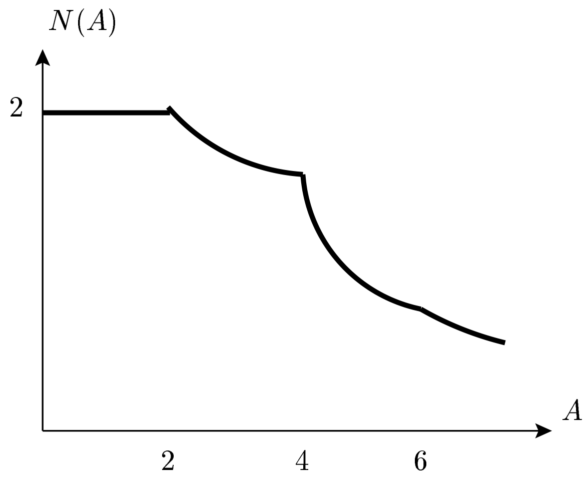
**Solutions to Exam in 2E1262 Nonlinear Control, April 5, 2002**

1. (a) The describing function represents an amplitude depending gain $N(A)$. Note that $N(A)$ is real-valued (why?). A rough sketch is shown below:




(b) We need to choose $G$ such that its Nyquist curve intersects $-1/N(A)$. Note that $-1/N(A)$ belongs to the negative part of the real axis and that $\min N^{-1}(A) = 1/2$. A suitable candidate is

$$G(s) = \frac{2}{s(s+1)^2}.$$

(c) The gains are $\gamma(f) = 2$, $\gamma(f \circ f) = 4$, and $\gamma(f+f) = \gamma(2f) = 4$.

(d) The system is globally stable with Lyapunov function $V(x) = x^2$ for all $K > 0$. For example, note that $\dot{V} = -2xf(x) = -2|xf(x)| \leq 0$, for all $x \neq 0$, and that $V$ is radially unbounded.

2. (a) Introduce $f$ through the equation $\ddot{\theta} = f(\theta, \dot{\theta}, \beta, \dot{\beta})$ and let $c_1, c_2, c_3$ denote the (positive) constants in $f$. Linearizing $f$ around $\theta = \beta = \dot{\beta} = 0$ gives

$$\begin{aligned} f(\theta, \dot{\theta}, \beta, \dot{\beta}) &\approx \frac{\partial f}{\partial \theta}\theta + \frac{\partial f}{\partial \dot{\theta}}\dot{\theta} + \frac{\partial f}{\partial \beta}\beta + \frac{\partial f}{\partial \dot{\beta}}\dot{\beta} \\ &= c_1\theta + c_2\beta + c_3\dot{\beta} \end{aligned}$$

where $\dot{\beta} = u$. Hence, the linearized system is given by

$$\ddot{\theta} = c_1\theta + c_2\beta + c_3 u.$$

It follows that in the Laplace domain

$$s^2\theta = c_1\theta + c_2\beta + c_3 u = c_1\theta + (c_2/s + c_3)u$$

and thus

$$G(s) = \frac{c_3 s + c_2}{s(s^2 - c_1)}.$$

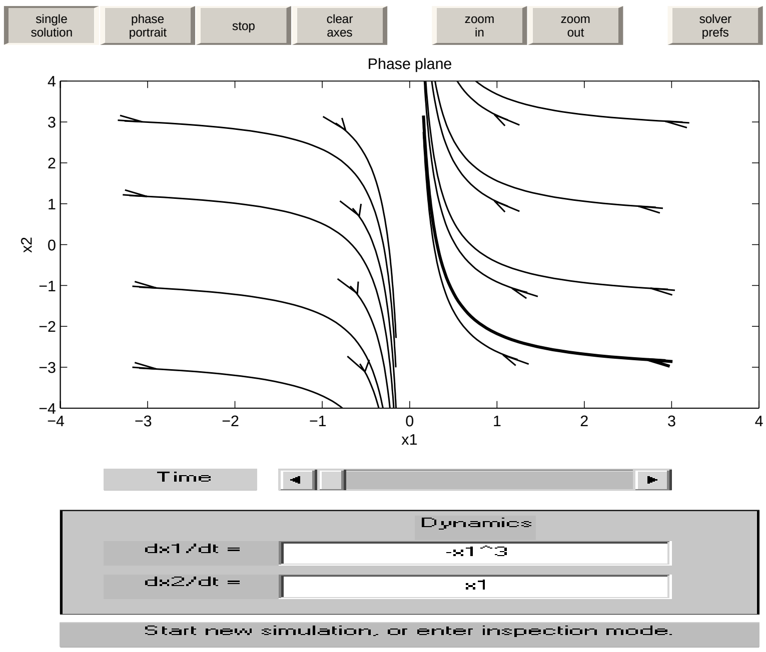
The poles are located in $0, \pm\sqrt{c_1}$ and the zero in $-c_2/c_3$. The bicycle is, as expected, unstable.

(b) See the minimum-time control example in Lecture 13. From that it follows that the optimal control is given by

$$u^*(t) = -C \operatorname{sgn} \lambda_2(t) = C \operatorname{sgn}(c_1 t - c_2)$$

Hence, $p(t) = c_1 t - c_2$ is a first-order polynomial.

3. (a) ICTools gives the phase portrait below:




(b)

$$\dot{V} = \frac{dV}{dx} f(x) = \begin{pmatrix} 2x_1 & 2x_2 \end{pmatrix} \begin{pmatrix} -x_1^3 + u \\ x_1 \end{pmatrix} = -2x_1^4 + 2x_1 u + 2x_1 x_2$$

If we choose $u = -x_2$, then $\dot{V} = -2x_1^4$. Hence, $x_1$ will tend to zero. It follows from the equation $\dot{x}_2 = x_1$ that $x_2$ will tend to a constant $\bar{x}_2$, say. Suppose $\bar{x}_2 \neq 0$. Then, $\dot{x}_1 = -x_1^3 - x_2$ implies that $\dot{x}_1 \to \bar{x}_2 \neq 0$, which contradicts that $x_1 \to 0$. Hence, $\bar{x}_2 = 0$. Global stability follows from that $V$ is radially unbounded.

(c) Choosing $u = x_1^3 + v$ gives the system

$$\begin{aligned} \dot{x}_1 &= v \\ \dot{x}_2 &= x_1, \end{aligned}$$

which is globally stabilized, for example, by $v = -2x_1 - x_2$. Hence,

$$u = x_1^3 - 2x_1 - x_2$$

is a possible control law.

4. (a) The equilibria are given by the equation

$$(F + Hu_e)x_e + Gu_e = 0.$$

If $F + Hu_e$ is nonsingular, we have $x_e = -(F + Hu_e)^{-1} G u_e$. Introduce the deviation variables $\delta x(t) = x(t) - x_s$ and $\delta u(t) = u(t) - u_s$. Then the linearization is given by

$$\delta \dot{x} = (F + Hu_e)\delta x + (G + Hx_e)\delta u$$

(b) Since $\dot{z} = Fz + Gu$ is asymptotically stable, we have that $F < 0$. If $u_e = 0$, it follows directly from (a) that the bilinear system is stable. If $u_e \neq 0$, then by choosing $|H| < |F|/u_e$, we have $F + Hu_e < 0$ and thus the bilinear system is asymptotically stable.

(c) Using the notation of Lecture 7, we get $a_1 = 0$ and $b_1 = 3A^3/4$. Hence,

$$N(A) = \frac{b_1 + ia_1}{A} = \frac{3A^2}{4}.$$

(d) $G(s)$ is passive if and only if $\operatorname{Re} G(i\omega) \geq 0$ for all $\omega > 0$. Here,

$$\begin{aligned} G(i\omega) &= \frac{i\omega + a}{(i\omega + 2)(i\omega + 10)} = \frac{(i\omega + a)(-i\omega + 2)(-i\omega + 10)}{(\omega^2 + 4)(\omega^2 + 100)} \\ &= \frac{20a + \omega^2(12 - a)}{(\omega^2 + 4)(\omega^2 + 100)} + i\frac{\omega(20 - \omega^2 - 12a)}{(\omega^2 + 4)(\omega^2 + 100)}. \end{aligned}$$

Hence, $0 \leq a \leq 12$.

5. (a) The linearization is given by

$$A = \frac{df}{dx}(0, 0) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

with characteristic polynomial $s^2 + s + 1$. Hence, the system is asymptotically stable.

(b)

$$P = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$$

(c)

$$\dot{V} = \frac{dV}{dx} f(x) = 2 \begin{pmatrix} x_1 & x_2 \end{pmatrix} P \begin{pmatrix} -x_2 \\ x_1 + (x_1^2 - 1)x_2 \end{pmatrix}$$

gives the expression.

(d) $c = 4$ corresponds to that largest $\Omega_c$ contained in $\Pi$.