Automatic Control

Department of Signals, Sensors & Systems

Nonlinear Control, 2E1262

Exam 8:00-13:00, April 5, 2002

Aid: • Lecture notes. (Textbooks, exercises, solutions, calculators etc. may **not** be used.)

Observandum:

- Name and social security number (*personnummer*) on every page.
- Only one solution per page.
- A motivation must be attached to every answer.
- Specify number of handed in pages on cover.
- Each subproblem is marked with its maximum credit.

Grading:

Grade 3: ≥ 23

Grade 4: ≥ 33

Grade 5: ≥ 43

Results: The results will be posted on the department's board on second floor, Osquldasväg 10.

Responsible: Karl Henrik Johansson, `kallej@s3.kth.se`

Good Luck!

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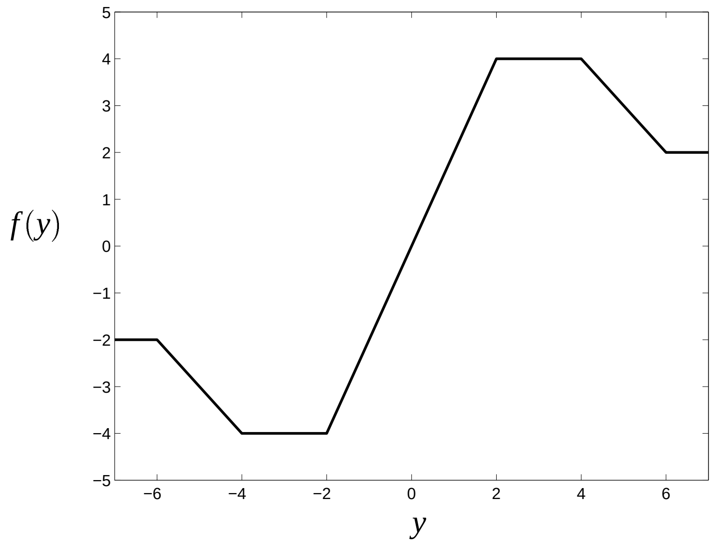
1. Consider the odd static nonlinearity f below.



(a) [3p] Sketch the describing function for f. You only need to draw an approximate sketch.

(b) [2p] Consider a feedback system that consists of a linear system $G(s)$ in negative feedback connection with f, that is,

$$y = Gu = -Gf(y).$$

Based on the describing function method, specify a transfer function $G(s)$ such that the closed-loop system is likely to have an oscillation.

(c) [2p] What is the gain of $f(y)$? What is the gain of the series connection $f(f(y))$? What is the gain of the parallell connection $f(y) + f(y)$?

(d) [3p] Consider a feedback system that consists of an integrator K/s in negative feedback connection with f, that is,

$$\dot{y} = -Kf(y)$$

For what values of $K > 0$ is the closed-loop system stable? Is it globally stable?

2.

(a) [6p] A regular bicycle is mainly controlled by turning the handle bars.[1] Let the tilt sideways of the bicycle be θ radians and the turning angle of the front wheel be β radians. The tilt of the bike obeys the following nonlinear differential equation:

$$\begin{aligned}\ddot{\theta} &= \frac{mg\ell}{J}\sin\theta + \frac{m\ell V_0^2}{bJ}\cos\theta\tan\beta + \frac{am\ell V_0}{bJ}\cdot\frac{\cos\theta}{\cos^2\beta}u \\ \dot{\beta} &= u,\end{aligned}$$

where $V_0 > 0$ is the (constant) velocity of the bicycle, and m, g, ℓ, J, a, and b are other positive constants. The control u is the angular velocity applied at the handle bars. To gain some understanding of the principal behavior of the bicycle, we study its linearization. Linearize the tilt equation around the equilibrium point $(\theta, \beta, u) = (0, 0, 0)$.

Derive the transfer function $G(s)$ from u to θ. Determine the poles and zeros of $G(s)$. Is the bike locally stable?

(b) [4p] The double integrator

$$\ddot{y} = u$$

should be controlled from $(y, \dot{y}) = (10, 0)$ to $(y, \dot{y}) = (0, 0)$ as fast as possible given the constraint $|u| \le C$, $C > 0$. Show that the optimal control law is given by

$$u(t) = C \operatorname{sgn} p(t),$$

where p is a polynomial. Detemine the degree of p. [You do not have to derive p explicitly.]

[1]The rider can also change the center of mass, but we will ignore that control actuation here.

3. Consider the control system

$$\dot{x}_1 = -x_1^3 + u$$
$$\dot{x}_2 = x_1$$

(a) [2p] Sketch the phase portrait for $u = 0$.

(b) [5p] Based on the Lyapunov function candidate $V(x) = x_1^2 + x_2^2$, suggest a control law $u = g(x_1, x_2)$, such that the origin is globally asymptotically stable. You need to clearly motivate why the control law of your choice makes the system globally asymptotically stable.

(c) [3p] Derive a globally stabilizing control law $u = h(x_1, x_2)$ based on exact feedback linearization.

4.

(a) [2p] A bilinear control system is given by the equation

$$\dot{x} = Fx + Gu + Hxu,$$

where $F, H \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{n\times 1}$ are constant matrices. Determine all equilibria (x_e, u_e) and linearize the system around them.

(b) [1p] Suppose the bilinear system in (a) is one-dimensional, that is, $n = 1$. Suppose that the corresponding linear system

$$\dot{z} = Fz + Gu$$

is asymptotically stable. Show that if the nonlinear term is sufficiently small (that is, $|H|$ is sufficiently small), then also the bilinear system is asymptotically stable.

(c) [4p] Derive the describing function for $f(x) = x^3$.
*[**Hint:** $2\sin^2\alpha = 1 - \cos 2\alpha$ and $\sin 2\alpha = 2\sin\alpha\cos\alpha$.]*

(d) [3p] Consider the transfer function

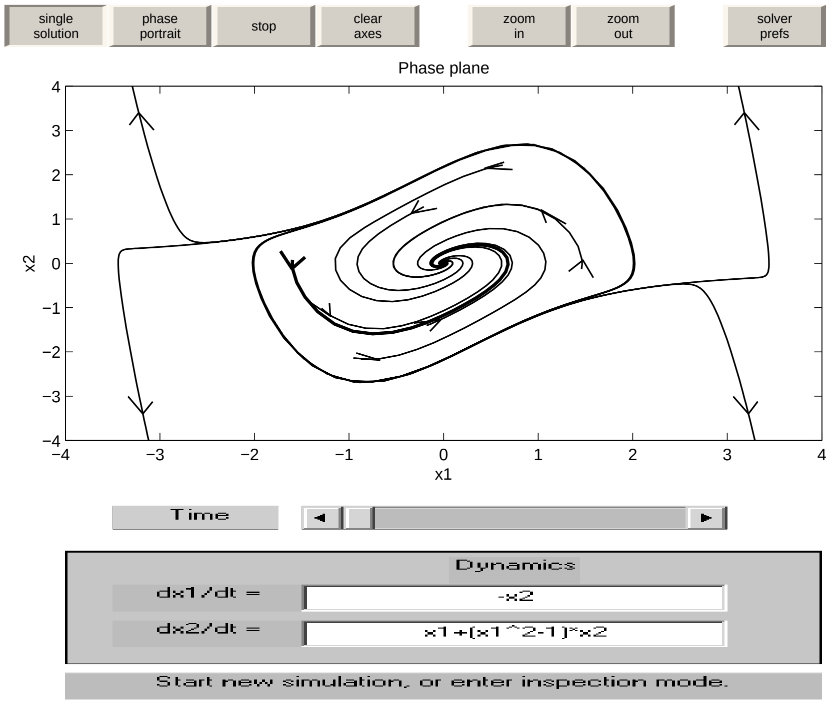
$$G(s) = \frac{s+a}{(s+2)(s+10)}.$$

For which values of a is G passive?

5. We wish to estimate the domain of attraction D of the origin for the system

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = x_1 + (x_1^2 - 1)x_2,$$

see phase portrait below:



(a) [2p] Show that the origin is a locally asymptotically stable equilibrium. Linearize the system at the origin and determine the system matrix A.

(b) [2p] Find a Lyapunov function of the form $V(x) = x^T P x$ by solving the Lyapunov equation

$$PA + A^T P = -2I,$$

for an unknown positive definite symmetric matrix P, where A is the matrix in (a).

(c) [2p] We want to find an as large region $\Pi \subset \mathbb{R}^2$ as possible such that for all $x \in \Pi$,

$$\dot{V} = \frac{dV}{dx} \cdot \frac{dx}{dt} < 0.$$

Show that

$$\dot{V} = -2x_1^2 - 2x_2^2 - 2x_1^3 x_2 + 4x_1^2 x_2^2.$$

(d) [4p] Let

$$\Omega_c = \{z \in \mathbb{R}^2 : V(z) \leq c\}.$$

We want to find $c > 0$, as large as possible, such that $\Omega_c \subset \Pi$, where Π is given in (c). Conclude that Ω_c is an estimate of the domain of attraction D. You may base your argument on the plot below. The ellipsoids are the boundaries of Ω_c for $c = 1, 2, \ldots, 10$. The other curves show the level curve for $\dot{V} = 0$.