ERRATA
EXERCISES 2011 - Modeling of dynamical systems

1.3) The last sentence should read, “[…] and the **concentration** of nutrition […]”.

1.5) In the solutions, numerical values and expressions are wrong. Correct solution included with this document.

3.2 d) In the solution, $\frac{dx_1}{dt} = x_3 - x_4$

5.7 a) In the solution, $\dot{x_1} = \frac{1}{m_1}\left(u(t) - \frac{2}{d}\left(J(z) + \frac{2b}{d}x_1(t) + glm_2\cos\left(x_2(t)\right)\right)\right)$

The third observation gives ut that

$$\dot{\eta}(t) = -k_3\dot{b}(t).$$

The resulting state space equations are

$$\begin{aligned}\dot{b}(t) &= k_2\frac{\eta(t)}{\eta(t)+\frac{k2}{k1}}b(t)\\ \dot{\eta}(t) &= -k_3k_2\frac{\eta(t)}{\eta(t)+\frac{k2}{k1}}b(t).\end{aligned}$$

1.4

The electric circuit gives us the following equations:

$$\begin{aligned}u(t) &= R_1i(t) + v_{C_1} \text{ (Kirchoff's voltage law)}\\ C_1\dot{v}_{C_1}(t) &= i_{C_1}\\ i(t) &= i_{C_1} + \frac{v_{C_1}}{R_2} \text{ (Kirchoff's current law),}\end{aligned}$$

which implies that

$$R_1\left(C_1\dot{v}_{C_1}(t) + \frac{v_{C_1}(t)}{R_2}\right) + v_{C_1} = u(t).$$

Energy balance gives us the following equations:

$$\begin{aligned}P_{\text{in}}(t) &= \frac{v_{C_1}^2(t)}{R_2}\\ P_{\text{out}}(t) &= q(t) = k(T(t) - T_0(t))\\ C\dot{T}(t) &= P_{\text{in}}(t) - P_{\text{out}}(t).\end{aligned}$$

Putting it all together, the resulting state space system is:

$$\begin{aligned}\dot{v}_{C_1}(t) &= \frac{-(R_1+R_2)}{R_1R_2C_1}v_{C_1}(t) + \frac{1}{R_1C_1}u(t)\\ \dot{T}(t) &= \frac{1}{R_2C}v_{C_1}^2(t) + \frac{k}{C}(T_0(t) - T(t))\\ y(t) &= T(t).\end{aligned}$$

1.5

a) Denote the volume inflow w_{in}. Let the area of the holes at a, b, and c be α. The volume flow through hole a is $w_a = \alpha\sqrt{2g(h_1 - h_3)}$ and the flow through hole c is $w_c = \alpha\sqrt{2gh_2}$.

When $h_2 = 10$ cm, then the outflow is 40 cm^3/s. This gives

$$\alpha = \frac{w_C}{\sqrt{2g(h_2)}} = \frac{4 \cdot 10^{-5}}{\sqrt{2 \cdot 9.81 \cdot 0.1}} \approx 2.86 \cdot 10^{-5} m^2$$

The mass in the tank is $\rho V = \rho A h$, and therefore the change in mass in the tank is $\rho \dot{V} = \rho A \dot{h}$. Mass balance for the tanks yields

$$\begin{aligned} \rho A \dot{h}_1 &= \rho w_{\text{in}} - w_a = \rho w_{\text{in}} - \rho \alpha \sqrt{2g(h_1 - h_3)} \\ \rho A \dot{h}_2 &= \rho w_a - \rho w_c = \rho \alpha \sqrt{2g(h_1 - h_3)} - \rho \alpha \sqrt{2gh_2} \end{aligned}$$

which can be simplified to

$$\begin{aligned} \dot{h}_1 &= \frac{1}{A}\left(w_{\text{in}} - \alpha \sqrt{2g(h_1 - h_3)}\right) \\ \dot{h}_2 &= \frac{1}{A}\left(\alpha \sqrt{2g(h_1 - h_3)} - \alpha \sqrt{2g(h_2)}\right). \end{aligned}$$

b) Find an equilibrium point $(h_1^0, h_2^0, w_{\text{in}}^0)$ and linearize around this point. The equilibrium pump flow w_{in}^0 is found by setting the derivatives to zero. It is given that $h_1^0 = 30$ cm and $h_2^0 = 10$ cm. The resulting equilibrium point is

$$\begin{aligned} w_{\text{in}}^0 &= \alpha \sqrt{2g(h_1^0 - h_3)} = \alpha \sqrt{2gh_2^0} = 4 \cdot 10^{-5} m^3/s \\ h_1^0 - h_3 &= h_2^0 \end{aligned}$$

Linearization is based on Taylor series expansion, where higher order terms are neglected

$$\dot{x} = f(x) \approx f(x_0) + \nabla f(x)^{\intercal}(x - x_0),$$

where $f(x_0)$ **always** is zero in our case, since x_0 is an equilibrium point.
Define new variables as

$$\begin{aligned} \Delta h_1 &= h_1 - h_1^0 \\ \Delta h_2 &= h_2 - h_2^0 \\ \Delta w_{\text{in}} &= w_{\text{in}} - w_{\text{in}}^0. \end{aligned}$$

Note that $\Delta \dot{h}_1 = \dot{h}_1$ etc and that $\frac{d}{dx} k\sqrt{x-a} = \frac{k}{2\sqrt{x-a}}$. We have $f_1(h_1, h_2, w_{\text{in}}) = \dot{h}_1$ and $f_2(h_1, h_2, w_{\text{in}}) = \dot{h}_2$ and

$$\frac{\partial f_1}{\partial h_1} = -\frac{\alpha\sqrt{g}}{\sqrt{2}A\sqrt{h_1 - h_3}}, \ \frac{\partial f_1}{\partial h_2} = 0, \ \frac{\partial f_1}{\partial w_{\text{in}}} = \frac{1}{A}$$

$$\frac{\partial f_2}{\partial h_1} = \frac{\alpha\sqrt{g}}{\sqrt{2}A\sqrt{h_1 - h_3}}, \ \frac{\partial f_2}{\partial h_2} = -\frac{\alpha\sqrt{g}}{\sqrt{2}A\sqrt{h_2}}, \ \frac{\partial f_2}{\partial w_{\text{in}}} = 0.$$

Set $d = \frac{\alpha\sqrt{g}}{\sqrt{2}A\sqrt{h_1^0 - h_3}} = \frac{\alpha\sqrt{g}}{\sqrt{2}A\sqrt{h_2^0}}$, then the resulting linear system is

$$\begin{pmatrix} \Delta\dot{h}_1 \\ \Delta\dot{h}_2 \end{pmatrix} = \begin{pmatrix} -d & 0 \\ d & -d \end{pmatrix} \begin{pmatrix} \Delta h_1 \\ \Delta h_2 \end{pmatrix} + \begin{pmatrix} 1/A \\ 0 \end{pmatrix} \Delta w_{\text{in}}.$$

The transfer functions are

$$\Delta H_1(s) = \frac{1/A}{s+d} \Delta W_{\text{in}}(s)$$
$$\Delta H_2(s) = \frac{d}{s+d} \Delta H_1(s) = \frac{d/A}{(s+d)^2} \Delta W_{\text{in}}(s).$$

It is given that $h_1^0 = 30$ cm and $h_2^0 = 10$ cm, thus $d \approx 0.071$ and $1/A \approx 360$. Thus

$$\Delta H_1(s) = \frac{360}{s+0.071} \Delta W_{\text{in}}(s)$$
$$\Delta H_2(s) = \frac{6.0}{s+0.071} \Delta H_1(s) = \frac{2100}{(s+0.071)^2} \Delta W_{\text{in}}(s).$$

c) Mass balance for the tanks yields

$$\rho A\dot{h}_1 = \rho w_{\text{in}} - w_b = \rho w_{\text{in}} - \rho\alpha\sqrt{2g(h_1 - h_2)}$$
$$\rho A\dot{h}_2 = \rho w_b - \rho w_c = \rho\alpha\sqrt{2g(h_1 - h_2)} - \rho\alpha\sqrt{2gh_2}$$

which can be simplified to

$$\dot{h}_1 = \frac{1}{A}\left(w_{\text{in}} - \alpha\sqrt{2g(h_1 - h_2)}\right)$$
$$\dot{h}_2 = \frac{1}{A}\left(\alpha\sqrt{2g(h_1 - h_2)} - \alpha\sqrt{2g(h_2)}\right).$$

The equilibrium point is

$$h_2^0 = \frac{1}{2}h_1^0 = 10\text{cm}$$
$$w_{\text{in}}^0 = \alpha\sqrt{2g(h_1^0 - h_2^0)} = \alpha\sqrt{2gh_2^0} = 4 \cdot 10^{-6}\text{m}^3/\text{s}$$

The resulting linearized system (see b) for details) is

$$\begin{pmatrix} \Delta\dot{h}_1 \\ \Delta\dot{h}_2 \end{pmatrix} = \begin{pmatrix} -e & e \\ e & -2e \end{pmatrix} \begin{pmatrix} \Delta h_1 \\ \Delta h_2 \end{pmatrix} + \begin{pmatrix} 1/A \\ 0 \end{pmatrix} \Delta W_{\text{in}},$$

with $e = \alpha\sqrt{2g(h_1^0 - h_2^0)} = \alpha\sqrt{2gh_2^0} \approx 0.071$. The transfer function from the pump flow to h_2 is

$$\Delta H_2(s) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s+e & -e \\ -e & s+2e \end{pmatrix}^{-1} \begin{pmatrix} 1/A \\ 0 \end{pmatrix} \Delta W_{\text{in}}(s) =$$
$$= \frac{1}{(s+e)(s+2e) - e^2} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s+2e & e \\ e & s+e \end{pmatrix} \begin{pmatrix} 1/A \\ 0 \end{pmatrix} \Delta W_{\text{in}}(s) =$$
$$= \frac{e/A}{s^2 + 3es + e^2} \Delta W_{\text{in}}(s) \approx \frac{26}{s^2 + 0.21s + 0.0051} \Delta W_{\text{in}}(s)$$