is

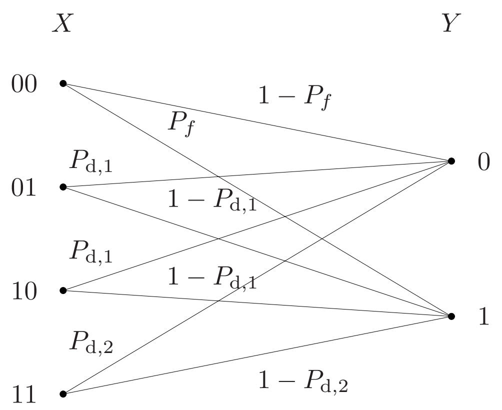
$$\begin{aligned}
p_e^a &= p_0\epsilon_0 + (1-p_0)\epsilon_1 = 3p_0\epsilon_1 + \epsilon_1 - p_0\epsilon_1 \\
&= \frac{1}{3}(1+2p_0)\epsilon_0 = \frac{1}{2}\epsilon_0
\end{aligned}$$

(b) The average error probability for the negative decision law is

$$\begin{aligned}
p_e^b &= p_0(1-\epsilon_0) + p_1(1-\epsilon_1) = p_0 - p_0\epsilon_0 + p_1 - p_1\epsilon_1 \\
&= 1 - \frac{1}{2}\epsilon_0
\end{aligned}$$

When $\epsilon_0 > 1$, we will have $p_e^b < p_e^a$, which is not possible.

3–3 There are four different bit combinations, $X \in \{00, 01, 10, 11\}$, output from the two encoders in the transmitter, where $X = 01$ denotes that the second, but not the first, encoder has sent a pulse over the channel. Similarly, two different bit values can be received, $Y \in \{0, 1\}$, where 1 denotes a pulse detected and 0 denotes the absence of a pulse. Based on this, the following transition diagram can be drawn (other possibilities exist as well, for example three inputs and a non-uniform X):



The channel capacity is given by $C = \max_{p(x)} I(X;Y)$. However, in this problem, the probabilities of the transmitted bits are fixed and equal. Hence, the full capacity of the channel might not be used, but nevertheless the amount of information transmitted per channel use is given by

$$I(X;Y) = H(Y) - H(Y|X) = H(Y) - \sum p(x)H(Y|X = x) \ ,$$

where $p(x) = 1/4$,

$$H(Y) = h([1 - P_f + 2P_{\mathrm{d},1} + P_{\mathrm{d},\,2}]/4)$$

and $h(a) = -a \log a - (1 - a)\log(1 - a)$ denotes the binary entropy function. The conditional entropies are obtained from the transition diagram as

$$\begin{aligned} H(Y|X = 00) &= h(P_f) \\ H(Y|X = 01) &= h(P_{\mathrm{d},1}) \\ H(Y|X = 10) &= h(P_{\mathrm{d},1}) \\ H(Y|X = 11) &= h(P_{\mathrm{d},2}) \ . \end{aligned}$$

Hence, the amount of information transmitted per channel use and the answer to the first part is

$$h([1 - P_f + 2P_{\mathrm{d},1} + P_{\mathrm{d},\,2}]/4) - [h(P_f) + 2h(P_{\mathrm{d},1}) + h(P_{\mathrm{d},2})]/4$$

which in the idealized case $P_{\mathrm{d},1} = P_{\mathrm{d},2} = P_f = 0$ in the second part of the problem equals approximately 0.81 bits/channel use. If the two transmitters would not interfere with each other at all, 2 bits/channel use could have been transfered (1 bit per user). The advantage with a scheme similar to the above is of course that the two users share the same bandwidth. Schemes similar to the above are the foundation to CDMA techniques, used in WCDMA, a recently developed 3rd generation cellular communication system for both voice and data.

3–4 Given the figures in the problem, a simple link budget in dB is given by

$$P_{\mathrm{rx}} = P_{\mathrm{tx}} + G_{\mathrm{rx\ antenna}} - L_{\mathrm{free\ space}} = 20\,\mathrm{dBW} + 5\,dB - 20\log_{10}(4\pi d/\lambda) = -263\,\mathrm{dBW} \quad (3.10)$$

where $\lambda = 3 \cdot 10^8/10 \cdot 10^9$ m and $d = 6.28 \cdot 10^{11}$ m. The temperature in free space is typically a few degrees above 0 K. Assume $\vartheta = 5$ K, then $N_0 = k\vartheta = 6.9 \cdot 10^{-23}$ W/Hz. It is known that E_b/N_0 must be larger that -1.6 dB for reliable communication to be possible. Solving for R_b results in $R_\mathrm{b} = P_{\mathrm{rx}}/N_0 10^{-1.6/10} = .001$ bit/s, which is far less than the required 1 kbit/s. Hence, the space probe should not be launched in its current design. One possible improvement is to increase the antenna gain, which is quite low.

3–5 (a) To determine the entropy $H(Y)$ we need the output probabilities of the channel. If we use the formula

$$f_Y(y) = \sum_{k=1}^{3} f_X(x_k) f_{Y|X}(y|x_k)$$

we obtain

$$\begin{bmatrix} f_Y(y_1) \\ f_Y(y_2) \\ f_Y(y_3) \end{bmatrix} = \begin{bmatrix} 1-\epsilon & \delta & 0 \\ \epsilon & \delta & \gamma \\ 0 & \delta & 1-\gamma \end{bmatrix} \begin{bmatrix} f_X(x_1) \\ f_X(x_2) \\ f_X(x_3) \end{bmatrix} .$$

It is well known that the entropy of a variable is maximum if the variable is uniformly distributed. Can we in our case select $f_X(x)$ so that Y is uniform? This is equal to check if the following linear equation system has (at least) one solution that satisfies $f_X(x_1) + f_X(x_2) + f_X(x_3) = 1$,

$$\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1-\epsilon & \delta & 0 \\ \epsilon & \delta & \gamma \\ 0 & \delta & 1-\gamma \end{bmatrix} \begin{bmatrix} f_X(x_1) \\ f_X(x_2) \\ f_X(x_3) \end{bmatrix}.$$

First we neglect the constraint on the solution and concentrate on the linear equation system without constraint. The determinant of the matrix is

$$(1-\epsilon)\delta(1-\gamma) - (1-\epsilon)\gamma\delta - \delta\epsilon(1-\gamma)$$

which is (check this!) zero in our case. The solution to the unconstrained system is thus not unique. To see this add/subtract rows to obtain

$$\begin{bmatrix} 1/3 \\ \frac{1-2\epsilon}{3(1-\epsilon)} \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1-\epsilon & \delta & 0 \\ 0 & \frac{\delta(1-2\epsilon)}{1-\epsilon} & \gamma \\ 0 & \delta & 1-\gamma \end{bmatrix} \begin{bmatrix} f_X(x_1) \\ f_X(x_2) \\ f_X(x_3) \end{bmatrix},$$

$$\begin{bmatrix} 1/3 \\ \frac{1-2\epsilon}{3(1-\epsilon)} \\ 0 \end{bmatrix} = \begin{bmatrix} 1-\epsilon & \delta & 0 \\ 0 & \frac{\delta(1-2\epsilon)}{1-\epsilon} & \gamma \\ 0 & 0 & \frac{1-2\gamma-2\epsilon+3\gamma\epsilon}{1-2\epsilon} \end{bmatrix} \begin{bmatrix} f_X(x_1) \\ f_X(x_2) \\ f_X(x_3) \end{bmatrix}.$$

where we have assumed that $\epsilon \neq 1$ and $\epsilon \neq 1/2$. If we "plug" in the values given for the transition probabilities the equation system becomes

$$\begin{bmatrix} 1/3 \\ 1/6 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 0 & 1/6 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_X(x_1) \\ f_X(x_2) \\ f_X(x_3) \end{bmatrix}.$$

From the last equation system we obtain

$$f_X(x_1) = f_X(x_3) = \frac{1 - f_X(x_2)}{2}.$$

If we note that

$$f_X(x_1) + f_X(x_2) + f_X(x_3) = \frac{1 - f_X(x_2)}{2} + \frac{1 - f_X(x_2)}{2} + f_X(x_2) = 1,$$

we realize that the maximum entropy of the output variable Y is given by $\log_2 3$ and that this entropy is obtained for the input probabilities characterized by

$$f_X(x_1) = f_X(x_3) = \frac{1 - f_X(x_2)}{2}.$$

(b) As usual we apply the identity

$$H(Y|X) = \sum_{k=1}^{3} f_X(x_k) H(Y|X = x_k).$$

The entropy for Y, given $X = x_k$, is in our case

$$\begin{aligned} H(Y|X = x_1) &= -\sum_{l=1}^{3} f_{Y|X}(y_l|x_1) \log_2 f_{Y|X}(y_l|x_1) \\ &= -(1-\epsilon)\log_2(1-\epsilon) - \epsilon \log_2 \epsilon = H_b(\epsilon) \\ H(Y|X = x_2) &= \log_2 3 \\ H(Y|X = x_3) &= H_b(\delta) \end{aligned}$$

The answer to the question is thus

$$H(Y|X) = f_X(x_1)H_b(\epsilon) + f_X(x_2)\log_2 3 + f_X(x_3)H_b(\delta)$$

(c)
$$C \leq \max_{f_X(x)} H(Y) - \min_{f_X(x)} H(Y|X).$$

The first term in this expression has already been computed and is equal to $\log_2 3$. We now turn to the second term

$$H(Y|X) = f_X(x_1)(H_b(\epsilon) - H_b(\delta)) + f_X(x_2)(\log_2 3 - H_b(\epsilon)) + H_b(\delta),$$

where we have used the identity $f_X(x_1) + f_X(x_2) + f_X(x_3) = 1$. If we "plug" in the values given for ϵ and δ we have

$$H(Y|X) = f_X(x_2)(\log_2 3 - H_b(1/3)) + H_b(1/3).$$

From this expression we clearly see that $H(Y)$ is minimized when $f_X(x_2) = 0$ and the minimum is $H_b(1/3)$. That is,
$$C \leq \log_2 3 - H_b(1/3)$$

That this upper bound is attainable is realized by noting that the maxima of $H(Y)$ is attained for $f_X(x_1) = f_X(x_3) = 1/2$ and $f_X(x_2) = 0$. The minima of $H(Y|X)$ is obtained for the same input probabilities and the bound is thus attainable. Finally, the channel capacity is

$$C = \log_2 3 - H_b(1/3) = \frac{2}{3}\log_2 2 = \frac{2}{3} \text{ bits/symbol.}$$

3–6 (a) Let $X \in \{0,1\}$ be the input to channel 1, and let $Y \in \{0,1\}$ be the corresponding output. We get

$$p_{Y|X}(0|0) = 1/2 \qquad p_{Y|X}(1|0) = 1/2 \qquad p_{Y|X}(0|1) = 0 \qquad p_{Y|X}(1|1) = 1\,.$$

To determine the capacity we need an expression for the mutual information $I(X;Y)$. Assume that
$$p_X(0) = q \qquad p_X(1) = 1 - q\,,$$
then
$$p_Y(0) = q/2 \qquad p_Y(1) = 1 - q/2\,.$$
We have $I(X;Y) = H(Y) - H(Y|X)$ where

$$H(Y|X) = H(Y|X=0)p_X(0) + H(Y|X=1)p_X(1) = H_b\left(\frac{1}{2}\right)q + 0\cdot(1-q) = qH_b\left(\frac{1}{2}\right)$$
$$H(Y) = H_b\left(q/2\right)$$

Here $H_b(\epsilon)$ is the binary entropy function

$$H_b(\epsilon) = -\epsilon\log_2\epsilon - (1-\epsilon)\log_2(1-\epsilon).$$

We get
$$I(X;Y) = H(Y) - H(Y|X) = H_b\left(q/2\right) - q$$
The capacity is, by definition

$$C = \max_{p_X(x)} I(X;Y) = \max_q \big\{ \underbrace{H_b\left(q/2\right) - q}_{=f(q)} \big\}$$

Differentiating $f(q)$ gives

$$\frac{df(q)}{dq} = \frac{1}{2}\log_2\frac{2-q}{q} - 1 = 0 \qquad \rightarrow \qquad \max_q f(q) = f(2/5) = H_b(0.2) - 0.4 \approx 0.3219$$

(b) Consider first the equivalent concatenated channel. Let $Z \in \{0,1\}$ be the output from channel 2. Then we get

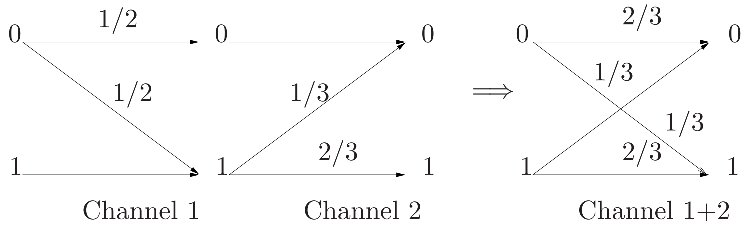
$$p_{YZ|X} = \frac{p_{YZX}}{p_X} = \frac{p_{YZX}}{p_X}\frac{p_{XY}}{p_{XY}} = \frac{p_{YZX}}{p_{XY}}\frac{p_{XY}}{p_X} = p_{Z|XY}p_{Y|X} = p_{Z|Y}p_{Y|X}$$

where the last step follows from the fact $p_{Z|YX} = p_{Z|Y}$. Note that $p_{Z|X} = \sum_y p_{ZY|X}$ and we get

$$p_{Z|X}(0|0) = \sum_{y=0,1} p_{Z|Y}(0|y)p_{Y|X}(y|0) = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{1}{3} = \frac{2}{3}$$

$$p_{Z|X}(1|0) = p_{Z|X}(0|1) = \frac{1}{3} \qquad p_{Z|X}(1|1) = \frac{2}{3}$$

The equivalent channel is illustrated below.



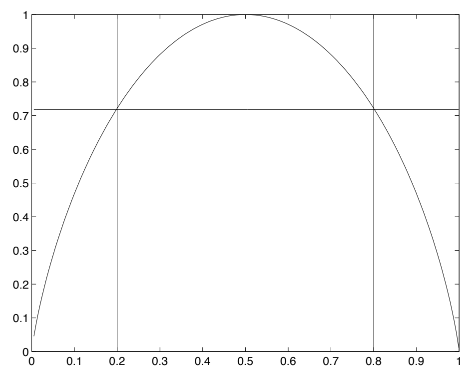
As we can see, this is the "Binary Symmetric Channel" with crossover probability 1/3. Hence we know that

$$C = 1 - H_b(1/3) \approx 0.0817$$

3–7 Let $\{A_i\}_{i=1}^4$ be the 2-bit output of the quantizer. Then two of the possible values of A_i have probabilities $a^2/2$ and the other two have probabilities $1/2 - a^2/2$, where $a = 1 - b$. Since the input process is memoryless, the output process of the quantizer hence has entropy rate $\bar{H} = 1/2(1 + H_b(a^2))$ [bits per binary symbol], where $H_b(x) = -x\log x - (1-x)\log(1-x)$ [bits]. The binary symmetric channel has capacity $C = 1 - H_b(0.02) \approx 0.859$ [bits]. In order for the source-channel coding to be able to give asymptotically perfect transmission, it must hold that $\bar{H} < C$. Hence, we must have $H_b(a^2) < 0.718$, which gives (approximately) $a^2 < 0.198$ or $a^2 > 0.802$ (c.f. the figure below). Hence we have that $1 - b < 0.4450$ or $0.8955 < 1 - b$, giving that error-free transmission is possible only if

$$0 < b < 0.104, \text{ or } 0.555 < b < 1,$$

which is the sought-for result.



3–8 The capacity of the described AWGN channel is

$$C = W\log\left(1 + \frac{P}{WN_0}\right) = 1000\log 1.7 \approx 765.53 \quad \text{[bits/s]}$$

The source can be transmitted without errors as long as its entropy-rate H, in bits/s, satisfies $H < C$. Since the source is binary and memoryless we have

$$H = \frac{1}{T_s}\big(- p \log p - (1-p)\log(1-p)\big) \quad \text{[bits/s]}$$

Solving for p in $H < C$ gives

$$-p\log p - (1-p)\log(1-p) < CT_s \approx 0.7655 \Rightarrow b < p < 1-b$$

with $b \approx 0.2228$.

3–9 Let $p(x)$ be a possible pmf for X, then the capacity is

$$C = \max_{p(x)} I(X;Y) = \max_{p(x)} \big(H(Y) - H(Y|X)\big)$$

The output entropy $H(Y)$ is maximized when the Y's are equally likely, and this will happen if $p(x)$ is chosen to be uniform over $\{0,1,2,3\}$. For the conditional output entropy $H(Y|X)$ it holds that

$$H(Y|X) = \sum_x p(x) H(Y|X=x)$$

Since $H(Y|X=x)$ has the same value for any $x \in \{0,1,2,3\}$, namely

$$H(Y|X=x) = -\varepsilon\log\varepsilon - (1-\varepsilon)\log(1-\varepsilon) = h(\varepsilon)$$

(for any fixed x there are two possible Y's with probabilities ε and $(1-\varepsilon)$), the value of $H(Y|X)$ cannot be influenced by chosing $p(x)$ and hence $I(X;Y)$ is maximized by maximizing $H(Y)$ which happens for $p(x) = 1/4$, any $x \in \{0,1,2,3\} \implies$

$$C = \log 4 - h(\varepsilon) = 2 - h(\varepsilon)$$

bits per channel use.

3–10 Let X denote the input and Y the output, and let $\pi = \Pr(X=0)$. Then

$$p_0 = \Pr(Y=0) = \pi(1-\alpha) + (1-\pi)\beta, \quad p_1 = \Pr(Y=1) = \pi\alpha + (1-\pi)(1-\beta)$$

and

$$\begin{aligned} I(X;Y) &= H(Y) - \pi\, H(Y|X=0) - (1-\pi)\, H(Y|X=1) \\ &= -p_0\log p_0 - p_1\log p_1 - \pi g(\alpha) - (1-\pi)g(\beta) \end{aligned}$$

where

$$g(x) = -x\log x - (1-x)\log(1-x)$$

is the binary entropy function. Taking derivative of $I(X;Y)$ w.r.t π and putting the result to zero gives

$$\log\frac{p_1}{p_0} = g(\alpha) - h(\beta) \iff \pi = \frac{\beta(1+f)-1}{(\alpha+\beta-1)(1+f)}$$

where $f = 2^{g(\alpha)-g(\beta)}$. Using $\beta = 2\alpha$ gives

$$\pi = \frac{2\alpha(1+f)-1}{(3\alpha-1)(1+f)}$$

with

$$f = 2^{g(\alpha)-g(2\alpha)}$$

The capacity is then given by the above value for π used in the expression for $I(X;Y)$.

3–11 (a)

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} P^T & I \end{bmatrix}$$

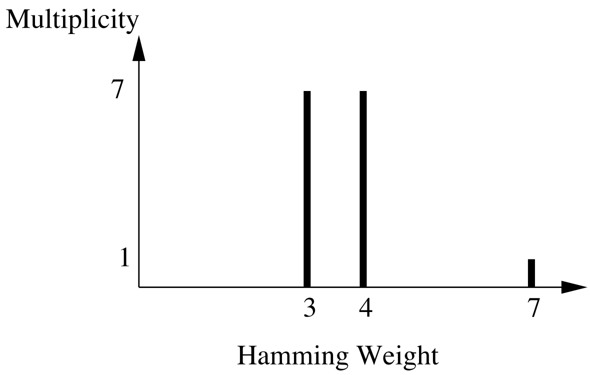
$$\mathbf{G} = \begin{bmatrix} I & P \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

You can check that $\mathbf{GH}^T = 0$

(b) The weights of all codewords need to be examined to get the distance profile.

data	**codeword**	**weight**
0000	0000000	0
0001	0001011	3
0010	0010101	3
0011	0011110	4
0100	0100110	3
0101	0101101	4
0110	0110011	4
0111	0111000	3
1000	1000111	4
1001	1001100	3
1010	1010010	3
1011	1011001	4
1100	1100001	3
1101	1101010	4
1110	1110100	4
1111	1111111	7

The minimum Hamming weight is 3. The distance profile is shown below



(c) This code can always detect up to 2 errors because the minimum Hamming distance is 3.

(d) This code always detects up to 2 errors on the channel and corrects 1 error. If there are 2 errors on the channel the correction may make a mistake although the detection will indicate correctly that there was an error on the channel.

(e) The syndrome can be calculated using the equation given below:

$$\mathbf{s} = \mathbf{eH}^T$$

Error e	Syndrome s
0000000	000
0000001	001
0000010	010
0000100	100
0001000	011
0010000	101
0100000	110
1000000	111

Syndrome Table

3–12 By studying the 8 different codewords of the code it is straightforward to conclude that the minimum distance is 3.

3–13 Binary block code with generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

(a) The given generator matrix directly gives $n = 7$ (number of columns), $k = 3$ (number of rows) and rate $R = 3/7$. The codewords are obtained as $\mathbf{c} = \mathbf{x}\mathbf{G}$ for all different $\mathbf{x}$ with three information bits. This gives

$$\mathcal{C} = \{0000000, 1110100, 0111010, 1001110, 0011101, 1101001, 0100111, 1010011\}$$

Looking at the codewords we see that $d_{\min} = 4$ (weight of non-zero codeword with least number of 1's)

(b) The generator matrix is clearly given in cyclic form and we can hence identify the generator polynomial as

$$g(p) = p^4 + p^3 + p^2 + 1$$

One way to get the parity check polynomial is to divide $p^7 + 1$ with $g(p)$. This gives

$$h(p) = \frac{p^7 + 1}{g(p)} = p^3 + p^2 + 1$$

(c) The generator matrix in systematic form is obtained from the given $\mathbf{G}$ (in cyclic form) as

$$\mathbf{G}_{\text{sys}} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

(Add 1st and 2nd rows and put the result in 1st row. Add 2nd and 3rd rows and put the result in 2nd row. Keep the 3rd row.) The systematix generator matrix then gives the systematic parity check matrix as

$$\mathbf{H}_{\text{sys}} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

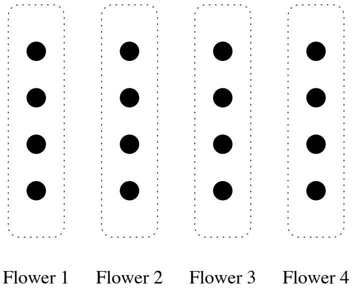
(d) The dual code has parity check matrix

$$\mathbf{H}_{\text{dual}} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

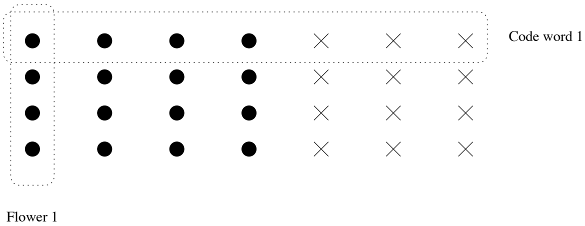
and we see that this is the parity check matrix of the Hamming $(7, 4)$ code. (All different non-zero combinations of 3 bits as columns.) Hence we know that the dual code has minimum distance 3 and can correct all 1-bit error patters *and no other error patterns.* Consequently

$$p_e = \Pr(> 1 \text{ error}) = 1 - \Pr(0 \text{ or } 1 \text{ error}) = 1 - (1 - \varepsilon)^n - n\varepsilon(1 - \varepsilon)^{n-1} \approx 0.002$$

3–14 There are several possible solutions to this problem. One way is to encode the 16 species by four bits. C picks four flowers first, which results in four words of four bits each. Put these in a matrix as illustrated below, where each column denotes one flower.



Now, use a *systematic* Hamming(7,4) block code to encode each *row* and store the three parity bits in the three rightmost columns, resulting in



A makes his choice of flowers according to the Hamming(7,4) code. When C replaces one of the flowers with a new one, i.e., changes one of the columns above, he introduces a (possible) single-bit error in each of the code words. Since the code has a minimum distance of 3, the Hamming(7,4) code can correct this and B can tell the original flower sequence.

Note that, by distributing the bits in a smart way, the total encoding scheme can recover from a 4-bit error even though each Hamming code only can correct single bit errors. The idea of reordering the bits in a bit sequence before transmitting them over a channel and doing a reverse reordering before decoding is used in real communication systems and is known as *interleaving.* This way, single bit error correcting codes can be used for correcting burst errors. Interleaving will be discussed in more detail in the advanced course.

3–15 The inner code words, each consisting of three coded bits, are 000, 101, 011, 110, and the mutual Hamming distance between any two code words is $d_H = 2$. Hence, the squared Euclidean distance is $d_{\mathrm{E}}^2 = 4 d_H E_{\mathrm{c}}$. The decoder chooses the code word closest (in the Euclidean sense) to the received word. The *word* error probability for soft decision decoding of a block code is hard to derive, but can be upper bounded by union bound techniques. At high SNRs, this bound is a good approximation. Each code word has three neighbors, all of them at distance d_{E}. Hence,

$$P_{\mathrm{e,inner}} \approx 3Q\left(\sqrt{\frac{d_{\mathrm{E}}^2/4}{N_0/2}}\right) = 3Q\left(\sqrt{\frac{2d_H E_{\mathrm{c}}}{N_0}}\right) .$$

If the wrong inner code word is chosen, with probability $2/3$ the first information bit is in error (similar for the other infomration bit). Hence,

$$p = \frac{2}{3} P_{\mathrm{e}} ,$$

where p is the bit error probability after the inner decoder. After deinterleaving, all the bits are independent of each other. The Hamming code has $d_{\min} = 3$ and can correct at most one bit error. The word error probability is thus

$$P_{\mathrm{e,outer}} = 1 - \Pr\{\text{no errors}\} - \Pr\{\text{one error}\} = 1 - (1-p)^7 - \binom{7}{1} p(1-p)^6 .$$

With the given SNR of 10 dB, $P_{\text{e,inner}} \approx 1.16 \cdot 10^{-5}$, $p \approx 7.73 \cdot 10^{-6}$, and $P_{\text{e,outer}} \approx 1.26 \cdot 10^{-9}$ are obtained. The inner code chosen above is not a good one and in a practical application, a powerful (convolutional) code would probably be used rather than the weak code above.

3–16 The code consists of four codewords: 000, 011, 110 and 101. Since the full ability of the code to detect errors is utilized, an undetected error pattern must result in a different codeword from the transmitted one.

Let $\mathbf{c}$ be the transmitted codeword. Then

$$\begin{aligned}
&\Pr(\text{undetected error}|\mathbf{c} = 000) \\
&\quad = \Pr(\mathbf{r} = 011|\mathbf{c} = 000) + \Pr(\mathbf{r} = 110|\mathbf{c} = 000) + \Pr(\mathbf{r} = 101|\mathbf{c} = 000) \\
&\quad = 3\epsilon^2(1-\epsilon)
\end{aligned}$$

$$\begin{aligned}
&\Pr(\text{undetected error}|\mathbf{c} = 011) \\
&\quad = \Pr(\mathbf{r} = 000|\mathbf{c} = 011) + \Pr(\mathbf{r} = 110|\mathbf{c} = 011) + \Pr(\mathbf{r} = 101|\mathbf{c} = 011) \\
&\quad = (1-\epsilon)\gamma^2 + 2\epsilon(1-\gamma)\gamma
\end{aligned}$$

$$\begin{aligned}
&\Pr(\text{undetected error}|\mathbf{c} = 101) \\
&\quad = \Pr(\mathbf{r} = 000|\mathbf{c} = 101) + \Pr(\mathbf{r} = 011|\mathbf{c} = 101) + \Pr(\mathbf{r} = 110|\mathbf{c} = 101) \\
&\quad = \gamma^2(1-\epsilon) + 2\gamma\epsilon(1-\gamma)
\end{aligned}$$

$$\begin{aligned}
&\Pr(\text{undetected error}|\mathbf{c} = 110) \\
&\quad = \Pr(\mathbf{r} = 000|\mathbf{c} = 110) + \Pr(\mathbf{r} = 011|\mathbf{c} = 110) + \Pr(\mathbf{r} = 101|\mathbf{c} = 110) \\
&\quad = \gamma^2(1-\epsilon) + 2\gamma\epsilon(1-\gamma)
\end{aligned}$$

That is, we get the average probability of undetected error as

$$\begin{aligned}
\Pr(\text{undetected error}) &= \frac{3}{4}\left(\epsilon^2(1-\epsilon) + \gamma^2(1-\epsilon) + 2\gamma\epsilon(1-\gamma)\right) \\
&= \frac{3}{4}\left((\epsilon^2+\gamma^2)(1-\epsilon) + 2\gamma\epsilon(1-\gamma)\right)
\end{aligned}$$

3–17 (a) With the given generator polynomial $g(x) = 1+x+x^3$ we obtain the following non-systematic generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

To obtain the systematic generator matrix, add the fourth row to the first and second rows to obtain

$$\mathbf{G}_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Now, to compute the systematic generator matrix add the third row to the first row

$$\mathbf{G}_{\text{SYS}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

The parity check matrix is given by

$$\mathbf{H} = \begin{bmatrix} \mathbf{P}^T & \mathbf{I}_3 \end{bmatrix}$$

where $\mathbf{P}$ are the three last columns of $\mathbf{G}_{\text{SYS}}$, i.e.,

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

This matrix is already on a systematic form.

(b) We calculate the mutual information of X and Y with the formula

$$I(X;Y) = H(Y) - H(Y|X)$$

First, the entropy of Y given X is computed using the two expressions below.

$$\begin{aligned} H(Y|X) &= \sum_{j=1}^{2} f_X(x_j) H(Y|X = x_j) \\ H(Y|X = x_j) &= -\sum_{i=1}^{2} f_{Y|X}(y_i|x_j) \log(f_{Y|X}(y_i|x_j)) \end{aligned}$$

These expressions result in

$$\begin{aligned} H(Y|X = x_1) &= -\sum_{i=1}^{2} f_{Y|X}(y_i|x_1) \log(f_{Y|X}(y_i|x_1)) \\ &= -(1-\varepsilon)\log(1-\varepsilon) - \varepsilon\log(\varepsilon) \\ &= h(\varepsilon) \\ H(Y|X = x_2) &= h(\delta) \\ H(Y|X) &= p(h(\varepsilon) - h(\delta)) + h(\delta) \end{aligned}$$

To calculate the entropy of Y, we have to determine the probability density function.

$$\begin{aligned} f_Y(y_1) &= \sum_{j=1}^{2} f_X(x_j) f_{Y|X}(y_1|x_j) \\ &= p(1-\varepsilon) + (1-p)\delta = \delta + p(1-\varepsilon-\delta) \\ f_Y(y_2) &= 1 - f_Y(y_1) \end{aligned}$$

With the function $h(x)$ the entropy can now be written

$$H(Y) = h(\delta + p(1-\varepsilon-\delta)).$$

Finally, the mutual information can be written

$$I(X;Y) = h(\delta + p(1-\varepsilon-\delta)) - p(h(\varepsilon) - h(\delta)) - h(\delta)$$

The channel capacity is obtained by first maximizing $I(X;Y)$ with respect to p and then calculating the corresponding maximum by plugging the maximizing p into the formula for the mutual information. That is,

$$C = \max_p I(X;Y)$$

(c) According to the hint, there will be no block error as long as no more than $t = (d_{min} - 1)/2 = (3-1)/2 = 1$ bit error occur during the transmission. This holds for exactly all code words. We denote the bit error probability with p_e and the block error probability with p_b.

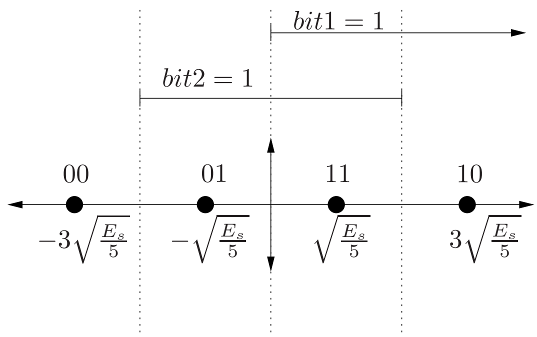
$$p_b = \Pr\{\text{block error}\} = 1 - \Pr\{\text{no block error}\} = 1 - \Pr\{0 \text{ or } 1 \text{ bit error}\}$$

Using well known statistical arguments p_b can be written

$$p_b = 1 - \left[\binom{7}{0}(1-p_e)^7 + \binom{7}{1}p_e(1-p_e)^6\right] = 1 - (1-p_e)^7 - 7p_e(1-p_e)^6.$$

By numerical evaluation (Newton-Raphson) the bit error rate corresponding to $p_b = 0.001$ is easily calculated to $p_e = 0.00698$.

3–18 (a) The 4-AM Constellation with decision regions is shown in the figure below.



The probability of each constellation point being transmitted is 1/4. the probability of noise perturbing the transmit signal by a distance greater than $\sqrt{E_s/5}$ in one direction is given by

$$P = Q\left(\sqrt{\frac{2E_s}{5N_0}}\right)$$

An error in bit1 is most likely caused by the 01 symbol being transmitted, perturbed by noise, and the receiver interpreting it as 11 or vice versa. Hence the probability of an error in bit1 is

$$P_{b1} = \frac{1}{4}P + \frac{1}{4}P = \frac{1}{2}Q\left(\sqrt{\frac{2E_s}{5N_0}}\right)$$

An error in bit2 could be cause by transmit symbol 00 being received as 01 or symbol 11 being received as 10 or 10 being received as 11 or 01 being received as 00. Hence bit2 has twice the error probability as bit1

$$P_{b2} = Q\left(\sqrt{\frac{2E_s}{5N_0}}\right)$$

It has been assumed that the cases where there noise perturbs the transmitted signal by more than $3\sqrt{E_s/5}$ are not significant.

(b) The probability of e errors in a block of size n is given by the binomial distribution,

$$P_e(e) = \binom{n}{e} P_{cb}^e (1-P_{cb})^{n-e}$$

where P_{cb} is the channel bit error probability.
When there are t errors or less then the decoder corrects them all. When there are more than t errors it is assumed that the decoder passes on the systematic part of the bit stream unaltered, so the output error rate will be e/n. The post decoder error probability P_d can then be calculated as a sum according to

$$P_d = \sum_{e=t+1}^{n} \frac{e}{n} P_e(e)$$

3–19 Assume that one of the 2^k possible code words is transmitted. To design the system, the worst case needs to be considered, i.e., probability of bit error is 1/2. Therefore, at the receiver, any one of 2^n possible combinations is received with equal likelihood. The three possible events at the receiver are

- the received word is decoded as the transmitted code word (1 possibility)
- the received word is corrupted and detected as such ($2^n - 2^k$ possibilities)
- the received word is decoded as another code word than transmitted, i.e., a undetectable error ($2^k - 1$ possibilities)

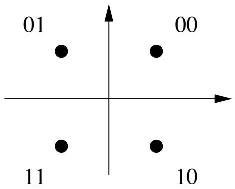
From this, it is deduced that the probability of accepting a corrupted received word is

$$P_{fd} = \frac{2^k - 1}{2^n} \approx 2^{-(n-k)} \ ,$$

i.e., the probability of undetectable errors depends on the number of parity bits appended by the CRC code. A probability of false detection of less than 10^{-10} is required.

$$P_{fd} \approx 2^{-(n-k)} < 10^{-10} \ ,$$
$$(n-k)\log_{10}(2) > 10 \ ,$$
$$k \leq 94 \ .$$

3–20 QPSK with Gray labeling is illustrated below.



The energy per *symbol* is given as E_s. Let the energy per *bit* be denoted $E_b = E_s/2$. Studying the figure, we then see that the signaling is equivalent to two independent uses of BPSK with bit-energy E_b. To convince ourselves that this is indeed the case, note that for example the first bit is 0 in the first and fourth quadrant and 1 in the second and third. Hence, with ML-detection the y-axis works as a "decision threshold" for the first bit. Similarly, the x-axis works as a decision threshold for the second bit, and the two "channels" corresponding to the two different bits are independent binary symmetric channels with bit-error probability

$$q \triangleq \Pr(\text{bit error}) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{E_s}{N_0}}\right) = Q\big(\sqrt{10^{0.7}}\big) \approx 0.012587$$

(a) There exist codes that can achieve $p_e \to 0$ at all rates below the channel capacity C of the system. Since we have discovered that the transmission is equivalent to using a binary symmetric channel (BSC) with crossover probability $q \approx 0.0126$ we have

$$C = 1 - H_b(q) \approx 0.9025 \quad \text{[bits per channel use]}$$

where $H_b(x)$ is the binary entropy function. All rates $R_c < C \approx 0.9$ are hence achievable. In words this means that as long as the fraction of information bits in a codeword (of a code with very long codewords) is less than about 90 % it is possible to convey the information without errors.

(b) The systematic form of the given generator matrix is

$$\mathbf{G}_{\text{sys}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

and we hence get the systematic parity check matrix as

$$\mathbf{H}_{\text{sys}} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Since all different non-zero 3-bit patterns are columns of $\mathbf{H}_{\text{sys}}$ we see that the given code is a Hamming code, and we thus know that it can correct all 1-bit error patterns *and no other error patterns.* (To check this carefully one may construct the corresponding standard array and note that only 1-bit errors can be corrected and no other error patterns will be listed.) The *exact* block error rate is hence

$$p_e = \Pr(> 1 \text{ error}) = 1 - \Pr(0 \text{ or } 1 \text{ error}) = 1 - (1-q)^n - nq(1-q)^{n-1} \approx 0.0032$$

(c) The energy *per information bit* is $E_b' = (n/k)E_b$. With

$$q' = Q\left(\sqrt{\frac{2E_b'}{N_0}}\right) = Q(\sqrt{7 \cdot 10^{0.7}/4}\,) \approx 0.00153$$

the "block error probability" obtained when transmitting $k = 4$ uncoded information bits is

$$p_e' = 1 - \Pr(\text{all 4 bits correct}) = 1 - (1-q')^4 \approx 0.0061 > p_e$$

Hence there is a gain, in the sense that using the given code results in a higher probability of transmitting 4 bits of information without errors.

3–21 (a) Systematic generator matrix

$$\mathbf{G}_{\text{sys}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Systematic parity-check matrix

$$\mathbf{H}_{\text{sys}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Minimum distance is $d_{\min} = 5$ since < 5 columns in $\mathbf{H}_{\text{sys}}$ cannot sum to zero, while 5 can (e.g., columns 1, 8, 9, 10, 12).

(c) The code can correct at least $t = 2$ errors, since $d_{\min} = 5 \implies$

$$\Pr(\text{block error}) \leq 1 - \Pr(\leq 2 \text{ errors}) = 1-(1-\varepsilon)^n - n\varepsilon(1-\varepsilon)^{n-1} - \binom{n}{2}\varepsilon^2(1-\varepsilon)^{n-2} \approx 0.0362$$

(d) The code can surely correct all erasure-patterns with less than $d_{\min}$ erasures, since such patterns will always leave the received word with at least $n - d_{\min} + 1$ correct bits that can uniquely identify the transmitted codeword. Hence

$$\Pr(\text{block error}) \leq 1 - \sum_{i=0}^{4} \binom{n}{i} \alpha^i (1-\alpha)^{n-i} \approx 0.00061468$$

3–22 We will need the systematic generator and parity check matrices of the $(15, 7)$ code. Based on $g(x)$ we can derive the systematic H-matrix as

$$\mathbf{H} = \begin{bmatrix} 1&0&0&0&1&0&1&1&0&0&0&0&0&0&0\\ 1&1&0&0&1&1&1&0&1&0&0&0&0&0&0\\ 1&1&1&0&1&1&0&0&0&1&0&0&0&0&0\\ 0&1&1&1&0&1&1&0&0&0&1&0&0&0&0\\ 1&0&1&1&0&0&0&0&0&0&0&1&0&0&0\\ 0&1&0&1&1&0&0&0&0&0&0&0&1&0&0\\ 0&0&1&0&1&1&0&0&0&0&0&0&0&1&0\\ 0&0&0&1&0&1&1&0&0&0&0&0&0&0&1 \end{bmatrix}$$

corresponding to the generator matrix

$$\mathbf{G} = \begin{bmatrix} 1&0&0&0&0&0&0&1&1&1&0&1&0&0&0\\ 0&1&0&0&0&0&0&0&1&1&1&0&1&0&0\\ 0&0&1&0&0&0&0&0&0&1&1&1&0&1&0\\ 0&0&0&1&0&0&0&0&0&0&1&1&1&0&1\\ 0&0&0&0&1&0&0&1&1&1&0&0&1&1&0\\ 0&0&0&0&0&1&0&0&1&1&1&0&0&1&1\\ 0&0&0&0&0&0&1&1&1&0&1&0&0&0&1 \end{bmatrix}$$

(a) The output codeword from the $(7, 4)$ code is

$$0011101$$

Via the systematic G-matrix for the $(15, 7)$ code we then get the transmitted codeword

$$001110100010000$$

(b) Studying the H-matrix one can conclude that the minimum distance of the $(15, 7)$ code is $d_{\min} = 5$ since at least 5 columns need to be summed up to produce a zero column.

Based on the H-matrix we can compute the syndrome of the received word for the $(15, 7)$ decoder as

$$\mathbf{s}^T = (00000011)$$

Since adding the last 2 columns from $\mathbf{H}$ to $\mathbf{s}$ gives a zero column, the syndrome corresponds to a double error in the last 2 positions. Since $d_{\min} = 5$ this error is corrected to the codeword

$$000101110111111$$

Since the codeword is in systematic form, the corresponding input bits to the $(15, 7)$ code are

$$0001011$$

which is the codeword in the $(7, 4)$ code corresponding to $\hat{\mathbf{a}} = (0001)$.

(c) We know that the $(7, 4)$ code is cyclic. From its G-matrix we get its generator polynomial as

$$g_1(x) = x^3 + x + 1$$

The generator polynomial of the overall code then is

$$(x^8 + x^7 + x^6 + x^4 + 1)(x^3 + x + 1) = x^{11} + x^{10} + x^7 + x^6 + x^5 + x^4 + x^3 + x + 1$$

and the cyclic generator matrix of the overall code is easily obtained as

$$\begin{bmatrix} 1&1&0&0&1&1&1&1&1&0&1&1&0&0&0\\ 0&1&1&0&0&1&1&1&1&1&0&1&1&0&0\\ 0&0&1&1&0&0&1&1&1&1&1&0&1&1&0\\ 0&0&0&1&1&0&0&1&1&1&1&1&0&1&1 \end{bmatrix}$$

3–23 (a) The systematic generator matrix has the form $\mathbf{G} = [\mathbf{I}\ \mathbf{P}]$, where the polynomial describing the i:th row of $\mathbf{P}$ can be obtained from $p^{n-i} \bmod g(p)$, where $n = 7$ and $i = 1, 2, 3$.

$$\begin{aligned}
p^6 \bmod p^4 + p^2 + p + 1 &= p^2 p^4 = p^2(p^2 + p + 1) = p^4 + p^3 + p^2 = p^2 + p + 1 + p^3 + p^2 = p^3 + p + 1 \\
p^5 \bmod p^4 + p^2 + p + 1 &= pp^4 = p(p^2 + p + 1) = p^3 + p^2 + p \\
p^4 \bmod p^4 + p^2 + p + 1 &= p^2 + p + 1
\end{aligned}$$

Hence, the generator matrix in systematic form is

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

(b) To find the minimum distance we list the codewords.

Input	Codeword
000	0000000
001	0010111
010	0101110
011	0111001
100	1001011
101	1011100
110	1100101
111	1110010

The minimum distance is $d_{min} = 4$.

(c) The received sequence consists of three independent strings of length 7 ($\mathbf{y} = [\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3]$), since the input bits are independent and equiprobable and since the channel is memoryless. Also the encoder is memoryless, unlike the encoder of a convolutional encoder. Hence, the maximum likelihood sequence estimate is the maximum likelihood estimates of the three received strings. If we call the three transmitted codewords $[\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3]$, the ML estimates of $[\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3]$ are the codewords with minimum Hamming distances from $[\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3]$. The estimated codewords and corresponding information bit strings are obtained from the table above.

$$\begin{aligned}
\hat{\mathbf{c}}_1 &= 1011100 \\
\hat{\mathbf{c}}_2 &= 1100101 \\
\hat{\mathbf{c}}_3 &= 1110010 \\
&\Rightarrow \\
\hat{\mathbf{x}}_1 &= 101 \\
\hat{\mathbf{x}}_2 &= 110 \\
\hat{\mathbf{x}}_3 &= 111
\end{aligned}$$

To conclude, $\hat{\mathbf{x}} = [\hat{\mathbf{x}}_1\hat{\mathbf{x}}_2\hat{\mathbf{x}}_3] = [101110111]$.

3–24 (a) The generator matrix is for example:

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(b) The coset leader is [0000000010]

(c) The received vectors of weight 5 that have syndrome [0 0 0 1 0] are

$$\begin{matrix} 1&0&0&0&0&1&1&1&1&0\\ 1&0&1&0&0&1&1&0&0&1\\ 0&1&1&0&0&0&1&0&1&1\\ 0&0&0&0&1&1&1&1&0&1\\ 0&1&0&0&1&1&0&0&1&1\\ 0&0&1&0&1&1&1&0&1&0\\ 1&0&1&0&1&0&0&1&1&0\\ 0&1&1&0&1&1&0&1&0&0\\ 1&1&1&0&1&0&1&0&0&0 \end{matrix}$$

The received vectors of weight 7 that have syndrome [0 0 0 1 0] are

$$\begin{matrix} 1&1&1&0&0&1&0&1&1&1\\ 1&1&0&0&1&0&1&1&1&1 \end{matrix}$$

(d) The syndrome is [1 0 1 0 1]. The coset leader is [0 0 0 1 0 0 0 1 0 0].

(e) For example

$$\begin{matrix} 0&0&1&0&0&0&0&0&1&0\\ 0&0&0&1&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&1&0&0\\ 0&0&0&1&0&0&0&0&1&0\\ 1&1&0&0&0&0&1&0&0&0\\ 1&0&1&0&0&1&0&0&0&0 \end{matrix}$$

The coset leaders should be associated to followed syndromes

$$\begin{matrix} 0&0&1&0&1\\ 1&1&0&0&1\\ 1&0&0&1&1\\ 1&1&0&1&0\\ 0&1&0&1&1 \end{matrix}$$

3–25 For any $m \geq 2$ the $n = 2^m$, $k = 2^m - m - 1$ and $r = n - k = m + 1$ extended Hamming code has as part of its parity check matrix the parity check matrix of the corresponding Hamming code in rows $1, 2, \ldots, r-1$ and columns $1, 2, \ldots, n-1$. The presence of this matrix in the larger parity check matrix of the extended code guarantees that all non-zero codewords have weight at least 3 (since the minimum distance of the Hamming code is 3). The zeros in column n from row 1 to row $r-1$ do not influence this result.

In addition, the last row of the parity check matrix of the extended code consists of all 1's. Since the length $n = 2^m$ of the code is an even number, this forces all codewords to have even weight. (The sum of an even number of 1's is 0 modulo 2.)

Hence the weight of any non-zero codeword is at least 4. Since the Hamming code has codewords of weight 4 there are codewords of weight 4 also in the extended code $\Rightarrow d_{\min} = 4$.

3–26 (a) Since the generator matrix is written in systematic form, the parity check matrix is easily obtained as

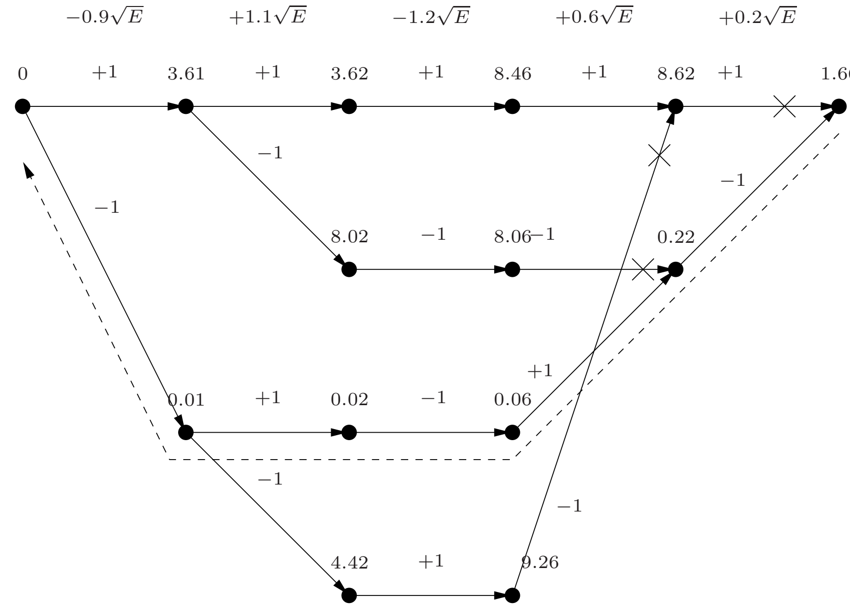
$$\mathbf{H} = \begin{bmatrix} 1&1&1&0&0\\ 0&1&0&1&0\\ 1&1&0&0&1 \end{bmatrix}$$

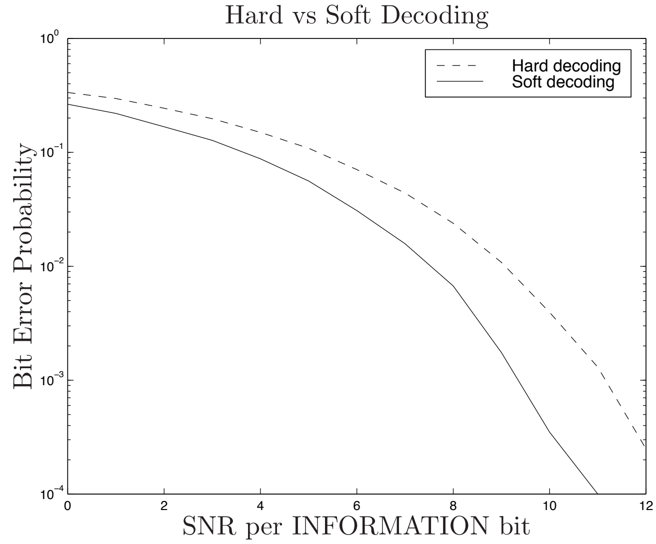
and the standard array, with the addition of the syndrome as the rightmost column, is given by (actually, only the first and last columns are needed)

00000	01111	10101	11010	000
00001	01110	10100	11011	001
00010	01101	10111	11000	010
00100	01011	10001	11110	100
01000	00111	11101	10010	111
10000	11111	00101	01010	101
00011	01100	10110	11001	011
00110	01001	10011	11100	110

Hard decisions on the received sequence results in $\mathbf{x} = 10100$ and the syndrome is $\mathbf{s} = \mathbf{x}\mathbf{H}^T = 001$, which, according to the table above, corresponds to the error sequence $\mathbf{e} = 00001$. Hence, the decoded sequence is $\hat{\mathbf{c}} = \mathbf{x} + \mathbf{e} = 10101$.

(b) Soft decoding is illustrated in the figure below. The decoded code word is 10101, which is identical to the one obtained using hard decoding. However, on the average an asymptotic gain in SNR of approximately 2 dB is obtained with soft decoding.





3–27 To solve the problem we need to compute $\Pr(\mathbf{r}|\mathbf{x})$ for all possible $\mathbf{x}$. Since $\mathbf{c} = \mathbf{x}\mathbf{G}$ we have $\Pr(\mathbf{r}|\mathbf{x}) = \Pr(\mathbf{r}|\mathbf{c})$. The decoder chooses the information block $\mathbf{x}$ that corresponds to the codeword $\mathbf{c}$ that maximizes $\Pr(\mathbf{r}|\mathbf{c})$ for a given $\mathbf{r}$. Since the channel is memoryless we get

$$\Pr(\mathbf{r}|\mathbf{c}) = \prod_{i=1}^{7} \Pr(r_i|c_i) \; .$$

With $\mathbf{r} = [1, 0, 1, \triangle, \triangle, 1, 0]$ we thus get

Information block $\mathbf{x}$ $x_1x_2x_3$	Codeword $\mathbf{c}$ $c_1c_2c_3c_4c_5c_6c_7$	Metric $\Pr(\mathbf{r}\vert\mathbf{c})$
000	0000000	$0.1^3 \cdot 0.6^2 \cdot 0.3^2$
001	0101101	$0.1^5 \cdot 0.3^2$
010	0010111	$0.1^2 \cdot 0.6^3 \cdot 0.3^2$
011	0111010	$0.1^2 \cdot 0.6^3 \cdot 0.3^2$
100	1100011	$0.1^3 \cdot 0.6^2 \cdot 0.3^2$
101	1001110	$0.1^1 \cdot 0.6^4 \cdot 0.3^2$
110	1110100	$0.1^2 \cdot 0.6^3 \cdot 0.3^2$
111	1011001	$0.1^2 \cdot 0.6^3 \cdot 0.3^2$

The "erasure symbols" give the same contribution, 0.1^2, to all codewords. Because of symmetry we hence realize that these symbols should simply be ignored. Studying the table we see that the codeword $[1, 0, 0, 1, 1, 1, 0]$ maximizes $\Pr(\mathbf{r}|\mathbf{c})$. The corresponding information bits are $[1, 0, 1]$.

3–28 Based directly on the given generator matrix we see that $n = 9$, $k = 3$ and $r = n - k = 6$.

(a) The general form for the generator polynomial is

$$g(p) = p^r + g_2 p^{r-1} + \cdots + g_r p + 1$$

We see that the given $\mathbf{G}$-matrix is the "cyclic version" generated based on the generator polynomial, so we can identify

$$g(p) = p^6 + p^3 + 1$$

The parity check polynomial can be obtained as

$$h(p) = \frac{p^n + 1}{g(p)} = \frac{p^9 + 1}{p^6 + p^3 + 1} = p^3 + 1$$

(b) We see that in this problem the cyclic and systematic forms of the generator matrix are equal, that is, the given $\mathbf{G}$-matrix is already in the systematic form. Hence we can immediately get the systematic $\mathbf{H}$-matrix as

$$\mathbf{G}_{\text{sys}} = \begin{bmatrix}\mathbf{I}_k | \mathbf{P}\end{bmatrix} \Rightarrow \mathbf{H}_{\text{sys}} = \begin{bmatrix}\mathbf{P}^T | \mathbf{I}_r\end{bmatrix}$$

with

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The codewords of the code can be obtained from the generator matrix as

000000000
100100100
010010010
001001001
110110110
011011011
101101101
111111111

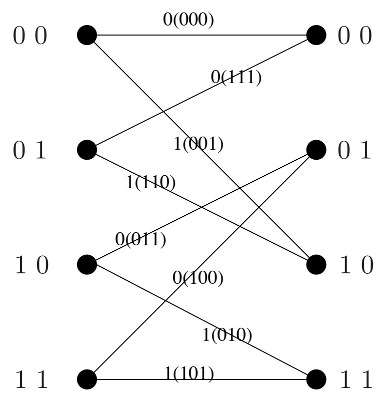
Hence we see that $d_{\min} = 3$.

(c) The given bound is the union bound to block-error probability. Since the code is linear, we can assume that the all-zeros codeword was transmitted and study the neighbors to the all-zero codeword. The N_i terms correspond to the number of neighboring codewords at different distances. There are 3 codewords at distance 3, 3 at distance 6 and one at distance 9, hence $N_1 = N_2 = 3$ and $N_3 = 1$. The M_i terms correspond to the distances according to $M_1 = 3, M_2 = 6, M_3 = 9$.

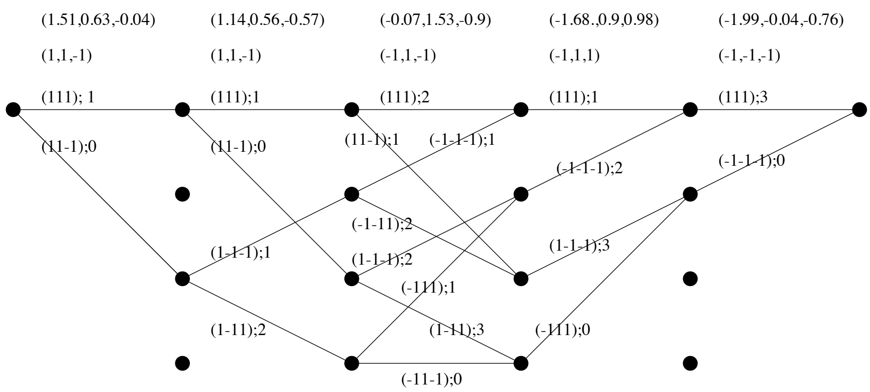
3–29 First we study the encoder and determine the structure of the trellis. For each new bit we put into the encoder there are two old bits. The memory of the encoder is thus 2 bits, which tells us that the encoder consists of 2^2 states. If we denote the bits in the register with x_k, x_{k-1} and x_{k-2} with x_k being the leftmost one, the following table is easily computed.

x_k	Old state of encoder (x_{k-1} and x_{k-2})	c_1	c_2	c_3	New state (x_k and x_{k-1})
0	0 0	0	0	0	0 0
0	0 1	1	1	1	0 0
0	1 0	0	1	1	0 1
0	1 1	1	0	0	0 1
1	0 0	0	0	1	1 0
1	0 1	1	1	0	1 0
1	1 0	0	1	0	1 1
1	1 1	1	0	1	1 1

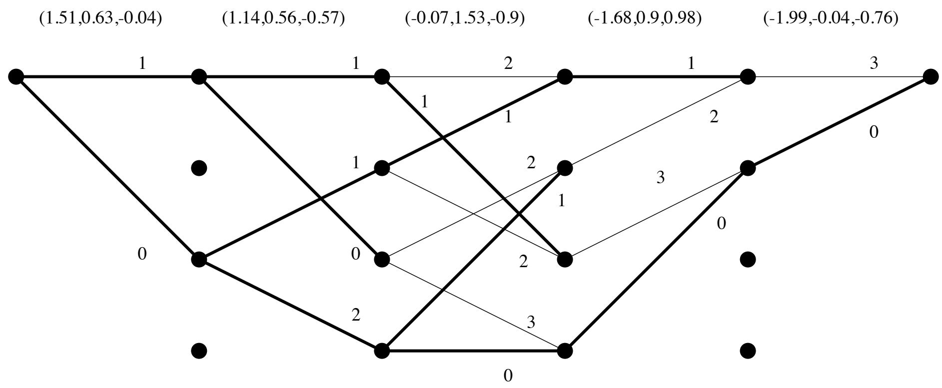
The corresponding one step trellis looks like



In the next figure the decoding trellis is illustrated. The output from the transmitter and the corresponding metric is indicated for each branch. The received sequence is indicated at the top with the corresponding hard decisions below. The branch metric indicated in the trellis is the number of bits that differ between the received sequence and the corresponding branch.



Now, it only remains to choose the path that has the smallest total metric. The survivor paths at each step are in the below figure indicated with thick lines. From this trellis we see that the path with the smallest total metric is the one at the "bottom".



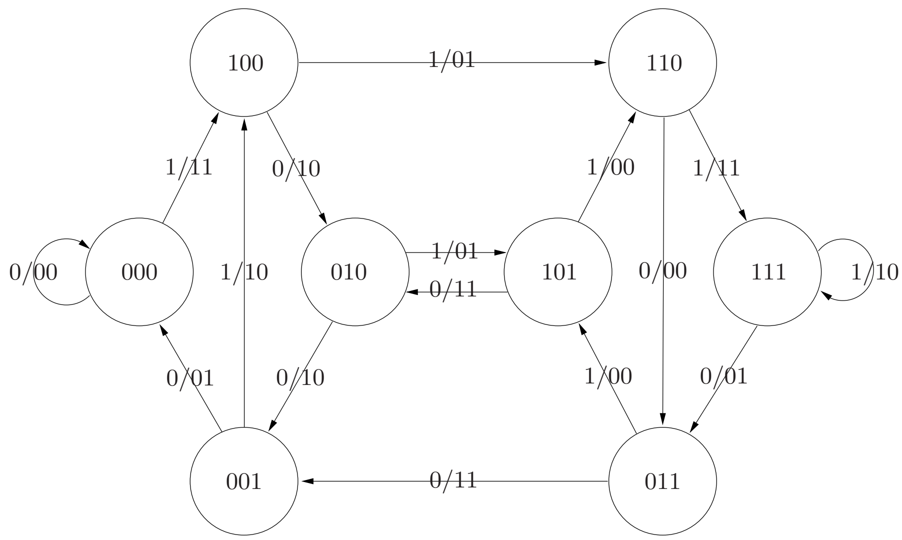
Note that when we go down in the trellis the input bit is one and when we go up the input bit is zero. Thus, the decoder output is $\{1, 1, 1, 0, 0\}$.

3–30 To check if the encoder generates a catastrophic code we need to check if the encoder graph contains a circuit in which a nonzero input sequence corresponds to an all-zero output sequence.

Assume that the right-most bit in the shift register is denoted with $x(0)$ and the input bit with $x(3)$. Then we obtain the following table for the encoder

$x(3)$	$x(2)$	$x(1)$	$x(0)$	$y^{(0)}(3)$	$y^{(1)}(3)$
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	1	1
0	1	0	0	1	0
0	1	0	1	1	1
0	1	1	0	0	0
0	1	1	1	0	1
1	0	0	0	1	1
1	0	0	1	1	0
1	0	1	0	0	1
1	0	1	1	0	0
1	1	0	0	0	1
1	1	0	1	0	0
1	1	1	0	1	1
1	1	1	1	1	0

The corresponding encoder graph looks like:



We see that the circuit $011 \to 101 \to 110 \to 011$ generates an all-zero output for a nonzero input. The encoder is thus catastrophic.

3–31 First we show that if we choose as partial path metric $a(\log(p(r(k)|c(k)) + b)$, $a > 0$, then the estimate using this metric is still the maximum likelihood estimate. The ML-estimate is equal to the code word that minimizes

$$\log p(\mathbf{r}|\mathbf{c}) = \sum_{k=0}^{N} \log p(r(k)|c^i(k)).$$

With the new path metric we have

$$\sum_{k=0}^{N} a(\log p(r(k)|c^i(k)) + b) = abN + a\sum_{k=0}^{N} \log p(r(k)|c^i(k)),$$

which is easily seen to have the same minimizing/maximizing argument as the original ML-function. Now choose a and b according to

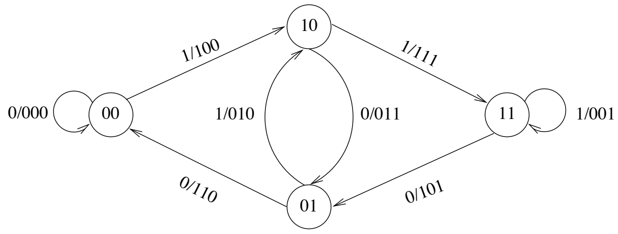
$$a = \frac{1}{\log\gamma - \log(1-\gamma)}, \qquad b = -\log(1-\gamma).$$

We now have the following for the partial path metric

$a(\log(p(r(k)\|c^i(k)) + b)$	r(k)=0	r(k)=1
c(k)=0	0	1
c(k)=1	1	0

This shows that the Hamming distance is valid as a path metric for all cross over probabilities γ.

3–32 (a) With $\big(x(n-1), x(n-2)\big)$ defining a state, we get the following state diagram.



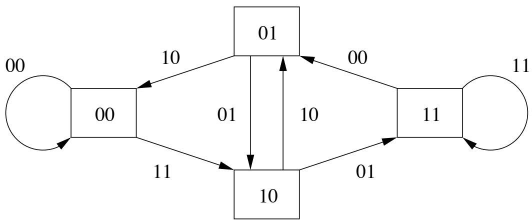
The minimum free distance is the minimum Hamming distance between two different sequences. Since the code is linear we can consider all codewords that depart from the zero-sequence. That is, we are interested in finding the lowest-weight non-zero sequence that starts and ends in the zero state. Studying the state diagram we are convinced that this sequence corresponds to the path $00 - 10 - 01 - 00$ and the coded sequence $100, 011, 110$, of weight five.

(b) The received sequence is of length 12, corresponding to two information bits and two zeros ("tail") to make the encoder end in the zero state. ML decoding corresponds to finding the closest code-sequence to the received sequence $\mathbf{r} = 001, 110, 101, 111$. Considering the following table

Info $\mathbf{x}$	Codeword $\mathbf{c}$	Metric $d_H(\mathbf{r},\mathbf{c})$
00 (00)	000,000,000,000	8
01 (00)	000,100,011,110	5
10 (00)	100,011,110,000	9
11 (00)	100,111,101,110	4

we hence get that the ML estimate of the encoded information sequence is $\hat{\mathbf{x}} = 1, 1, (0, 0)$.

3–33 The state diagram of the encoder is obtained as



(a) Based on the state diagram, the transfer function in terms of the "output distance variable D" can be computed as

$$T = \frac{D^4(2-D)}{1-4D-2D^4-D^6} = 2D^4 + \cdots$$

Hence there are two sequences of weight 4, and the free distance is thus 4. (This can also be concluded directly from the state diagram by studying all the different ways of leaving state 00 and getting back "as fast as possible.")

(b) The codewords are of the form

$$\mathbf{c} = (c_{11}, c_{12}, c_{21}, c_{22}, c_{31}, c_{32}, c_{41}, c_{42}, c_{51}, c_{52})$$

where we note that $c_{12} = x_1, c_{22} = x_2, c_{32} = x_3$. A generator matrix can thus be formed by the three codewords that correspond to $x_1x_2x_3 = 100, 010, 001$. Using the state diagram we hence get

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

From the $\mathbf{G}$-matrix we can conclude that

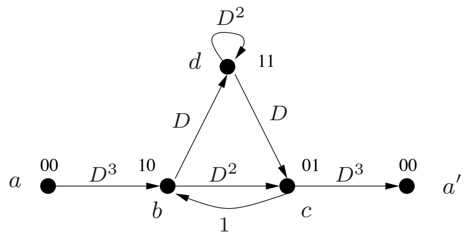
$$\begin{aligned} c_{11} &= x_1 \\ c_{12} &= x_1 \\ c_{21} &= x_1 + x_2 \\ c_{22} &= x_2 \\ c_{31} &= x_1 + x_2 + x_3 \\ c_{32} &= x_3 \\ c_{41} &= x_2 + x_3 \\ c_{42} &= 0 \\ c_{51} &= x_3 \\ c_{52} &= 0 \end{aligned}$$

These equations can be used to compute the corresponding $\mathbf{H}$-matrix as

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Note, e.g., that the first two equations give $c_{11} + c_{12} = 0$, reflected in the first row of $\mathbf{H}$, and the second, third and fourth equation give $c_{12} + c_{21} + c_{22} = 0$, as reflected in the second row of $\mathbf{H}$, and so on.

3–34 The encoder has four states. A graph that describes the output weights of different state-transitions in terms of the variable D, with state "00" split into two parts, is obtained as below.



From the graph we get the following equations that describe how powers of D evolve through the graph.

$$b = D^3a + c, \quad c = D^2b + Dd, \quad d = Db + D^2d, \quad a' = D^3c$$

Solving for a'/a gives the transfer function

$$T(D) = \frac{a'}{a} = \frac{2D^8 - D^{10}}{1 - 3D^2 + D^4} = 2D^8 + 5D^{10} + 13D^{12} + 34D^{14} + \cdots$$

Hence,

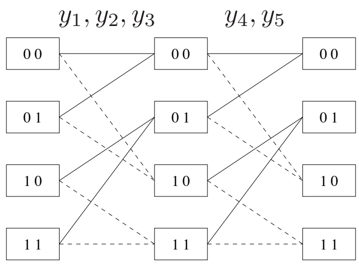
(a) $d_{\text{free}} = 8$ (and there are 2 different paths of weight 8)

(b) See above

(c) There are 34 different such sequences

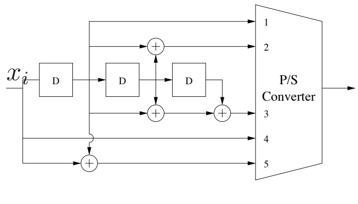
3–35 a) For each source bit the convolutional encoder produces 3 encoded bits. The puncturing device reduces 6 encoded bits to 5. Hence, the rate of the punctured code is $r = 2/5 = 0.4$.

b) Soft decision decoding is considered. Thus the Euclidian distance measure is the appropriate measure to consider. The trellis diagram of the punctured code is shown below. Note that

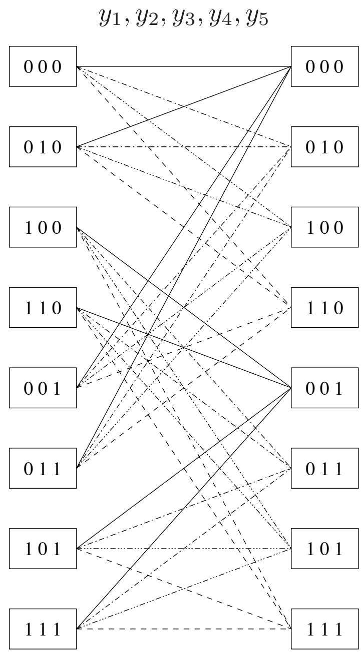


since every 6th bit is discarded, the trellis structure will repeat itself with a periodicity equal to two trellis time steps (corresponding to 5 sent BPSK symbols). During the first time step, the detector is to consider the 3 first input samples, $y1, y2, y3$. In the second, the two last i.e., $y4, y5$.

c) The correspondning conventional implementation of the punctured code is shown below. From the figure we recognice the previously considered rate 1/3 encoder as the rightmost



part of the new encoder. The output corresponding to the punctured part is generated by the network around the first memory element. Note that in order for the conventional implementation to be identical to the 1/3 encoder followed by puncturing, two input samples must be clocked into the delay line prior to generating the 5 output samples. The corrseponding trellis diagram isshown below.

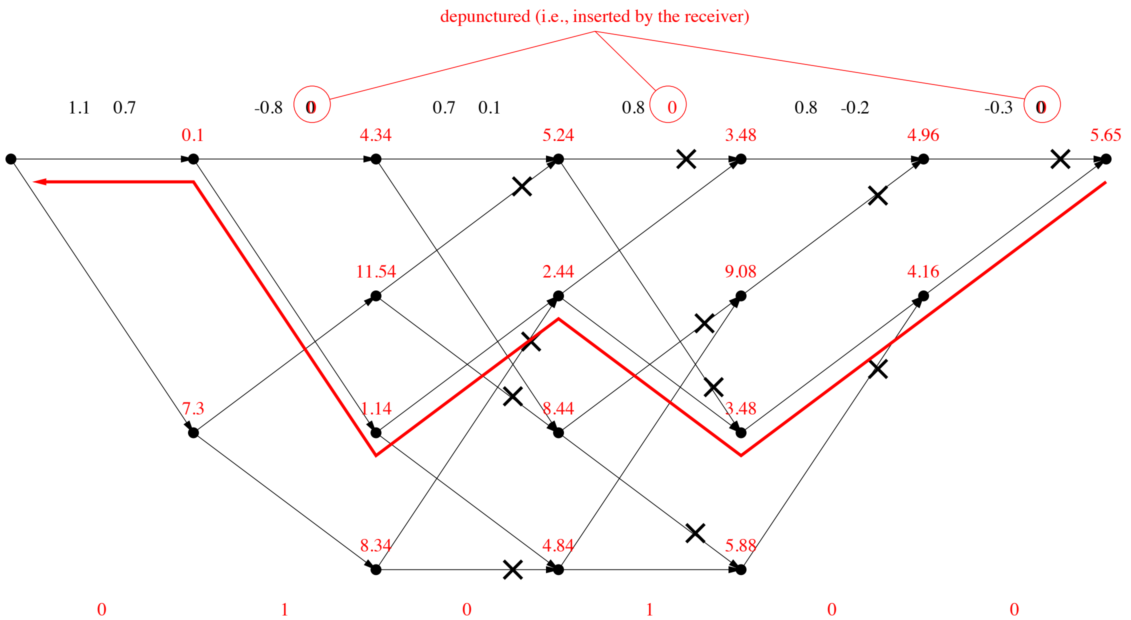


d) The representation in b) is better since in order to traverse the 2 time steps of the trellis corresponding to the single time-step in c), we need to perform 16 distance calculation and sum operations while considering the later trellis we need to carry out 32 operations.

3–36 For every two bits fed to the encoder, four bits result. Of those four bits, one is punctured. Hence, two bits in results in three bits being transmitted and the code rate is 2/3.

First, the receiver depunctures the received sequence by inserting zeros where the punctured bits should have appeared in absence of puncturing. Since the transmitted bits, $\{0, 1\}$, are mapped onto $\{+1, -1\}$ by the BPSK modulator, an inserted 0 in the received sequence of soft information corresponds to no knowledge at all about the particular bit (the Euclidean distances to $+1$ and -1 are equal). After depuncturing, the conventional Viterbi algorithm can be used for decoding, which results in the information sequence being estimated as 010100, where the last two bits is the tail.

Puncturing of convolutional codes is common. One advantage is that the same encoder can be used for several different code rates by simply varying the amount of puncturing. Secondly, puncturing is often used for so-called rate matching, i.e., reducing the number of encoded bits such that the number fits perfectly into some predetermined format. Of course, excessive puncturing will destroy the properties of the code and care must be taken to ensure a good puncturing pattern. As a rule of thumb, up to approximately 20% of puncturing works fairly well.



3–37 Since the two coded bits have different probabilities, a MAP estimator should be used instead of an ML estimator. The MAP algorithm maximizes

$$M = \Pr\{\mathbf{r}|\mathbf{c}\}\Pr\{\mathbf{c}\} = \prod_{i=1}^{4} \Pr\{r_{i,1}|c_{i,1}\}\Pr\{c_{i,1}\}\Pr\{r_{i,2}|c_{i,2}\}\Pr\{c_{i,2}\} \ .$$

The second equality above is valid since the coded bits are independent. In reality, it is more realistic to assume independent *information* bits, but with the coder used, this results in independent coded bits as well. It is common to use the logarithm of the metric instead, i.e.,

$$\mathcal{M} = \log(M) = \sum_i \left(\log\left[\Pr\{r_{i,1}|c_{i,1}\}\Pr\{r_{i,2}|c_{i,2}\}\right] + \log\left[\Pr\{c_{i,1}\}\Pr\{c_{i,2}\}\right]\right) \ .$$

The probabilities are given by

s_{i-1}	s_i	Trans. $c_{i,1}c_{i,2}$	Received $r_{i,1}r_{i,2}$ 00	01	10	11	$r_{i,1}r_{i,2}$ 00	01	10	11
0	0	00	$(1-\varepsilon)(1-\varepsilon)$	$(1-\varepsilon)\varepsilon$	$\varepsilon(1-\varepsilon)$	$\varepsilon\varepsilon$	.81	.09	.09	.01
1	0	01	$(1-\varepsilon)\varepsilon$	$(1-\varepsilon)(1-\varepsilon)$	$\varepsilon\varepsilon$	$\varepsilon(1-\varepsilon)$	.09	.81	.01	.09
1	1	10	$\varepsilon(1-\varepsilon)$	$\varepsilon\varepsilon$	$(1-\varepsilon)(1-\varepsilon)$	$(1-\varepsilon)\varepsilon$	.09	.01	.81	.09
0	1	11	$\varepsilon\varepsilon$	$(1-\varepsilon)\varepsilon$	$(1-\varepsilon)\varepsilon$	$(1-\varepsilon)(1-\varepsilon)$	.01	.09	.09	.81

Initially $(i = 1)$

$c_{i,1}$	$c_{i,2}$	Probability
0	0	$p_0 = .3$
0	1	—
1	0	—
1	1	$p_1 = .7$

Continously $(i = 2, 3)$

$c_{i,1}$	$c_{i,2}$	Probability
0	0	$p_0p_0 = .09$
0	1	$p_0p_1 = .21$
1	0	$p_1p_1 = .49$
1	1	$p_1p_0 = .21$

Tail $(i = 4)$

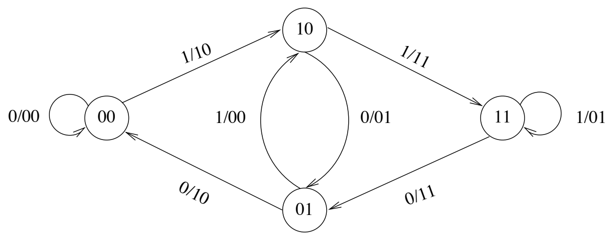
$c_{i,1}$	$c_{i,2}$	Probability
0	0	$p_0 = .3$
0	1	$p_1 = .7$
1	0	—
1	1	—

The maximization of $\mathcal{M}$ (or, equivalently, the minimization of $-\mathcal{M}$) is easily obtained by using the Viterbi algorithm. The correct path has a metric given by $\mathcal{M} = (\log .7 + \log .09) + (\log .21 + \log .81) + (\log .21 + \log .81) + (\log .7 + \log .09) = -9.07$, corresponding to the coded sequence 11 01 11 01 and the information sequence 1010. In summary:

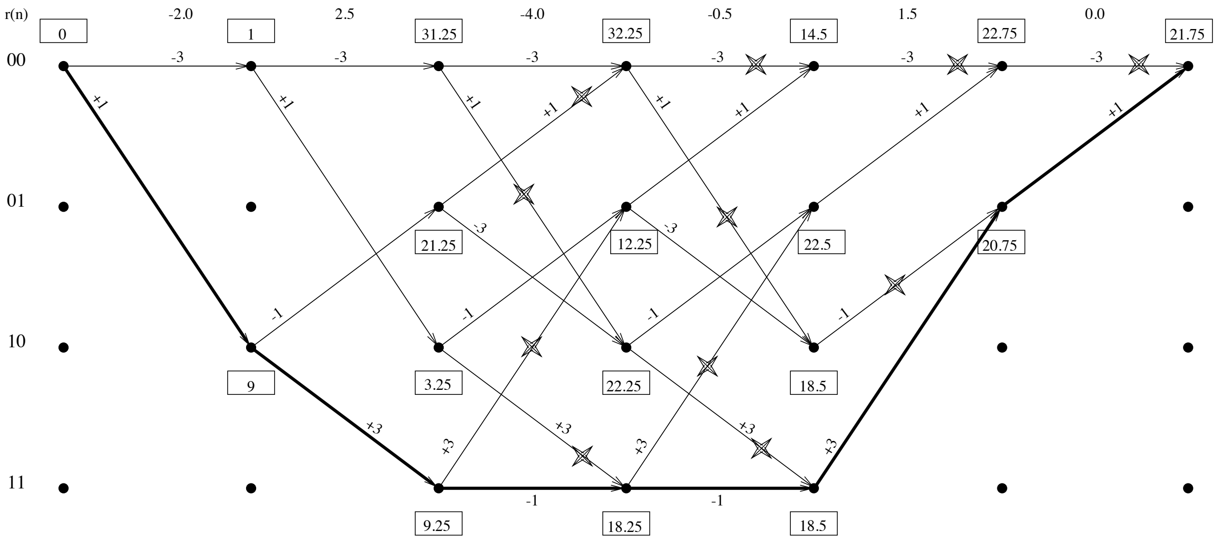
- Alice's answer is not correct (even if the decoded sequence by coincidence happens to be the correct one). She has, without motivation, used the Viterbi algorithm to find the ML solution. This does *not* give the best sequence estimate since the information bits have *different a priori probabilities*. Since the answer lacks all explanation and is fundamentally wrong, no points should be given.

- Bob's solution is correct. However, it is badly explained and therefore it warrants only four points. It is not clear whether the student has fully understood the problem or not.

3–38 Letting $\big(x(n-1), x(n-2)\big)$ determine the state, the following state diagram is obtained



The mapping of the modulator is $00 \to -3, 01 \to -1, 10 \to +1, 11 \to +3$, and this gives the trellis

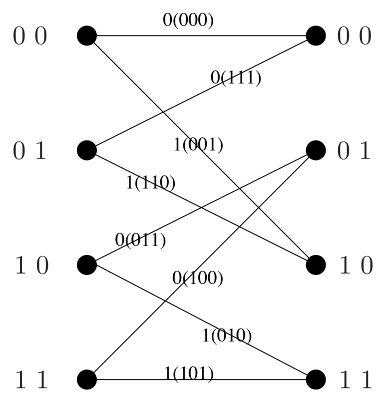


The sequence in the trellis closest to the received sequence is: $+1, +3, -1, -1, +3, +1$. This corresponds to the coded sequence: 10 11 01 01 11 10. The corresponding data sequence is: $1, 1, 1, 1, 0, 0$.

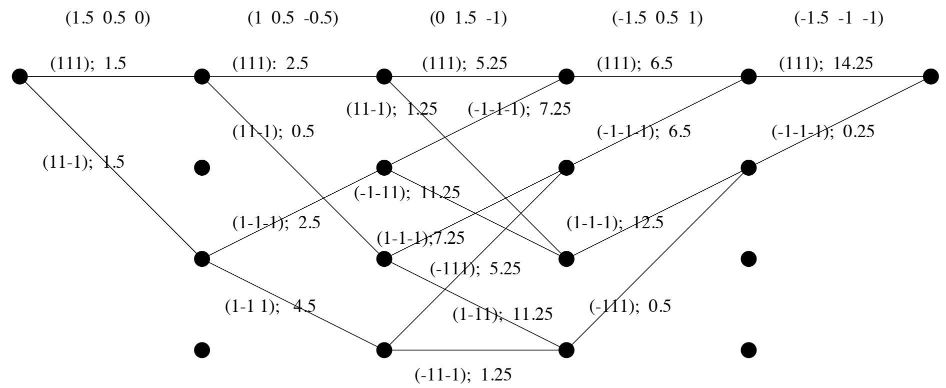
3–39 First we study the encoder and determine the structure of the trellis. For each new bit we put into the encoder there are two old bits. The memory of the encoder is thus 2 bits, which tells us that the encoder consists of 2^2 states. If we denote the bits in the register with x_k, x_{k-1} and x_{k-2} with x_k being the leftmost one, the following table is easily computed.

x_k	Old state of encoder (x_{k-1} and x_{k-2})	c_1	c_2	c_3	New state (x_k and x_{k-1})
0	0 0	0	0	0	0 0
0	0 1	1	1	1	0 0
0	1 0	0	1	1	0 1
0	1 1	1	0	0	0 1
1	0 0	0	0	1	1 0
1	0 1	1	1	0	1 0
1	1 0	0	1	0	1 1
1	1 1	1	0	1	1 1

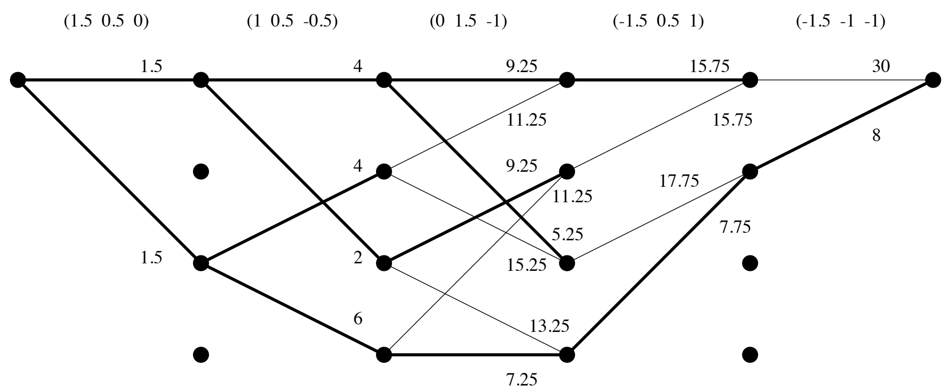
The corresponding one step trellis looks like



In the next figure the decoding trellis is illustrated. The output from the transmitter and the corresponding metric is indicated for each branch. The received sequence is indicated at the top.



Now, it only remains to choose the path that has the smallest total metric. The survivor paths at each step are in the below figure indicated with thick lines. From this trellis we see that the path with the smallest total metric is the one at the "bottom". Note that the two paths entering the top node in the last but one step have the same total metric. When this happens choose one of them at random, the resulting estimate at the end will still be maximum likelihood.

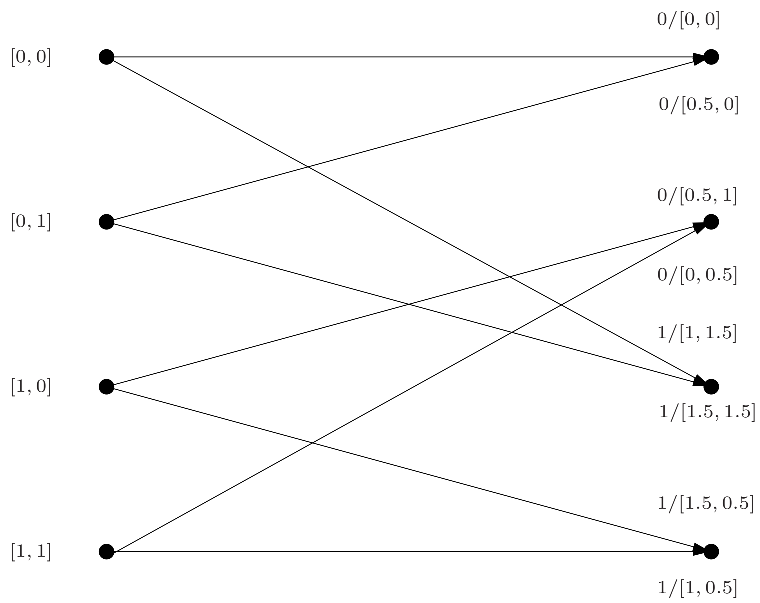


Note that when we go down in the trellis the input bit is one and when we go up the input bit is zero. Thus, the maximum likelihood decoder output is $\{1,1,1,0,0\}$.

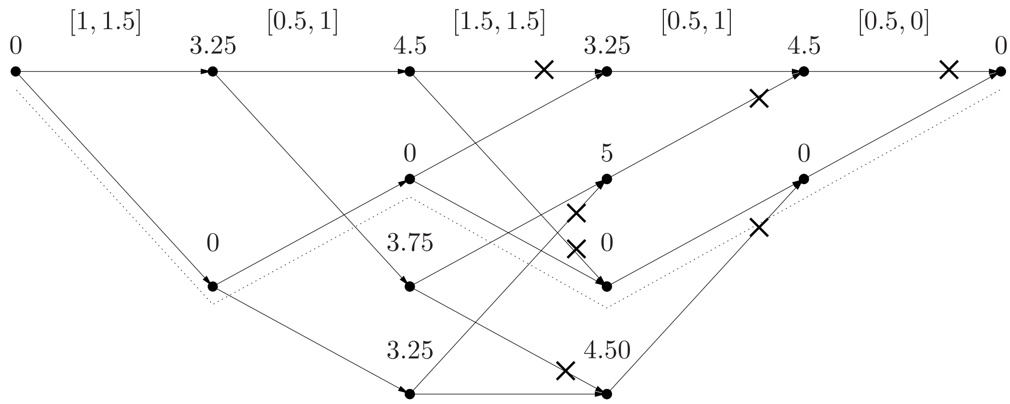
3–40 The encoder followed by the ISI channel can be viewed as a composite encoder where ($\oplus$ denotes addition modulo 2)

$$r_{i1} = d_i + \alpha\,(d_{i-1} \oplus \alpha d_{i-2})$$
$$r_{i2} = d_i \oplus d_{i-1} + \alpha d_i \quad .$$

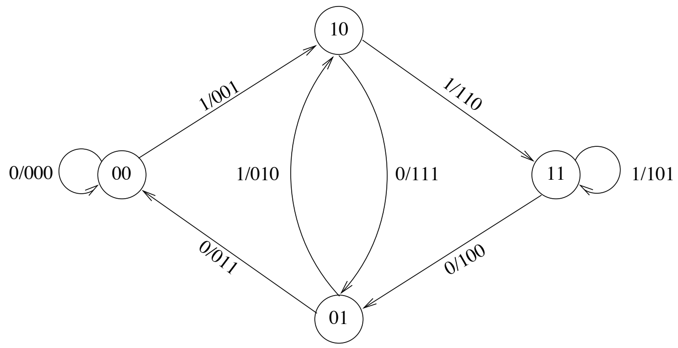
The trellis transitions, where the state vector is $[d_{i-1} d_{i-2}]$, are given by



ML-decoding of the received sequence yields 10100. Note that the final metric is zero, i.e., the received sequence is noise free. Since there is no noise, the problem is somewhat degenerated. A more realistic problem would have noise. If the noise is white Gaussian noise, the optimal metric is given by the Euclidean distance.

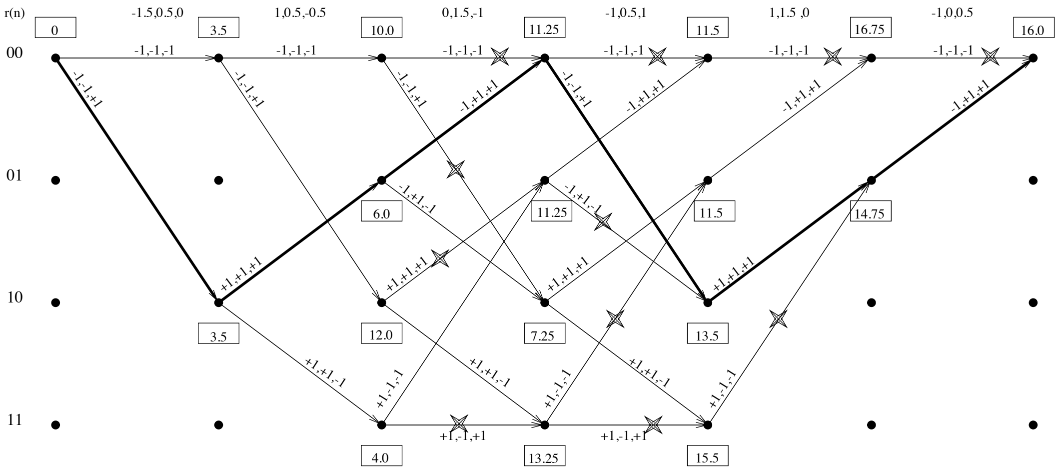


3–41 Letting $\big(x(n-1), x(n-2)\big)$ define the states, we get the following state diagram.



The rate of the code is 1/3. Hence 18 coded bits correspond to 6 information bits.

Implementing soft ML decoding is solved using the Viterbi algorithm and the trellis below.



The transmitted sequence closest to the received sequence is

$$\hat{\hat{\mathbf{s}}} = -1, -1, 1, \;\; 1, 1, 1, \;\; -1, 1, 1, \;\; -1, -1, 1, \;\; 1, 1, 1, \;\; -1, 1, 1 \;\; .$$

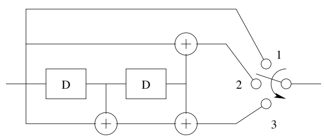
corresponding to the codeword

$$\hat{\hat{\mathbf{c}}} = 001\ 111\ 011\ 001\ 111\ 011\ .$$

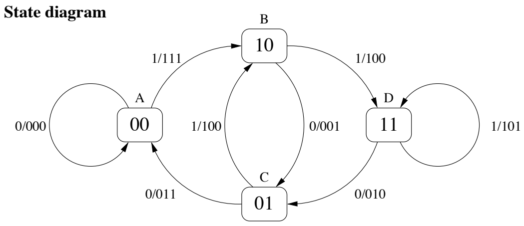
and the information bits

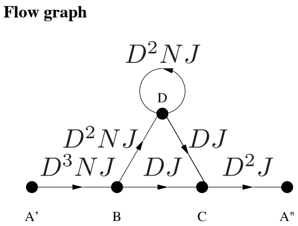
$$\hat{\hat{\mathbf{x}}} = 1, 0, 0, 1, 0, 0\,.$$

3–42 (a) The rate $\frac{1}{3}$ convolutional encoder may be drawn as below.



From the state diagram the flow graph may be established as shown below.





The free distance corresponds to the smallest possible number of binary ones generated when jumping from the start state A' to the return state A''. Thus, from the graph it is easily verified that $d_{\text{free}} = 6$.

(b) The pulse amplitude output signal may be written:

$$s(t) = \sum_m \sqrt{E_c}(2x_m - 1)p(t - mT).$$

The signal part of $y(t) = y_s(t) + y_w(t)$ is then given by $y_s(t) = (s \star q)(t) = \int s(\tau)q(t-\tau)d\tau$. Hence,

$$\begin{aligned} y_s(t = \frac{i}{T}) &= \sqrt{E_c}\sum_m (2x_m - 1)\int p(\tau - mT)q(\frac{i}{T} - \tau)d\tau \\ &= \sqrt{E_c}(2x_i - 1)\int |p(\tau - iT)|^2 d\tau = \sqrt{E_c}(2x_i - 1). \end{aligned}$$

The noise part of $y(t)$ is found as $y_w(t) = (w \star q)(t) = \int w(\tau)q(t-\tau)d\tau$. Since $e_i = y_w(iT)$ it follows directly that e_i will be zero mean Gaussian. The acf is

$$\begin{aligned}
E\{e_i e_{i+k}\} &= E\{y_w(iT)y_w((i+k)T)\} \\
&= \int_\tau \int_\gamma E\{w(\tau)w(\gamma)\}q(iT-\tau)q((i+k)T-\gamma)d\gamma d\tau \\
&= \frac{N_0}{2}\int_\tau q(iT-\tau)q((i+k)T-\tau)d\tau \\
&= \begin{cases} \frac{N_0}{2} & k=0 \\ 0 & k \neq 0 \end{cases}
\end{aligned}$$

Hence, $\alpha = \sqrt{E_c}$, $\beta = E\{e_i e_i\} = \frac{N_0}{2}$. e_i is a temporally white sequence since $E\{e_i e_{i+k}\} = 0$ for $k \neq 0$.

(c) i. The message sequence contains 5 unknown entries. The remaining two are known and used to bring the decoder back to a known state. Hence, since there is only one path through the flow graph generating d_{free} ones, there exist only 5 possible error events with $d_{\text{free}} = 5$. (This may easily be verified drawing the trellis diagram!) Thus, the size of the set $\mathcal{I}$ is 5.

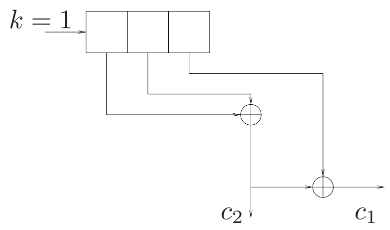
ii. Given that $i \in \mathcal{I}$ it follows directly that

$$\Pr\{\bar{\boldsymbol{b}}_i \text{ detected} \,|\, \bar{\boldsymbol{b}}_0 \text{ sent}\} = Q\left(\frac{d_E(\bar{\boldsymbol{s}}_i, \bar{\boldsymbol{s}}_0)}{2\sqrt{\frac{N_0}{2}}}\right) = Q\left(\sqrt{\frac{2E_c d_{\text{free}}}{N_0}}\right) \leq \left(e^{-\frac{2E_c}{N_0}}\right)^{\frac{d_{\text{free}}}{2}}.$$

Since $e^{-\frac{2E_c}{N_0}}$ is the Chernoff bound of the channel bit error probability we may thus approximate

$$P_{\text{seq. error}} \approx P_{\text{ch. bit error}}^{\frac{d_{\text{free}}}{2}}.$$

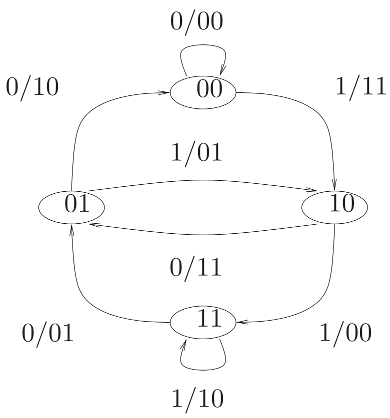
3–43 (a) The encoder is $g_1 = (1\;\;1\;\;1)$ and $g_2 = (1\;\;1\;\;0)$. The shift register for the code is:



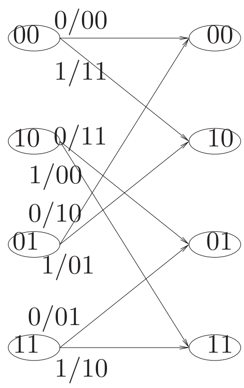
The state diagram is obtained by:

old state					*new state*	
x_k	x_{k-1}	x_{k-2}	c_1	c_2	x_k	x_{k-1}
0	0	0	0	0	0	0
0	0	1	1	0	0	0
0	1	0	1	1	0	1
0	1	1	0	1	0	1
1	0	0	1	1	1	0
1	0	1	0	1	1	0
1	1	0	0	0	1	1
1	1	1	1	0	1	1

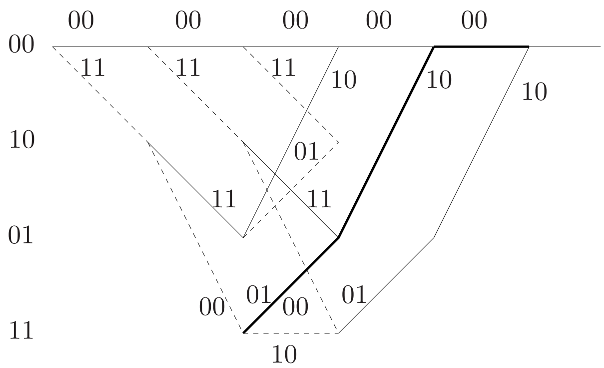
and



(b) The corresponding one step trellis looks like:

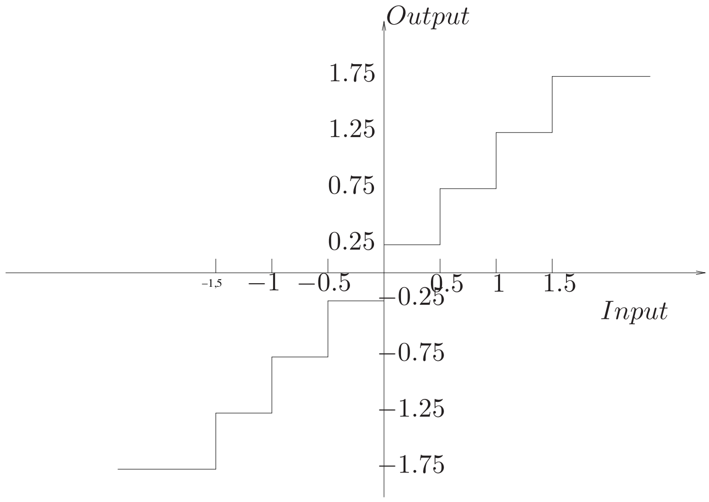


The terminated trellis is given in the followed figure and $d_{free} = 4$.



(c) The reconstruction levels of the quantizer are

$$\{-1.75 \ -1.25 \ -0.75 \ -0.25 \ 0.25 \ 0.75 \ 1.25 \ 1.75\}$$



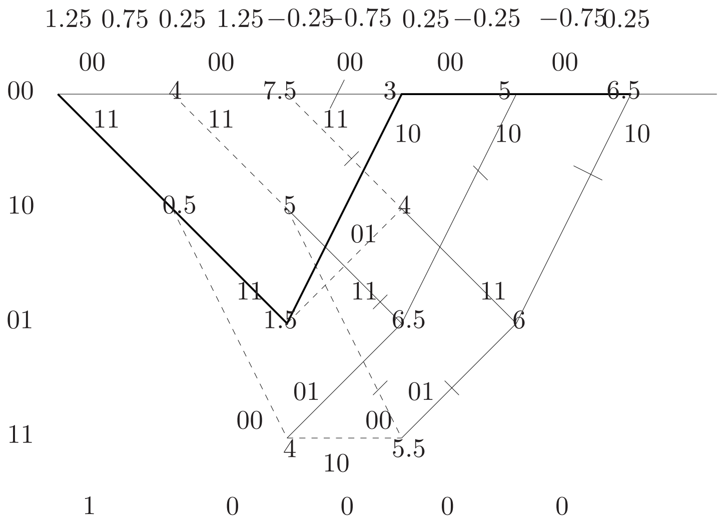
and the quantized sequence is

$$\{1.25 \;\; 0.75 \;\; 0.25 \;\; 1.25 \;\; -0.25 \;\; -0.75 \;\; 0.25 \;\; -0.25 \;\; -0.75 \;\; 0.25\}$$

Minimum Euclidean distance is calculated as:

$$\sum_{k=1}^{2} |s^{(k)} - \mathcal{Q}(r)^{(k)}|$$

where $\mathcal{Q}(r)$ is the quantized receiving bits. Decoding starts from and ends up at all zero state.



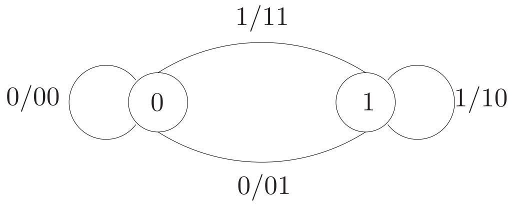
The estimated message is $\hat{s} = \{1\,0\,0\,0\,0\}$ and the first 3 bits are information bits $\hat{I} = \{1\,0\,0\}$

3–44 (a) To find the free distance, d_F, the state transition diagram can be analyzed. The input bit, s_i, and state bit, s_D, are mapped to coded bits according to the following table.

s_i	s_D	a_i	b_i
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

The two-state encoder is depicted below, where the arrow notation is s_i/a_ib_i and the state is s_D.

The free distance is equal to the minimum Hamming weight of any non-zero codeword, which is obtained by leaving the '0'-state and directly returning to it. This gives the free distance $d_F = 3$.



(b) First consider the hard decoding. The optimal decision unit is simply a threshold device, since the noise is additive, white and Gaussian. For $r_i \geq 0$, $\hat{c}_i = 1$, and otherwise, $\hat{c}_i = 0$. For the particular received vector in this problem, $\hat{\mathbf{c}} = [11010100]$. By applying the Viterbi algorithm, we obtain the codeword [11010000] with minimum Hamming distance 1 from $\hat{\mathbf{c}}$, which corresponds to the incorrect information bits $\hat{\mathbf{s}}_h = [100]$.

Considering the soft decoder, we again use the Viterbi algorithm to find the most likely transmitted codeword [1 1 0 1 1 1 0 1] with minimum squared Euclidean distance 3.73 from $\mathbf{r}$, which corresponds to the correct information bits $\hat{\mathbf{s}}_s = [101]$.

(c) Yes, it is possible. For example, if $\mathbf{s} = [101]$ and $\mathbf{z} = [0.0\ 0.0\ 0.9\ -0.9\ -0.9\ -0.9\ 0.9\ -11]$, the hard decoder correctly estimates $\hat{\mathbf{s}}_h = [101]$ and the soft decoder incorrectly estimates $\hat{\mathbf{s}}_s = [110]$.

3–45 Larry uses what is known as *Link Adaptation*, i.e., the coding (and sometimes the modulation) scheme is chosen to match the current channel conditions. In a high-quality channel, less coding can be used to increase the throughput, while in a noise channel, more coding is necessary to ensure reliable decoding at the receiver. For a fixed E_b/N_0, the block error rate is found from the plots. From the plot, it is seen that the switching point between the two coding schemes, γ, should be set to $\gamma \approx 6.7$ dB as the uncoded scheme has a block error rate less than 10% at this point. Since one of the objectives with the system was to provide as high bit rate as possible, there is no need to waste valuable transmission time on transmitting additional redundancy if uncoded transmission suffices. The average block error probability is found as

$$P_{\text{block, LA}} = 0.1 \cdot P_{\text{cod.}}(2\text{ dB}) + 0.4 \cdot P_{\text{cod.}}(5\text{ dB}) + 0.5 \cdot P_{\text{unc.}}(8\text{ dB}) \approx 5.2 \cdot 10^{-2}$$

and the average number of 100-bit blocks transmitted per information block is

$$n_{\text{LA}} = 2 \cdot 0.1 + 2 \cdot 0.4 + 1 \cdot 0.5 = 1.5\ .$$

Irwin uses a simple form of *Incremental Redundancy*. Whenever an error is detected, a retransmission is requested and soft combining of the received samples from the previous and the current block is performed before decoding. In case of a retransmission, this is identical to repetition coding, i.e., each bit has twice the energy (+3 dB) compared to the original transmission (a repeated bit can be seen as one single bit with twice the duration). Hence, for a fixed transmitter power, the probability of error is $P_\mathrm{e}(E_\mathrm{b}/N_0)$ for the original transmission and $P_\mathrm{e}(E_\mathrm{b}/N_0 + 3\text{ dB})$ after one retransmission. Hence, the resulting block error probability is, as a maximum of one retransmission is allowed, given by

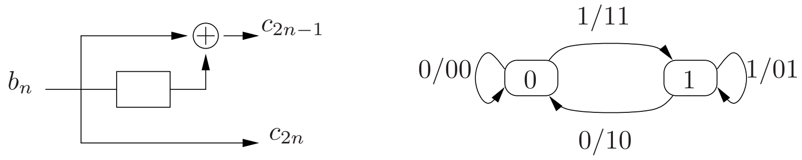
$$P_{\text{block, IR}} = 0.1 \cdot P_{\text{unc.}}(2\text{ dB} + 3\text{ dB}) + 0.4 \cdot P_{\text{unc.}}(5\text{ dB} + 3\text{ dB}) + 0.5 \cdot P_{\text{unc.}}(8\text{ dB} + 3\text{ dB}) \approx 4.8 \cdot 10^{-2}$$

(the last term hardly contributes) and the average number of blocks transmitted is

$$\begin{aligned} n_{\text{IR}} &= 0.1 \cdot [1 \cdot (1 - P_{\text{unc.}}(2\text{ dB})) + 2 \cdot P_{\text{unc.}}(2\text{ dB})] \\ &\quad + 0.4 \cdot [1 \cdot (1 - P_{\text{unc.}}(5\text{ dB})) + 2 \cdot P_{\text{unc.}}(5\text{ dB})] \\ &\quad + 0.5 \cdot [1 \cdot (1 - P_{\text{unc.}}(8\text{ dB})) + 2 \cdot P_{\text{unc.}}(8\text{ dB})] \\ &\approx 1.27 \end{aligned}$$

The incremental redundancy scheme above can be improved by, instead of transmitting uncoded bits, encode the information with a, e.g., rate 1/2 convolutional code, but only transmit half the encoded bits in the first attempt (using puncturing). If a retransmission is required, the second half of the coded bits are transmitted and used in the decoding process. Both incremental redundancy and (slow) link adaptation are used in EDGE, an evolution of GSM supporting higher data rates.

3–46 The encoder and state diagram of the convolutional code are given below.



Let $\mathbf{b} = (b_1, \ldots, b_5)$ (where b_1 comes before b_2 in time) be a codeword output from the encoder of the $(5, 2)$ block code. One tail-bit (a zero) is added and the resulting 6 bits are encoded by the convolutional encoder. Based on the state-diagram it is straightforward to see that this operation can be described by the generator matrix

$$\mathbf{G}_{\text{conv}} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

in the sense that $\mathbf{c} = \mathbf{b}\,\mathbf{G}_{\text{conv}}$ where $\mathbf{c} = (c_1, \ldots, c_{12})$ is the corresponding block of output bits from the encoder of the convolutional code. Consequently, since $\mathbf{b} = \mathbf{G}_{\text{block}}\mathbf{a}$, where $\mathbf{a} = (a_1, a_2)$ contains the two information bits and where $\mathbf{G}_{\text{block}}$ is the generator matrix of the given $(5, 2)$ block code, we get

$$\mathbf{c} = \mathbf{a}\,\mathbf{G}_{\text{block}}\mathbf{G}_{\text{conv}} = \mathbf{a}\mathbf{G}_{\text{tot}}$$

where

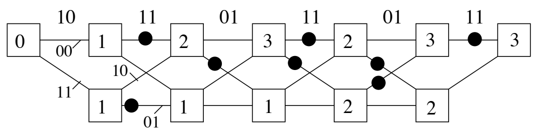
$$\mathbf{G}_{\text{tot}} = \mathbf{G}_{\text{block}}\mathbf{G}_{\text{conv}} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

is the equivalent generator matrix that describes the concatenation of the block and convolutional encoders.

(a) Four information bits produce two 12-bit codewords at the output of the convolutional encoder, according to

$$(\mathbf{c}_1, \mathbf{c}_2) = \big((00)\mathbf{G}_{\text{tot}}, (11)\mathbf{G}_{\text{tot}}\big) = (000000000000, 110110110110)$$

(b) The received 12-bit sequence is decoded in two steps. It is first decoded using the Viterbi algorithm to produce a 5-bit word $\hat{\mathbf{b}}$, as illustrated below



resulting in $\hat{\mathbf{b}} = (011110)$. Then $\hat{\mathbf{b}}$ is decoded by an ML decoder for the block code. Since the codewords of the block code are

00000
10100
01111
11011

we see that the closest codeword to $\hat{\mathbf{b}}$ is (01111) resulting in $\hat{\mathbf{a}} = (01)$.

(c) We have already specified the equivalent generator matrix $\mathbf{G}_{\text{tot}}$. The simplest way to calculate the minimum distance of the overall code is to look at the four codewords

000000000000
111011100000
001101010110
110110110110

computed using $\mathbf{G}_{\text{tot}}$. We see that the total minimum distance is six. Studying the trellis diagram of the convolutional code, we see that its free distance is three (corresponding to the state-sequence $0 \to 1 \to 0$). Also, looking at the codewords of the $(5, 2)$ block code we see that its minimum distance is two. And since $6 = 2 \cdot 3$ we can conclude that the statement is true.