Automatic Control
Department of Signals, Sensors & Systems

Nonlinear Control, 2E1262

Exam 14:00-19:00, December 19, 2002

Aid: • Lecture notes and textbook from basic course ("Reglerteknik" by Glad & Ljung or similar text by Lunze). Other textbooks, handbooks, exercises, solutions, calculators etc. may **not** be used.

Observandum:

- Name and social security number (*personnummer*) on every page.
- Only one solution per page.
- A motivation must be attached to every answer.
- Specify number of handed in pages on cover.
- Each subproblem is marked with its maximum credit.

Preliminary Grading:

Grade 3: ≥ 23

Grade 4: ≥ 33

Grade 5: ≥ 43

Results: The results will be posted on the department's board on second floor, Osquldasväg 10.

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Remember to fill in the course evaluation linked on the course homepage.

Good Luck!

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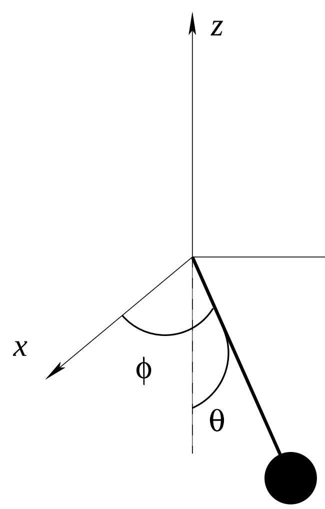
1. Consider the spherical pendulum depicted below (similar to the one Professor Calculus uses in Tintin):



Here θ denotes the angular position of the pendulum with respect to the z-axis and ϕ denotes the angular position in the x-y plane. A (normalized) model of the spherical pendulum is given by

$$\begin{aligned}\ddot{\theta} - \dot{\phi}^2 \sin\theta\cos\theta + \sin\theta &= 0 \\ \ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta &= 0\end{aligned}$$

(a) [3p] Specify the pendulum dynamics on first-order form

$$\dot{x} = f(x)$$

and give condition on when the first-order form is equivalent to the original equations.

(b) [2p] Determine all equilibria for the pendulum system. (You don't have to determine stability properties.) *[Hint: Consider the original equations.]*

(c) [2p] Show that $(\theta(t), \phi(t)) = (\pi/3, t\sqrt{2})$ is a trajectory of the pendulum system.

(d) [3p] Linearize the pendulum system about the trajectory in (c).

2.

(a) [2p] Consider the scalar nonlinear control system

$$\dot{x} = xu \tag{1}$$

where the $u = u(x)$ is a state-feedback control. Determine a condition that ensures that the origin is an asymptotically stable equilibrium.

(b) [2p] Does there exist a linear control law $u(x) = kx$ that ensures that (1) is asymptotically stable?

(c) [2p] Does there exist an affine control law $u(x) = k_1 x + k_2$ that ensures that (1) is asymptotically stable?

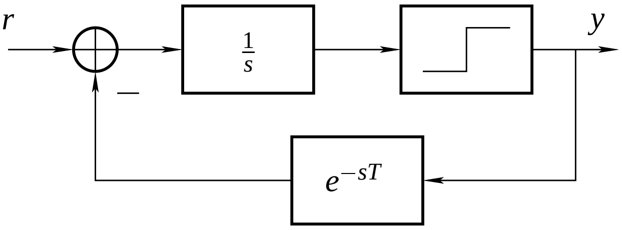
(d) [2p] Show that $u(x) = -x^2$ gives that (1) is globally asymptotically stable.

(e) [2p] We want to analyze an n-dimensional dynamical system of the form

$$\dot{x}(t) = (A + B\Delta C)x(t) \tag{2}$$

where Δ is a scalar but unknown constant. The matrices A, B, C are constant and of appropriate dimension. Show that the differential equation (2) can be analyzed as a linear system with a constant gain Δ in the feedback loop. Draw an illustrative block diagram.

3. A simple model of a ΣΔ-modulator is shown below.



It works as an AD-converter and converts the analog signal $r : [0, \infty) \mapsto \mathbb{R}$ to the digital signal $y : [0, \infty) \mapsto \{-1, 1\}$. The relay block (one-bit quantizer) represents the sign function $\text{sgn}(\cdot)$ and e^{-sT} represents a time delay of T units.

(a) [1p] Let $r(t) = 0.5$ and $T = 1$. Then the average of y will converge to 0.5. Illustrate this by plotting r and y as functions of time in steady state.

(b) [3p] Let $r(t) = 0$ and $T = 1$. Use describing function analysis to show that the system is likely to have a (symmetric) periodic solution. Determine an estimate of the period time of the oscillation.

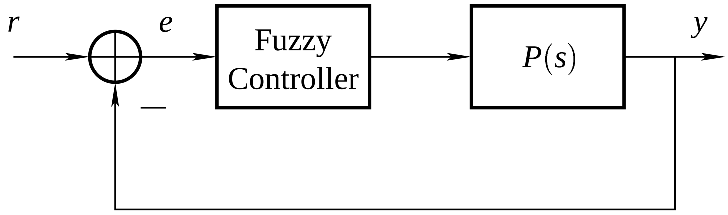
(c) [3p] Let $r(t) = 0$ and $T > 0$. Show that (for any non-zero initial condition of the integrator) the period time of the oscillation will be equal to $4T$.

(d) [3p] Let $r(t) = 0$ and $T = 1$ (as in (b)). Suppose that in an implementation the relay is replaced by a saturation with slope $k > 0$, that is,

$$\text{sat}(kx) = \begin{cases} 1, & kx > 1 \\ kx, & k|x| \leq 1 \\ -1, & kx < -1 \end{cases}$$

where x denotes the output of the integrator. For which values of $k > 0$ does describing function analysis suggest that there will be an oscillation?

4. Consider the fuzzy control system below.



A fuzzy control design for the plant

$$P(s) = \frac{1}{(s+1)^2}$$

led to a piece-wise linear (static) controller given by

$$u = \begin{cases} e+3, & e > 1 \\ 4e, & |e| \leq 1 \\ e-3, & e < -1 \end{cases}$$

(a) [3p] Use the Circle Criterion to show that the closed-loop system (from r to y) is bounded-input bounded-output (BIBO) stable.

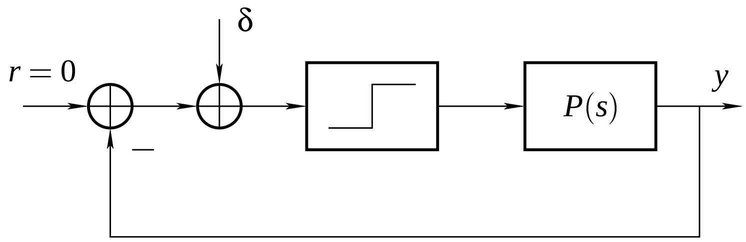
(b) [2p] Derive the gain γ of the fuzzy controller. What conclusions can you draw about BIBO stability using the Small Gain Theorem?

(c) [2p] Introduce the state $x = (y, \dot{y})^T$ and denote the fuzzy controller as $u = F(e)$. Then specify the closed-loop system on state-space form

$$\dot{x} = f(x, r)$$

(d) [3p] For $r = 0$, find a Lyapunov function and show that the closed-loop system is locally asymptotically stable.

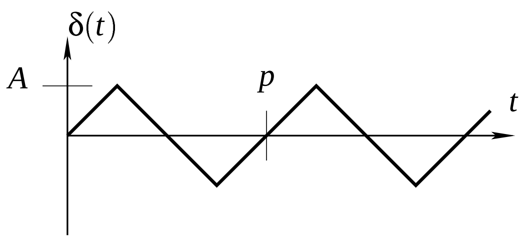
5. High-frequency dither is commonly used in many applications. In this problem we discuss the relay feedback system with dither shown below:



The system is given by the equation

$$\dot{x}(t) = Lx(t) + b\,\mathrm{sgn}(cx(t) + \delta(t)), \qquad x(0) = x_0, \tag{3}$$

where L is a stable matrix. The dither signal δ is a triangle wave:



As we will discuss further below, the system (3) can be approximated by the so called averaged system

$$\dot{w}(t) = Lw(t) + bN(cw(t)), \qquad w(0) = x_0, \tag{4}$$

where $N(z) = p^{-1}\int_0^p \mathrm{sgn}(z+\delta(t))\,dt$. Note that N depends on the amplitude A of the dither.

(a) [1p] Specify the expression for $P(s)$ in the block diagram.

(b) [3p] Show that

$$N(z) = \mathrm{sat}(z/A)$$

for each constant z.

(c) [2p] One can show that $x \approx w$ if δ has sufficiently large frequency. (You don't have to do that.) Hence, one can study (4) and draw conclusions about (3). Discuss how one can derive a condition on the dither amplitude $A > 0$, which guarantees that (4) is stable. (You don't have to explicitly derive such a condition.) In general, will a large or a small A give a more stable system? Motivate.

(d) [2p] With no dither the relay feedback system may oscillate. Argue using describing function analysis, how we can check if the system (3) is likely to have a stable oscillation when $\delta(t) = 0$ for all $t > 0$.

(e) [2p] From the discussion above it seems like A large and p small are desirable. Argue about practical disadvantages with such choices for a real control system.