Solutions to Exam in 2E1262 Nonlinear Control, December 19, 2002

1. (a) With $x = (\theta, \dot\theta, \phi, \dot\phi)^T$, we have if $\sin\theta \neq 0$,

$$\dot x = \begin{pmatrix} x_2 \\ x_4^2 \sin x_1 \cos x_1 - \sin x_1 \\ x_4 \\ -2x_2 x_4 \cos x_1 / \sin x_1 \end{pmatrix}$$

(b) Setting all derivatives in the original equation equal to zero, yields $\sin\theta = 0$. Hence, the equilibria are determined by $\theta_k = k\pi$ with ϕ taking any value. (Note that the equilibria cannot be obtained directly from the first-order form in (a).)

(c) The solution $(\theta(t), \phi(t)) = (\pi/3, t\sqrt{2})$ fulfills the first pendulum equation, since

$$-2\sin\pi/3\cos\pi/3 + \sin\pi/3 = 0$$

The second equation is also satisfied.

(d) Denote the first pendulum equation by $f_1(z) = 0$ and the second by $f_2(z) = 0$ where $z = (\theta, \dot\theta, \ddot\theta, \phi, \dot\phi, \ddot\phi)^T$. To linearize these equations, we write

$$\begin{aligned} 0 = f_1(z(t)) = f_1(z^0(t) + \delta z(t)) &\approx f_1(z^0(t)) + \left.\frac{df_1}{dz}\right|_{z=z^0(t)} \delta z(t) \\ &= \left.\frac{\partial f_1}{\partial z_1}\right|_{z=z^0(t)} \delta z_1(t) + \cdots + \left.\frac{\partial f_1}{\partial z_6}\right|_{z=z^0(t)} \delta z_6(t) \end{aligned}$$

and similar for f_2. Deriving the partial derivatives and using the definition of z ($\delta z_1 = \delta\theta$ etc.), we get

$$\begin{aligned} \frac{3}{2}\delta\theta + \delta\ddot\theta - \sqrt{\frac{3}{2}}\delta\dot\phi &= 0 \\ \sqrt{2}\delta\dot\theta + \frac{\sqrt{3}}{2}\delta\ddot\phi &= 0 \end{aligned}$$

2. (a) The system $\dot x = f(x) = xu(x)$ is locally asymptotically stable at $x = 0$ if

$$\left.\frac{df}{dx}\right|_{x=0} = u(0) < 0$$

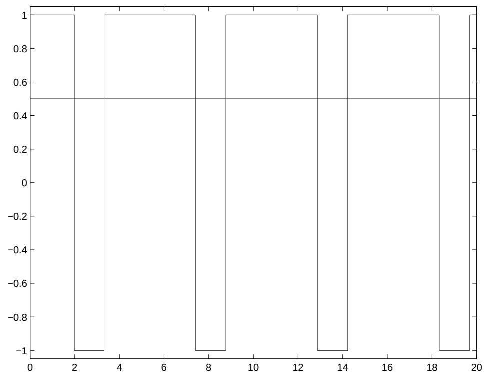
(b) $u = kx$ gives $\dot x = kx^2$. If $k > 0$ then $x \to \infty$ for $x(0) > 0$ and if $k < 0$ then $x \to -\infty$ for $x(0) < 0$; hence the system is unstable. For $k = 0$, we have the equation $\dot x = 0$ which is not asymptotically stable.

(c) $k_1 = 0$ and $k_2 < 0$ (e.g., $k_2 = -1$) gives $\dot x = k_2 x$, which is (globally) asymptotically stable.

(d) The Lyapunov function $V(x) = x^2/2$ can be used to prove that $\dot x = -x^3$ is globally asymptotically stable.

(e) Follows from the feedback system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$u = \Delta y$$



3. (a) See plot above.

   (b) The Nyquist curve of e^{-s}/s intersects the negative real axis for the first time at $\omega = \pi/2$. The intersection corresponds to a stable oscillation. The period time estimate is thus $2\pi/\omega = 4$.

   (c) Any trajectory starting in $x(0) \neq 0$ will at some instance intersect the switching surface $x = 0$. Suppose that time instance is at $t_0 = 0$ and the intersection is from the positive side. Then at time T the integrator shifts sign from $\dot{x} = -1$ to $\dot{x} = +1$. At $2T$ the state intersects the switching surface from the negative side, and the procedure repeats itself. Hence, the period is $2T + 2T = 4T$.

   (d) $-1/N(A)$ for the saturation $\mathrm{sat}(\cdot)$ (without k) starts at $(-1, 0)$ and follows the negative real axis towards $-\infty$. The Nyquist curve of ke^{-s}/s intersects $(-1, 0)$ when $k \sin\omega/\omega = 1$, see (b). Hence, for $k = \pi/2$. So the DF analysis predicts an oscillation for $k > \pi/2$.

4. (a) The sector in the Circle Criterion that specifies the nonlinearity is given by $k_1 = 1$ and $k_2 = 4$. Hence, the system is BIBO stable if $P(i\omega)$ does not encircle or intersect the circle defined by the two points $(-1, 0)$ and $(-1/4, 0)$. Since

$$\operatorname{Re} P(i\omega) = \frac{1 - \omega^2}{(1 + \omega^2)^2}$$

   which attain its minimum at $\omega^2 = 3$, we have that $\operatorname{Re} P(i\omega) \geq -1/8$. Therefore the system is BIBO stable.

   (b) The gain of the controller is equal to $\gamma(C) = 4$, see Lecture 5 for calculations. Since the gain of P is $\gamma(P) = 1$, we cannot draw any conclusions using the Small Gain Theorem.

(c)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - 2x_2 + F(r - x_1)$$

(d) For small y, we see that the closed-loop system is linear. For example, by using the state-space realization in (c), we get for $|x_1| \le 1$ that

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -5x_1 - 2x_2$$

$V(x) = x^T P x$ is a Lyapunov function with $PA + A^T P = -I$. For A as above, we have

$$P = \frac{1}{10}\begin{pmatrix} 17 & 1 \\ 1 & 3 \end{pmatrix}$$

Using this V, we can prove stability.

5. (a) $P(s) = c(sI - L)^{-1}b$

(b) Follows from evaluating the integral of N for various constant values of z.

(c) The averaged system is a linear system with saturation. This can be studied using the Circle Criterion, see for example the previous problem.

(d) When $\delta = 0$, the system becomes a relay feedback system. We can then draw the Nyquist curve of P and check intersections with $-1/N(A)$ where $N(A)$ is the describing function for the relay.

(e) Large A leads to that the dither signal is present in y (which might be undesirable if we want y to be close to r). Small p is expensive to implement, since that means that we need a high-frequency triangle-wave generator.