Automatic Control

Department of Signals, Sensors & Systems

Nonlinear Control, 2E1262

Exam 8:00-13:00, Dec 20, 2001

Aid:
- Lecture notes. (Textbooks, exercises, solutions, calculators etc. may **not** be used.)

Observandum:

- Name and social security number (*personnummer*) on every page.
- Only one solution per page.
- A motivation must be attached to every answer.
- Specify number of handed in pages on cover.
- Each subproblem is marked with its maximum credit.

Grading:

Grade 3: ≥ 23

Grade 4: ≥ 33

Grade 5: ≥ 43

Results: The results will be posted on the department's board on second floor, Osquldasväg 10. The marked exams are available for discussion Jan 16, 12:30-13:30, in Office A:607 (Johansson's office), 6th floor.

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Good Luck and Merry Christmas!

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1. Consider a second-order differential equation

$$\dot{x} = f(x), \tag{1}$$

where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a $\mathbf{C}^1$ (continuously differentiable) function such that $f(0) = 0$.

Determine if the following statements are true or false. You have to motivate your answers to get credits. The motivation can for example be a short proof, a counter example (Swedish: *motexempel*), or a reference to a result in the lecture notes.

(a) [2p] Suppose the differential equation (1) has more than one equilibria, then none of them can be globally asymptotically stable.

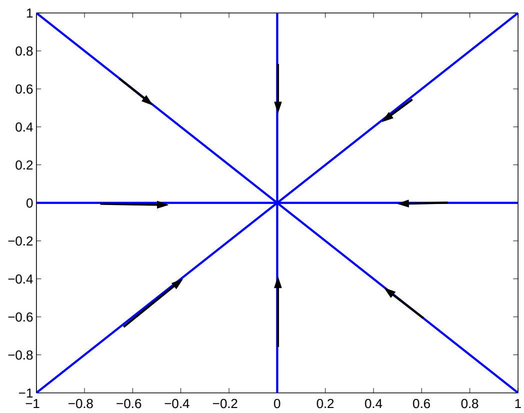
(b) [2p] The differential equation (1) cannot have a periodic solution.

(c) [2p] If the eigenvalues of

$$\frac{\partial f}{\partial x}(0)$$

are strictly in the left half-plane, then the nonlinear system (1) is asymptotically stable.

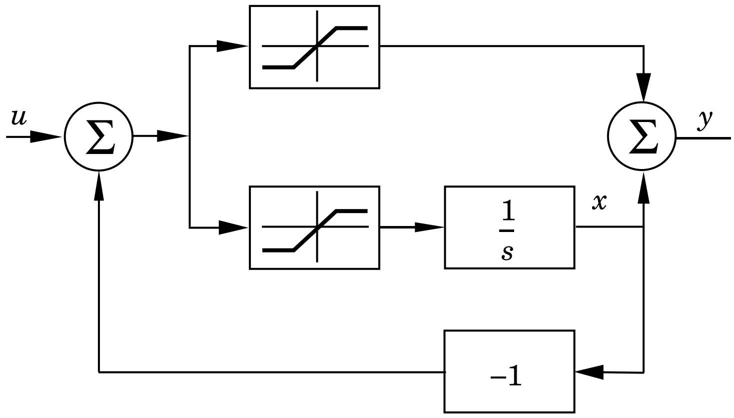
(d) [2p] There exists f such that the differential equation (1) have a phase portrait that looks like this:



(e) [2p] For any initial condition $x(0) = x_0 \neq 0$, the solution $x(t)$ of (1) cannot reach $x = 0$ in finite time, that is, there does not exist $0 < T < \infty$ such that $x(T) = 0$. *[**Hint:** Since f is $\mathbf{C}^1$, both the equations $\dot{x} = f(x)$ and $\dot{x} = -f(x)$ have unique solutions. What about a solution ending (for $\dot{x} = f(x)$) and starting (for $\dot{x} = -f(x)$) in $x_0 = 0$?]*

2.

(a) [2p] Consider the following block diagram:



The saturation blocks are described by the input–output map $v = \operatorname{sat} w$, where

$$v = \begin{cases} 1, & w > 1 \\ w, & -1 < w < 1 \\ -1, & w < -1. \end{cases}$$

Specify the block-diagram on state-space form

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u). \end{aligned} \tag{2}$$

Show that $x = 0$ is an asymptotically stable equilibrium of (2) if $u = 0$.

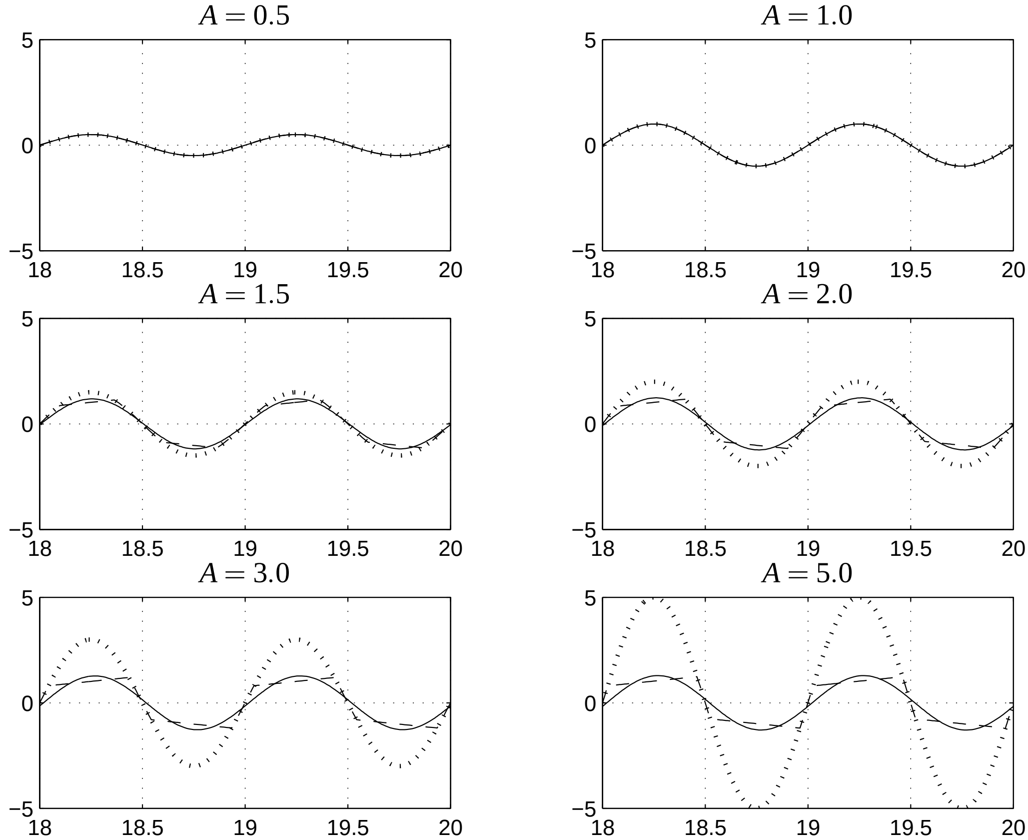
(b) [3p] The time responses below show simulations of the system in (a) with input

$$u(t) = A\sin(2\pi t).$$

The dotted lines correspond to u and the dashed lines to y. The solid lines are the first harmonics (Swedish: *grundton*) of y. Sketch an estimate of the describing function

$$N(A,\omega) = N(A,2\pi) = \frac{b_1 + ia_1}{A}$$

based on the plots, by drawing individual plots of the real part and the imaginary part of N as functions of A.



(c) [2p] Consider the control system

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2,$$

where $x(t) \in \mathbb{R}^3$ and $g_1, g_2 : \mathbb{R}^3 \to \mathbb{R}^3$. If $g_1(x) = B_1$ and $g_2(x) = B_2$ are constant vectors, then the control system is not controllable (as discussed in Lecture 12). Illustrate that if either g_1 or g_2 depend on x, then the system can be controllable. *[**Hint:** Let*

$$g_1(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad g_2(x) = \begin{pmatrix} 0 \\ x_1 \\ 1 \end{pmatrix}$$

and show that the system is controllable.]

(d) [3p] Solve the optimal control problem

$$\min_{u:[0,1]\to\mathbb{R}} \frac{1}{2}\int_0^1 \left[x^2(t) + u^2(t)\right] dt$$

with

$$\dot{x}(t) = u(t), \qquad x(0) = 0.$$

3. Consider the differential equation

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - bx_2 \cos x_1 - c \sin x_1\end{aligned} \tag{3}$$

(a) [3p] Suppose $a, b, c > 0$ are positive constants. Determine all equilibria of (3). Motivate why not all equilibria are asymptotically stable.

(b) [4p] Suppose a, b, c are positive constants. Use the function

$$V(x) = \frac{x_2^2}{2} + c - c \cos x_1$$

to show that all solutions of (3) tend to the origin if they start sufficiently close. *[**Hint:** LaSalle's Invariant Theorem.]*

(c) [3p] Suppose $a = a(t)$ is an uncertain time-varying function and $b = c = 1$. Let $x^0(t) = (x_1^0(t), x_2^0(t))^T$ together with $a = a^0(t)$ denote a nominal solution to (3). Linearize the differential equation about $(x^0(t), a^0(t))$ to obtain

$$\dot{\tilde{x}}(t) = A(x^0(t), a^0(t))\tilde{x}(t) + B(x^0(t), a^0(t))\tilde{a}(t),$$

where $(\tilde{x}, \tilde{a})$ denotes a small perturbation to the nominal solution. Be precise in your answer, specifying indices and which variables that depend on t.

4. Consider a prey (Swedish: *byte*) and a predator (*rovdjur*) population with densities x_1 and x_2, respectively. Assume that the predator population is harvested when they are sufficiently many, that is, when $x_2 > \alpha$ for some constant $\alpha > 0$. Think for instance on rabbits, foxes, and human hunters. The hunters only hunt if the density of foxes is sufficiently high. For a particular set of populations, if $x_2 < \alpha$ the dynamics is given by

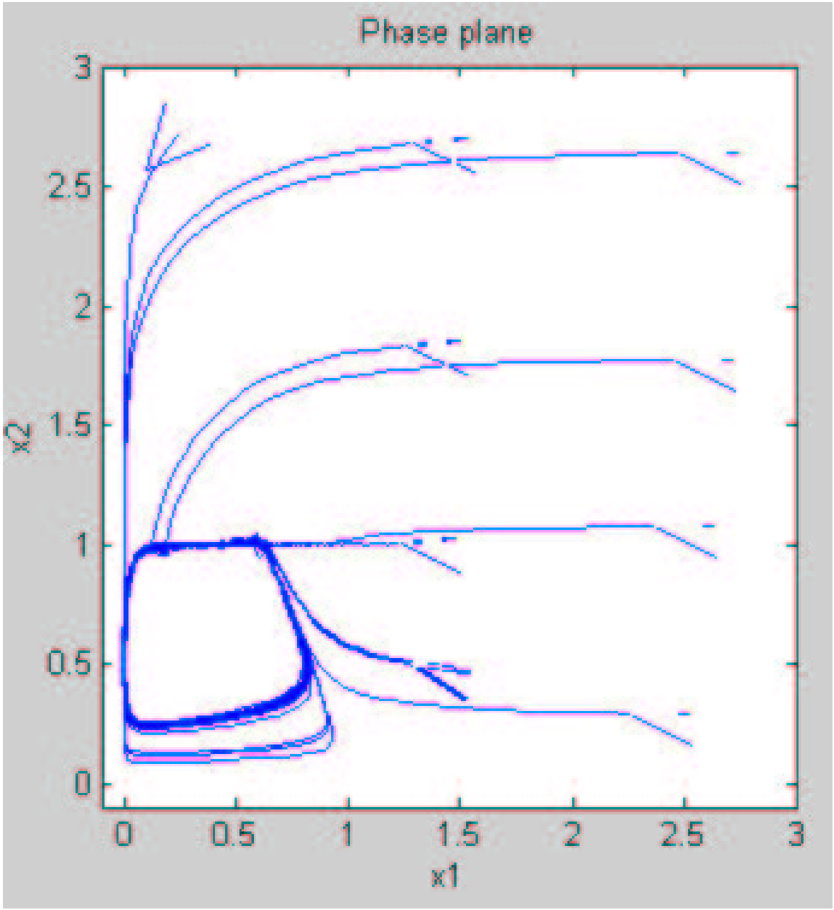
$$\dot{x}_1 = 5x_1(1-x_1) - \frac{20x_1x_2}{2+10x_1}$$
$$\dot{x}_2 = \frac{16x_1x_2}{2+10x_1} - \frac{6x_2}{10}$$

and if $x_2 > \alpha$ the dynamics is given by

$$\dot{x}_1 = 5x_1(1-x_1) - \frac{20x_1x_2}{2+10x_1}$$
$$\dot{x}_2 = \frac{16x_1x_2}{2+10x_1} - \frac{6x_2}{10} - \beta x_2$$

where $\beta > 0$ is a parameter reflecting harvesting effort. In the following, we assume that $\alpha = \beta = 1$.

A (low quality) phase plane plot in ICTools is shown below.



(a) [2p] Derive all equilibria in the region $\{x \in \mathbb{R}^2 : x_1 \geq 0, 0 \leq x_2 \leq 1\}$. Linearize the system about these equilibria.

(b) [2p] Derive all equilibria in the region $\{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 > 1\}$. Linearize the system about these equilibria.

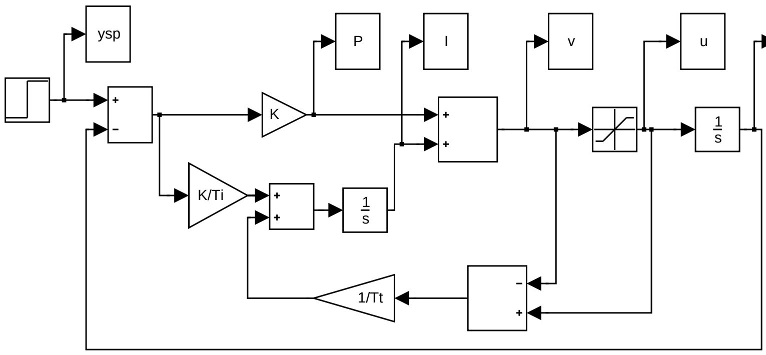
(c) [1p] Show that the population dynamics model can be written as

$$\begin{aligned} \dot{x} &= f(x,u) \\ u &= -\operatorname{sgn} \sigma(x). \end{aligned}$$

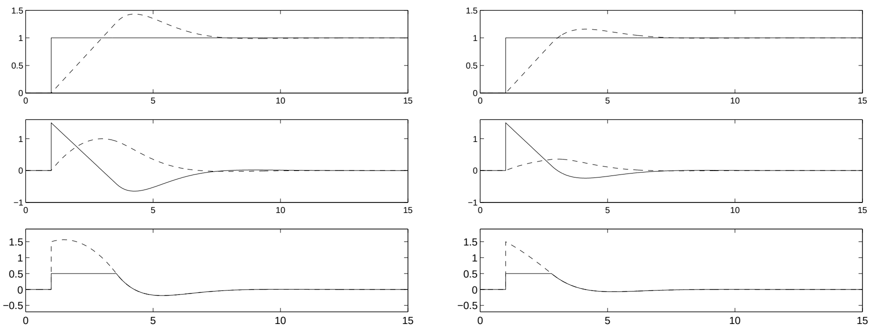
Determine f and σ.

(d) [5p] Derive the sliding mode at $x_2 = 1$ (see the phase plane plot), for example, using the equivalent control method. Describe what the sliding mode correspond to in physical terms (that is, in behavior of prey, predator, etc.).

5. The block diagram shows a control system with anti-windup. The process (as you see) is given by an integrator.



(a) [4p] The three left plots below show set-point experiments for the system without anti-windup ($T_t = 1000$). The three right plots show set-point experiments with anti-windup ($T_t = T_i$). Match the plots with the six signals indicated in the block-diagram. Also, specify the values of K, T_i, and the saturation level. (These parameters are the same in both simulations.) Motivate your answers.



(b) [3p] Let $y_{\text{sp}} = 0$. The block diagram can then be described as a linear part $v = G(s)u$ in feedback connection with a static nonlinear part $u = \text{sat}_\alpha(v)$, where $\alpha > 0$ is the saturation level from (a). Specify $G(s)$.

(c) [3p] Let again $y_{\text{sp}} = 0$. For certain values of T_i and T_t, the linear system in (b) is given by

$$G(s) = -\frac{(K-1)s + K}{s(s+1)}.$$

Use the Circle Criterion to determine for which $K > 1$ the closed-loop system is BIBO stable (from a perturbation in u to v).