Solutions to Exam in 2E1262 Nonlinear Control, April 24, 2003

1. We need to check the intersections between $G(i\omega)$ and $-1/N(A)$, where $-\infty < -1/N(A) < -1$.

   (a) No periodic solution since $-\pi/2 < \arg G(i\omega) < 0$.

   (b) No periodic solution since $-\pi < \arg G(i\omega) < -\pi/2$.

   (c) $\operatorname{Im} G(i\omega) = 0$ gives $\omega = \sqrt{3}$. No periodic solution since $G(i\sqrt{3}) = -1/4$.

   (d) From (c), it follows that $G(i\sqrt{3}) = -1/2$ and thus that there will be no periodic solution.

   (e) No periodic solution since $|G(i\omega)| \leq 1$.

2. (a) With $z = (z_1, z_2) = (x, \dot{x})$ we have

$$\dot{z} = f(z, u) = \begin{pmatrix} z_2 \\ 2z_2^2 - z_1 + u - 1 \end{pmatrix}$$

   (b) There is one equilibrium and it is given by $z^0 = (-1, 0)$. Linearizing the system in (a) about this point gives

$$\Delta\dot{z} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Delta z$$

   which has characteristic polynomial $\lambda^2 + 1$. The roots are $\lambda = \pm i$, so hence we cannot conclude if the equilibrium of the nonlinear system is stable or not.

   (c) The solution is satisfied since

$$\begin{aligned} \ddot{x} - 2(\dot{x})^2 + x &= -\cos t - 2\sin^2 t + \cos t = -2 + 2\cos^2 t \\ &= -2 + 1 + \cos(2t) = -1 + u \end{aligned}$$

   With the notation above, we have

$$\frac{\partial f}{\partial z} = \begin{pmatrix} 0 & 1 \\ -1 & 4z_2 \end{pmatrix}, \quad \frac{\partial f}{\partial u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

   The linearized system is hence

$$\delta\dot{z} = \begin{pmatrix} 0 & 1 \\ -1 & -4\sin t \end{pmatrix} \delta z + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta u$$

   where

$$\delta z = z(t) - \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \quad \delta u = u(t) - \cos(2t)$$

   (d) For example,

$$u(x, \dot{x}) = [-2(\dot{x})^2 + x + 1] - 2\dot{x} - x$$

   gives the closed-loop system

$$\ddot{x} + 2\dot{x} + x = 0$$

   which obviously is globally asymptotically stable.

3. (a) The equilibria are solutions to the equation

$$0 = -x_2$$
$$0 = x_1^3 + (x_1^2 - 1)x_2$$

Hence, $(x_1, x_2) = (0, 0)$ is the only one. Linearization yields

$$\Delta \dot{x} = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix} \Delta x$$

which does not give any information about the stability of the original nonlinear system.

(b) Phase portraits (small and large scale).

(c) $V(0,0) = 0$ and $V(x_1, x_2) > 0$, for all $(x_1, x_2) \neq 0$. Moreover,

$$\begin{aligned} \frac{dV}{dt} &= \begin{pmatrix} 4ax_1^3 & x_2 \end{pmatrix} \begin{pmatrix} -x_2 \\ x_1^3 + (x_1^2 - 1)x_2 \end{pmatrix} \\ &= -4ax_1^3 x_2 + x_1^3 x_2 + (x_1^2 - 1)x_2^2 = (x_1^2 - 1)x_2^2 \end{aligned}$$

if $a = 1/4$. Hence, $dV/dt < 0$ for $|x_1| < 1$ and $x_2 \neq 0$. Since small $x_1(0) \neq 0$ and $x_2(0) = 0$ gives $x_2(t) \neq 0$ for small $t > 0$, it follows that the origin is asymptotically stable.

(d) Global asymptotic stability cannot be proved using V, because for large x_2 the term $(x_1^2 - 1)x_2^2$ of dV/dt in (c) is dominating. Hence, if $|x_1| > 1$, this term is positive and thus $dV/dt > 0$.

4. (a) The maximum of $N(A)$ is given by

$$\max_{A>0} N(A) = \frac{2}{\pi a}$$

The Nyquist curve of $G(s)$ intersects the negative real axis at $\omega = \sqrt{2}$ corresponding to $G(i\omega) = -2/3$. Hence, if

$$-\frac{\pi a}{2} < -\frac{2}{3}$$

there will probably be no oscillations. Hence, choose $a > 4/(3\pi) \approx 0.42$.

(b) The response consists of one part from $y = 2$ to $y = 1$ and one from $y = 1$ to $y = 0$.

(c) The system is asymptotically stable for $|y(0)| \leq 1$, since in that case $f(y(0)) = 0$ and $G(s)$ is stable. For $y(0) > 1$, it is easy to see that $e = r - f(y) \leq -1$, which thus drives $y(t)$ towards -1. Then, at some instance τ, it must hold that $-1 < y(\tau) < 1$, so the previous stability argument applies. Since the case $y(0) < -1$ is similar, global asymptotic stability follows.

5. (a) With $u = \alpha$, we have $x(t) = x(0) + \alpha t = 1 + \alpha t$ and thus

$$J(u) = \int_0^1 (1 + \alpha^2 + 2\alpha t + \alpha^2 t^2) dt = 1 + \alpha + \frac{4}{3}\alpha^2$$

The minimum is attained for $\alpha^* = -3/8$, which yields $J(\alpha^*) = 13/16 \approx 0.81$.

(b) With $u = -kx$, we have $x(t) = e^{-kt}$ and thus

$$J(u) = \int_0^1 (1 + k^2)e^{-kt}dt = \frac{1 + k^2}{k}\left(1 - e^{-k}\right)$$

which has its minimum for $k^* \approx 1/4$. Moreover, $J(u^*) \approx 0.94$.

(c) $J(u^*) = p(0) = -\tanh(-1) \approx 0.76$, which of course is lower than the values obtained in (a) and (b).

(d) No, because $\dot{x} = -p(t)x(t)$ for large $t > 0$ yields an unstable system since $p(t) \to -1$ as $t \to \infty$.