Automatic Control
Department of Signals, Sensors & Systems

Nonlinear Control, 2E1262

Exam 14:00-19:00, April 24, 2003

Aid:
- Lecture notes and textbook from basic course ("Reglerteknik" by Glad & Ljung or similar text approved by course responsible). Other textbooks, handbooks, exercises, solutions, calculators etc. may **not** be used.

Observandum:

- Name and social security number (*personnummer*) on every page.
- Only one solution per page.
- A motivation must be attached to every answer.
- Specify number of handed in pages on cover.
- Each subproblem is marked with its maximum credit.

Preliminary Grading:

Grade 3: ≥ 23

Grade 4: ≥ 33

Grade 5: ≥ 43

Results: The results will be posted on the department's board on second floor, Osquldasväg 10.

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Good Luck!

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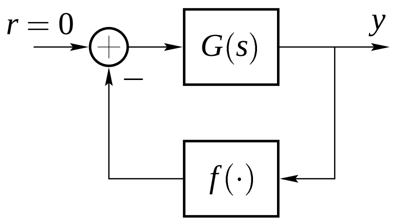
1. Consider a linear system $G(s)$ in feedback with a static nonlinearity $f(\cdot)$ as shown below:



The describing function $N(A)$ of the nonlinearity $f(\cdot)$ is real-valued and satisfies

$$0 < N(A) < 1, \quad A > 0$$

For which of the following $G(s)$ does the describing function method suggest that the system exhibit a periodic solution? Motivate.

(a) [2p]

$$G(s) = \frac{b}{s+a}, \quad a, b > 0$$

(b) [2p]

$$G(s) = \frac{b}{s(s+a)}, \quad a, b > 0$$

(c) [2p]

$$G(s) = \frac{2}{s^3 + 3s^2 + 3s + 1}$$

(d) [2p]

$$G(s) = \frac{4}{s^3 + 3s^2 + 3s + 1}$$

(e) [2p]

$$G(s) = \frac{1}{(s+1)^n}, \quad n \geq 1$$

2. Consider the control system

$$\ddot{x} - 2(\dot{x})^2 + x = u - 1 \tag{1}$$

(a) [2p] Write the system in first-order state-space form.

(b) [2p] Suppose $u(t) = 0$. Find all equilibria and determine if they are stable.

(c) [4p] Show that (1) satisfies the periodic solution $x(t) = \cos(t)$, $u(t) = \cos(2t)$. Linearize the system around this solution.

(d) [2p] Design a state-feedback controller $u = u(x, \dot{x})$ for (1), such that the origin of the closed-loop system is globally asymptotically stable.

3. Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1^3 + (x_1^2 - 1)x_2 \end{aligned} \tag{2}$$

(a) [2p] Show that the origin is the only equilibrium point of the system (2). What information does the linearization around the origin give about the stability of the origin?

(b) [2p] Sketch the phase portrait. Illustrate the behavior of the system both for small and for large $|x|$.

(c) [3p] Show that the origin is locally asymptotically stable by using

$$V(x_1, x_2) = ax_1^4 + \frac{1}{2}x_2^2 \tag{3}$$

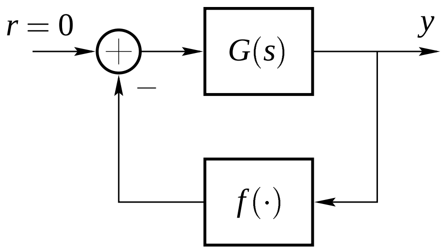
for some suitable choice of a.

(d) [3p] Can the function V in (3) be used to show global asymptotic stability of the origin? Motivate your answer.

4. Consider a servo with transfer function $G(s)$. The servo is controlled by a relay with deadzone $a > 0$ represented by the static nonlinearity

$$f(y) = \begin{cases} 1, & y > a \\ 0, & -a \leq y \leq a \\ -1, & y < -a \end{cases}$$

as illustrated by the figure below.



(a) [4p] A certain choice of sensor gives the servo transfer function

$$G(s) = \frac{4}{s(s+1)(s+2)}$$

The describing function $N(A)$ for the deadzone relay $f(\cdot)$ is equal to

$$N(A) = \frac{4}{\pi A}\sqrt{1 - \frac{a^2}{A^2}}, \quad A > a$$

How large should you choose the deadzone $a > 0$ in order to avoid oscillations according to the describing function method?

(b) [2p] Suppose another sensor is applied, so that the transfer function of the servo is equal to

$$G(s) = \frac{1}{s+1}$$

Moreover, suppose that the deadzone is equal to $a = 1$. Sketch the output y as a function of time when $y(0) = 2$.

(c) [4p] Consider the set-up in (b) again, but for an arbitrary initial state. Show that the closed-loop system is globally asymptotically stable.

5. Consider the integrator

$$\dot{x} = u, \qquad x(0) = 1$$

In this problem we will compare constant control, state-feedback control, and optimal control with respect to the performance index

$$J(u) = \int_0^1 \left[x^2(t) + u^2(t)\right] dt$$

(a) [4p] Suppose that the control is constant $u = \alpha \in \mathbb{R}$. Determine α^* that minimizes J. What is the minimal J equal to?

(b) [4p] Suppose that the control is a (stabilizing) state feedback $u = -kx$, $k > 0$. Show that $k^* \approx 1/4$ minimizes J. What is the minimal J equal to in this case?

(c) [1p] The optimal control $u : [0,1] \to \mathbb{R}$, which minimizes J, is given by

$$u(t) = -p(t)x(t)$$

where $p(t) = -\tanh(t-1)$. What is the minimal J equal to in this case?

Hint: The minimal J is equal to $J(u^*) = p(0)$. Recall that $\tanh(x) = (e^x - e^{-x})/(e^x + e^{-x})$.

(d) [1p] Does the optimal control in (b) give a stable closed-loop system?