Solutions to Exam in 2E1262 Nonlinear Control, Dec 18, 2000

1. (a) Plugging in the transformation and the control law gives

$$\dot{z} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} z$$

Since both eigenvalues of this matrix are -1, the system is asymptotically stable.

(b) Let

$$k(x) = u(z(x)) = -x_1 - 2x_2 - 2f(x_1) - [x_2 + f(x_1)]f'(x_1).$$

From the linear system in (a), we have the Lyapunov function $V(z) = z^T P z$, where $P = P^T > 0$ is the solution to the Lyapunov equation $PA + A^T P = -Q$. Choosing $Q = 2I$ (the identity matrix) gives

$$P = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

Hence,

$$V(x) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^T P \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + f(x_1) \end{pmatrix}^T \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 + f(x_1) \end{pmatrix},$$

which is easily checked to fulfill the conditions for the Lyapunov theorem for global stability (including $V(x) \to \infty$ as $\|x\| \to \infty$).

(c) For simplicity we include Δ in the nonlinearity, and thus consider $\hat{G}(s) = 1/[s(s+1)]$ in feedback with $\hat{f}(y) = \Delta K \arctan(y)$. Note that $\operatorname{Re} G(\hat{i\omega}) = -1/(1+\omega^2) > -1$ and that $\hat{f}(\cdot)$ is bounded by the sector $[0, \Delta K]$. The Circle Criterion now gives BIBO stability for $\Delta K < 1$.

(d) Small Gain Theorem is not applicable since the gain of $1/s$ is infinite.

2. (a) $x = (r, \dot{r}, \theta, \dot{\theta})^T$ gives

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ -g \sin x_3 - \beta x_2 + x_1 x_4^2 \\ x_4 \\ \dfrac{-2x_1x_2x_4 - gx_1 \cos x_3 + u}{x_1^2 + 1} \end{pmatrix} = f(x, u)$$

(b) $f(x_0, u_0) = 0$ gives $x_0 = (x_{10}, 0, k\pi, 0)^T$ with x_{10} and u_0 being the solutions to the equation $gx_{10} \cos k\pi = u_0$ and k being an integer.

(c) Take $x_0 = (0, 0, 0, 0)^T$ and $u_0 = 0$. Then,

$$\frac{\partial f}{\partial x}(x_0, u_0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\beta & -g & 0 \\ 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \end{pmatrix}, \qquad \frac{\partial f}{\partial u}(x_0, u_0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(d) Denote the linearization in (c) by

$$\delta\dot{x} = A\delta x + B\delta u.$$

Choosing $L = (\ell_1, \ell_2, \ell_3, \ell_4)$ such that the eigenvalues of $A - BL$ are in the left half-plane gives a control law $u = -Lx$, which is stabilizing also for $\dot{x} = f(x, u)$.

(e) The system $\ddot{r} + \beta\dot{r} = 0$ together with the suggested $V(r, \dot{r})$ gives

$$\begin{aligned} &V(0,0) = 0 \\ &V(r,\dot{r}) > 0, \qquad (r,\dot{r}) \neq (0,0) \\ &\dot{V}(r,\dot{r}) = -\beta\dot{r}^2 \leq 0 \\ &V(r,\dot{r}) \to \infty, \qquad \|(r,\dot{r})\| \to \infty, \end{aligned}$$

and, hence, global stability. Asymptotic stability does not follow because $\dot{V}(r,\dot{r}) \not< 0$ for all $(r,\dot{r}) \neq (0,0)$. (What's the physical interpretation?)

3. (a) Large T_t corresponds to low feedback of the error signal $v - u$ (see the lecture slides for notation). Hence, $T_t = 3$ corresponds to the largest overshoot in y, $T_t = 2$ corresponds to the slightly smaller overshoot etc. Similarly for u, $T_t = 3$ corresponds to the curve that is saturated the longest amount of time etc.

(b) The system is defined by

$$\begin{aligned} \dot{x} &= \operatorname{sat}(u - x) \\ y &= x + \operatorname{sat}(u - x) \end{aligned}$$

If $|u - x| \leq 1$ then $y = u$. If $u - x > 1$ then

$$\begin{aligned} \dot{x} &= 1 \\ y &= x + 1 \end{aligned}$$

and thus x will increase until $u - x < 1$ so that $y = u$ holds again. Similar for $u - x < -1$.

(c) The system consists of $G(s) = ab\exp(-sL)/s$ in negative feedback with a relay. The curve $-1/N(A)$ for the relay is the negative real axis, which $G(i\omega)$ intersects at points

$$-\frac{\pi}{2} - \omega L = \pi + 2\pi k, \qquad k = 0, 1, 2, \ldots,$$

that is, frequencies

$$\omega_k = \frac{\pi}{2L} + \frac{2\pi}{L}k, \qquad k = 0, 1, 2, \ldots.$$

Only $k = 0$ corresponds to a stable oscillation. The amplitude A of the relay input y follows from the equation $-1/N(A) = G(i\omega)$ or

$$\frac{\pi A}{4} = \frac{ab}{\omega_0},$$

that is,

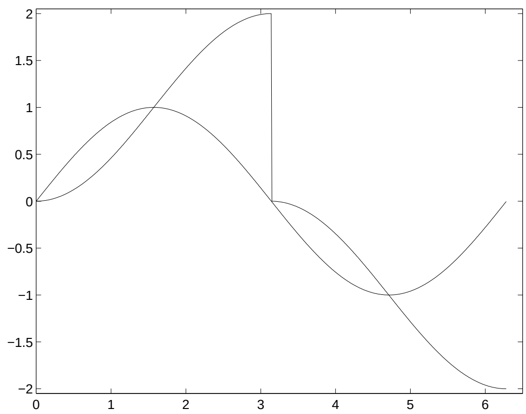
$$A_0 = \frac{4ab}{\pi\omega_0}.$$

The amplitude of the oscillations in x is thus

$$\frac{A_0}{b} = \frac{4a}{\pi\omega_0},$$

with ω_0 as given above.

4. (a) The output of the Clegg integrator over one period of the input is shown below:



(b) The output of an integrator with sinusoidal input is

$$\int_0^t A\sin\omega t\,dt = \frac{A}{\omega}\big(1-\cos\omega t\big).$$

Using this in the derivation of the describing function $N(A,\omega) = [b_1(\omega) + ia_1(\omega)]/A$ of the Clegg integrator gives

$$\begin{aligned} a_1(\omega) &= \frac{1}{\pi}\int_0^\pi x(\phi)\cos\phi\,d\phi = \frac{2A}{\pi\omega}\int_0^\pi (1-\cos\phi)\cos\phi\,d\phi = \cdots = -\frac{A}{\omega} \\ b_1(\omega) &= \frac{1}{\pi}\int_0^\pi x(\phi)\sin\phi\,d\phi = \frac{2A}{\pi\omega}\int_0^\pi (1-\cos\phi)\sin\phi\,d\phi = \cdots = -\frac{4A}{\pi\omega} \end{aligned}$$

(c) An advantage with the Clegg integrator is that it has a phase lag of 38 degrees, which is better than the 90 degrees for an ordinary integrator. However, a disadvantage is that the Clegg integrator is likely to induce oscillations.

5. (a)

$$H = L + \lambda^T f = \frac{1}{2}x^T Qx + \frac{1}{2}u^T Ru + \lambda^T(Ax + Bu)$$

(b)

$$\dot{\lambda}(t) = -\frac{\partial H^T}{\partial x}(x^*(t), u^*(t), \lambda(t)) = -Qx^*(t) - A^T\lambda(t)$$

with

$$\lambda(t_f) = \frac{\partial \phi^T}{\partial x}(x^*(t_f)) = 0$$

(c) Since H is a quadratic form in u and there is no constraints on u, the minimum is given by the solution to

$$0 = \frac{\partial H}{\partial u} = Ru + B^T\lambda,$$

which hence gives $u = R^{-1}B^T\lambda$.

(d) Combining

$$\dot{\lambda}(t) = \dot{S}(t)x(t) + S(t)[Ax^*(t) - BR^{-1}B^T S(t)x^*(t)]$$

and

$$\dot{\lambda}(t) = -Qx^*(t) - A^T\lambda(t) = -Qx^*(t) - A^T S(t)x^*(t)$$

gives the result.

(e) For $t_f = \infty$ we may set $\dot{S} = 0$. Then, the differential equation in $S(t)$ becomes an algebraic equation in the (constant) matrix S:

$$A^T S + SA - SBR^{-1}B^T S + Q = 0$$

See Lecture 13.

(f)

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial u}\dot{u} + \frac{\partial H}{\partial \lambda}\dot{\lambda} = \frac{\partial H}{\partial x}f - f^T\frac{\partial H}{\partial x} = 0$$