Automatic Control
Department of Signals, Sensors & Systems

Nonlinear Control, 2E1262

Exam 14:00-19:00, Dec 18, 2000

Aid: • Lecture notes. (Textbooks, exercises, solutions, calculators etc. may not be used.)

Observandum:

- Name and social security number (*personnummer*) on every page.
- Only one solution per page.
- A motivation must be attached to every answer.
- Specify number of handed in pages on cover.
- Each subproblem is marked with its maximum credit.

Grading:

Grade 3: ≥ 23

Grade 4: ≥ 33

Grade 5: ≥ 43

Results: The results will be posted on the department's board on second floor. **If you want your result emailed, please, state this and include your email address.** The marked exams are available for discussion Jan 16, 12:00-13:00, in S3's seminar room, 6th floor, Osquldasväg 10.

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Good Luck and Merry Christmas!

1.

(a) [2p] Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 + f(x_1) \\ \dot{x}_2 &= u, \end{aligned} \tag{1}$$

where $f(x_1)$ is a $\mathbf{C}^1$ function with $f(0) = 0$. Show that the coordinate transformation

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 + f(x_1), \end{aligned}$$

together with the control law

$$u = -z_1 - 2z_2 - z_2 f'(z_1)$$

gives an asymptotically stable linear system $\dot{z} = Az$.

(b) [3p] Find a state feedback controller $k : \mathbb{R}^2 \to \mathbb{R}$ for (1) such that the origin is asymptotically stable for the closed-loop system

$$\begin{aligned} \dot{x}_1 &= x_2 + f(x_1) \\ \dot{x}_2 &= k(x). \end{aligned} \tag{2}$$

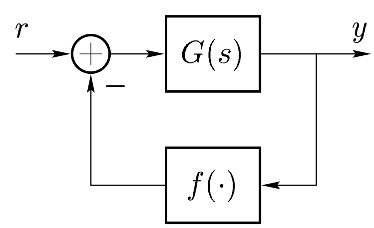
(You may use your result in (a).) Find a (Lyapunov) function $V : \mathbb{R}^2 \to \mathbb{R}$ and use it to prove that $x = 0$ is a globally asymptotically stable equilibrium for (2).

(c) [3p] Consider the feedback system below with

$$G(s) = \frac{\Delta}{s(s+1)}$$

and

$$f(y) = K \arctan(y)$$



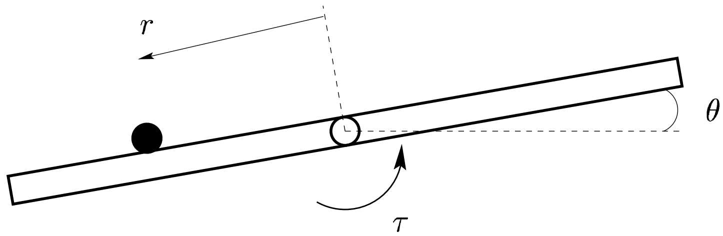
For what values of the uncertain (but constant) parameters $\Delta > 0$ and $K > 0$ does BIBO stability for the feedback system follow from the Circle Criterion?

(d) [2p] For which values of $\Delta > 0$ and $K > 0$ does the direct application of the Small Gain Theorem prove BIBO stability for the feedback system in (c)? (*Hint:* Is the Small Gain Theorem applicable?)

2. The ball-and-beam system is given by the equations

$$\begin{aligned} 0 &= \ddot{r} + g\sin\theta + \beta\dot{r} - r\dot{\theta}^2 \\ \tau &= (r^2+1)\ddot{\theta} + 2r\dot{r}\dot{\theta} + gr\cos\theta, \end{aligned} \tag{3}$$

where r is the position of the ball, θ the angle of the beam, τ the torque applied to the beam, $g > 0$ the gravity constant, and $\beta > 0$ the viscous friction constant.



(a) [1p] Transform the system into first-order state-space form $\dot{x} = f(x, u)$, where $x = (r, \dot{r}, \theta, \dot{\theta})^T$ and $u = \tau$.

(b) [2p] Determine all equilibria of (3).

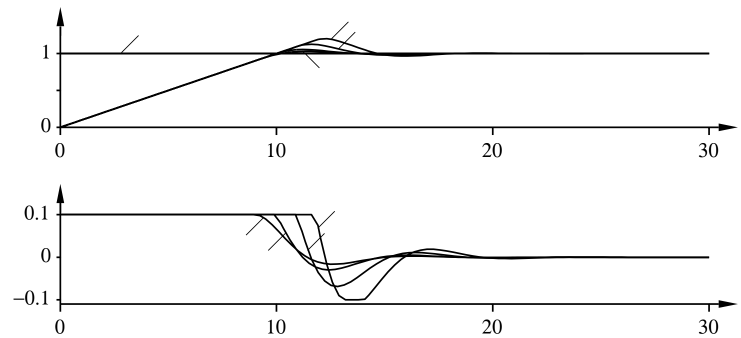
(c) [2p] Some of the equilibria in (b) correspond to that the beam is upside-down. Disregard these and linearize the system about one of the appropriate equilibria.

(d) [2p] Discuss how one can obtain a state feedback control law ($r, \dot{r}, \theta, \dot{\theta}$ are all measurable) that gives a locally asymptotically stable ball-and-beam system. You don't have to come up with an explicit solution.

(e) [3p] Consider only the first equation of (3) and assume that $\theta = \dot{\theta} = 0$. Show that $(r, \dot{r}) = (0, 0)$ is globally stable. What about asymptotic stability? (*Hint:* $V = (\beta r + \dot{r})^2/2 + \dot{r}^2/2$.)

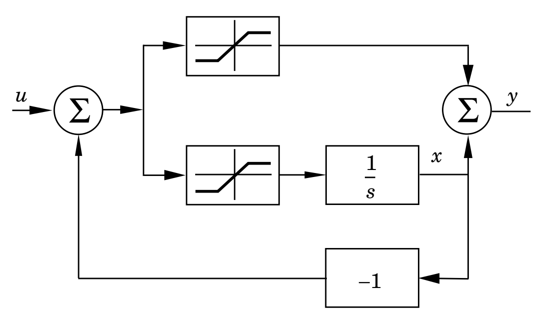
3.

(a) [3p] Consider a PID controller with anti-windup (see Lecture 7). The following plots illustrate control of an integrator process with four different choices of the tracking time constant T_t. The upper plot shows the desired set-point (the straight line) together with four curves for the output y (curves with more or less overshoot). The lower plot shows four curves for the control signal u. Redraw the curves in your solutions and combine $T_t = 0.1$, 1, 2, and 3 with the corresponding curves. (Note that y for two of the choices are almost identical.) Motivate.



(b) [3p] Write down the equations describing the following system, where the saturation blocks are defined as

$$\text{sat}(u-x) = \begin{cases} -1, & u-x \leq -1 \\ u-x, & |u-x| \leq 1 \\ 1, & u-x \geq 1. \end{cases}$$



The system is a jump and rate limiter: y is equal to u if u changes slowly. If u makes an abrupt change, then y converges to u after a while. Conclude this by simply deriving the equations for the system for $|u - x| \leq 1$ and for $|u - x| \geq 1$, when u makes a step change.

(c) [4p] The power output from a cellular phone is an important system variable, since the power consumption should be kept as low as possible to make the battery last longer. Information from the base station about the received power is sent back to the transmitting phone and is used to control the power output. A simple model for such a power control system is as follows:

$$\begin{aligned} \dot{x}(t) &= au(t) \\ u(t) &= -\operatorname{sgn} y(t-L) \\ y(t) &= bx(t). \end{aligned}$$

Here x is the power output of the cellular phone (normalized around zero) and u is the control signal, which either increase or decrease the power at a fixed rate $a > 0$. The measured power y at the base station is proportional to x with proportional constant $b > 0$. The measured power is being transmitted back to the cellular phone after a time delay $L > 0$.

Draw a diagram illustrating the system. Use describing function analysis to predict amplitude, frequency, and stability of possible power oscillations.

4. The Clegg integrator was invented by J. C. Clegg in 1958. It is simply an integrator with a state that is set to zero whenever the input crosses zero. Let e be the input to the Clegg integrator and x the integrator state. Then, the Clegg integrator can be described by the following two equations:

$$\begin{aligned} \dot{x}(t) &= e(t) \\ x(t+) &= 0, \qquad\qquad \text{if } e(t) = 0, \end{aligned}$$

where the plus sign in $x(t+)$ indicates that x is set to zero directly after e becomes zero.

(a) [1p] Sketch the output of the Clegg integrator for a sinusoidal input $e(t) = A \sin \omega t$. Assume that $x(0) = 0$.

(b) [6p] Show that the describing function for the Clegg integrator is

$$N(A, \omega) = \frac{4}{\pi \omega} - \frac{i}{\omega}$$

(c) [3p] The describing function gives (as you know) the amplification and phase shift of a sinusoidal input $e(t) = A \sin \omega t$. Draw the Nyquist diagram for the ordinary integrator ($G(s) = 1/s$) together with the describing function for the Clegg integrator. Comment on similarities and differences in their gain and phase characteristics. What is the main advantage of using the Clegg integrator instead of an ordinary integrator (for example, in a PID controller) and vice versa?

5. In this problem we study the linear quadratic optimal control problem

$$\min_{u:[0,t_f]\to\mathbb{R}^m} \frac{1}{2}\int_0^{t_f} \left[x(t)^T Q x(t) + u(t)^T R u(t)\right] dt$$

with

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0.$$

Suppose that $t_f > 0$ and x_0 are fixed and that the matrices $Q = Q^T$ and $R = R^T$ are positive definite. Then, the optimal control is given by

$$u^*(t) = -R^{-1}B^T S(t)x(t), \tag{4}$$

where the matrix $S(t) = S^T(t)$ satisfies the differential equation

$$-\dot{S}(t) = A^T S(t) + S(t)A - S(t)BR^{-1}B^T S(t) + Q, \tag{5}$$

where $S(t_f) = 0_{n\times n}$ is an $n \times n$ zero matrix.

(a) [1p] Determine the Hamiltonian function $H(x, u, \lambda)$.

(b) [2p] Derive the adjoint equation for the optimal control problem, that is, the differential equation for $\lambda(t)$ (including the final condition $\lambda(t_f)$).

(c) [2p] Show that the optimal control can be written as

$$u^*(t) = -R^{-1}B^T\lambda(t),$$

where λ is the solution to the adjoint equation in (b). (*Hint:* Derive the solution to $\partial H/\partial u = 0$.)

(d) [2p] Show that (4) is the optimal control with $S(t)$ given by (5). Do this by setting $\lambda(t) = S(t)x^*(t)$. Then derive

$$\dot{\lambda}(t) = \dot{S}(t)x^*(t) + S(t)\dot{x}^*(t) = \dot{S}(t)x(t) + S(t)[Ax^*(t) + Bu^*(t)]$$

and use (b) together with (c). You will end up with an equation

$$-\dot{S}(t)x^*(t) = [A^T S(t) + S(t)A - S(t)BR^{-1}B^T S(t) + Q]x^*(t)$$

from which you can conclude (5).

(e) [1p] What is the solution for the case $t_f = \infty$?

(f) [2p] Show that $H(x, u, \lambda)$ is constant along every optimal trajectory $(x^*(t), u^*(t))$. (*Hint:* Derive $\dot{H}(x^*(t), u^*(t), \lambda(t))$ and use that $\frac{\partial H}{\partial u}(x^*(t), u^*(t), \lambda(t)) = 0$ since u^* minimizes H.)