Solutions to Exam in 2E1262 Nonlinear Control, Dec 20, 2001

1. (a) True. Suppose $\hat{x}$ and $\tilde{x}$ are two equilibria. A trajectory that starts in either point, stays there forever since $f(\hat{x}) = f(\tilde{x}) = 0$.

   (b) False. As a counter example, take for instance the system from Lecture 3:

$$\dot{x}_1 = x_1 - x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 + x_2 - x_2(x_1^2 + x_2^2).$$

   (c) True. Follows from Lyapunov's linearization method (Lectures 3 and 4).

   (d) True. For example,

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} x$$

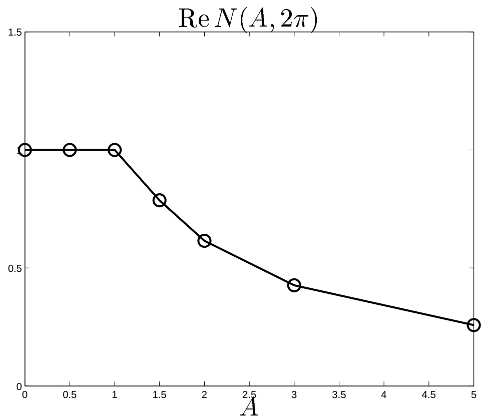
   (e) True. Suppose it was not true. Consider a solution of $\dot{x} = f(x)$ that starts in $x_0 \neq 0$ and ends at $x(T) = x_1 = 0$. Then, $\dot{x} = -f(x)$ has a solution that starts in $x_1 = 0$ and ends at $x_0 \neq 0$. However, $\dot{x} = -f(x)$ also has the solution $x(t) = 0$ for all $t \in (0, T)$, since $\dot{x} = 0$. This is a contradiction to that $\dot{x} = -f(x)$ has a unique solution, which holds because f is $\mathbf{C}^1$ (Lecture 1). Hence, the statement in the problem must be true.

2. (a)

$$\dot{x} = f(x, u) = \text{sat}(u - x)$$
$$y = h(x, u) = x + \text{sat}(u - x)$$

   If $u = 0$ and x small then $\dot{x} = -x$, so obviously $x = 0$ is asymptotically stable.

   (b) $\text{Re}\, N(A, 2\pi)$ is given below, while $\text{Im}\, N(A, 2\pi) \equiv 0$ (phase shift between u and first harmonic is equal to zero).



   (c) From the Controllability Theorem in Lecture 12, we have that the system is controllable if $g_1(x)$, $g_2(x)$, and $[g_1(x), g_2(x)]$ span $\mathbb{R}^3$. For the proposed example, this is the case because

$$g_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 0 \\ x_1 \\ 1 \end{pmatrix}, \qquad [g_1, g_2] = \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

(d) The control $u(t) = 0$ for all $t \in [0, 1]$ gives the cost

$$\frac{1}{2}\int_0^1 \left[x^2(t) + u^2(t)\right] dt = 0,$$

so $u = 0$ must be the optimal control. (This also follows from the usual calculations.)

3. (a) The equilibria are given by $x_0 = (k\pi, 0)$, $k \in \mathbb{Z}$ (integers). The linearized system is given by

$$\dot{z} = \begin{pmatrix} 0 & 1 \\ \mp c & -a \mp b \end{pmatrix} z,$$

where the choice of minus or plus sign depends on k. The characteristic polynomial is hence $s^2 + (a \pm b)s \pm c$. A second-order polynomial is asymptotically stable if and only if the coefficients are positive, which is not possible in this case since $c > 0$.

(b) We will apply LaSalle's Invariant Theorem (compare example in the end of Lecture 4). First note that $V(x)$

$$\dot{V}(x) = \frac{dV}{dx} f(x) = -ax_2^2 - bx_2^2 \cos x_1 < 0$$

if $x_2 \neq 0$. With the notation from the lectures, let $\Omega \subset \mathbb{R}^2$ be a small ball centered in the origin, let $E = \{x \in \Omega : \dot{V}(x) = 0\} = \{x : x_2 = 0\}$, and let $M \subset E$ denote the largest invariant set in E. Suppose there exists a point $\bar{x} = (\bar{x}_1, 0) \in M$ such that $\bar{x}_1 \neq 0$. A trajectory that starts in $x(0) = \bar{x}$ satisfies $\dot{x}_2(0) = -c \sin x_1(0) = -c \sin \bar{x}_1 \neq 0$, if Ω is sufficiently small. Hence, the trajectory must leave M. It follows that M is equal to the origin. The result now follows from LaSalle.[1]

(c) Follows from Lecture 3, if we treat $a(t)$ as an input. Then,

$$A(x^0(t), a^0(t)) = \begin{pmatrix} 0 & 1 \\ x_2^0(t)\sin x_1^0(t) - \cos x_1^0(t) & -a^0(t) - \cos x_1^0(t) \end{pmatrix}$$

$$B(x^0(t), a^0(t)) = \begin{pmatrix} 0 \\ -x_2^0(t) \end{pmatrix}.$$

4. (a) The equilibria in the specified region are $(0, 0)$, $(1, 0)$, and $(3/25, 88/125)$. The linearizations are given by

$$\dot{z} = \begin{pmatrix} 5 & 0 \\ 0 & -6/10 \end{pmatrix} z, \qquad \dot{z} = \begin{pmatrix} -5 & -5/3 \\ 0 & 11/5 \end{pmatrix} z, \qquad \dot{z} = \begin{pmatrix} 21/20 & -3/4 \\ 11/5 & 0 \end{pmatrix} z.$$

(b) There are no equilibria in the specified region.

(c)

$$f(x, u) = \begin{pmatrix} 5x_1(1 - x_1) - \dfrac{20x_1x_2}{2 + 10x_1} \\ \dfrac{16x_1x_2}{2 + 10x_1} - \dfrac{6x_2}{10} - \dfrac{x_2}{2}(1 - u) \end{pmatrix}$$

$$u = -\operatorname{sgn} \sigma(x) = -\operatorname{sgn}(x_2 - 1).$$

[1] Note that to be rigorous, it remains to show that Ω is invariant.

(d) The equivalent control $u_{\text{eq}} \in [-1, 1]$ follows from the equation $\dot{\sigma}(x) = 0$, that is,

$$\frac{16x_1}{2 + 10x_1} - \frac{6}{10} - \frac{1}{2}(1 - u_{\text{eq}}) = 0, \tag{1}$$

where we used that $x_2 = 1$ at the sliding mode. Solving for $u_{\text{eq}} = u_{\text{eq}}(x_1)$ and plugging into the system dynamics yields the sliding dynamics

$$\dot{x}_1 = 5x_1(1 - x_1) - \frac{20x_1}{2 + 10x_1}.$$

It follows from the constraint $u_{\text{eq}} \in [-1, 1]$ and Equation (1) that the sliding mode takes place at $\{x : x_1 > 3/25, x_2 = 0\}$ (compare the phase plane analysis).

The physical interpretation of the sliding mode is that the predator population x_2 is kept at the level $\alpha = 1$, while only the prey population x_1 may change. If the predator population increases above α, it decreases due to harvest. If the predator population decreases below α, it increases due to the excess of prey.

5. (a) The upper plots show y_{sp} (solid) and y (dashed). The middle plots show P (solid) and I (dashed). The lower plots show u (solid) and v (dashed).

From P in the middle plots we see that $K \approx 1.5$. From the left plots, it follows that the maximum of

$$I(t) = \frac{K}{T_i} \int_0^t [y_{\text{sp}}(s) - y(s)]\, ds$$

is $\max_t I(t) \approx 1$ and attained at $t \approx 2$. From the upper left plot we have $\int_0^2 [y_{\text{sp}}(s) - y(s)]\, ds \approx 1$. Hence, approximately $T_i = K = 1.5$. From the lower plots we see that the saturation level is equal to 0.5.

(b)

$$G(s) = -\frac{(KT_t - 1)T_i s + KT_t}{T_i s(T_t s + 1)}$$

(c) With notation from the lectures, the saturation satisfies $k_1 = 0$ and $k_2 = 1$. Note the negative feedback. The Circle Criterion is thus fulfilled if the Nyquist curve of $-G$ is to the right of the line $\{z \in \mathbb{C} : \operatorname{Re} z = -1\}$. Here $\operatorname{Re}(-G(i\omega)) = -1/(\omega^2 + 1)$, so the criterion is fulfilled. Hence, the closed-loop system is BIBO stable for all $K > 1$.