Solutions to Exam in 2E1262 Nonlinear Control, Dec 16, 2003

1. (a) All singular points are given by the solutions to

$$0 = -3x_1 + x_1^3 - x_2$$
$$0 = x_1 - x_2,$$

or

$$0 = x_1(x_1^2 - 4)$$
$$x_1 = x_2.$$

Hence, they are equal to $(x_1, x_2) = (0,0)$ and $\pm(2,2)$. The linearization about the equilibria is given by

$$A(x_1, x_2) = \begin{bmatrix} -3+3x_1^2 & -1 \\ 1 & -1 \end{bmatrix}$$

so that

$$A(0,0) = \begin{bmatrix} -3 & -1 \\ 1 & -1 \end{bmatrix}, \quad A(2,2) = A(-2,-2) = \begin{bmatrix} 9 & -1 \\ 1 & -1 \end{bmatrix}.$$

The characteristic polynomial for $A(0,0)$ is $\lambda^2 + 4\lambda + 4$ and for $A(2,2) = A(-2,-2)$ is $\lambda^2 - 8\lambda - 8$, so $(0,0)$ is locally (asymptotically) stable while $\pm(2,2)$ are unstable.

(b) We have

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = -3x_1^2 + x_1^4 - x_1x_2 + x_1x_2 = x_1^2(x_1^2 - 3) < 0$$

as long as $|x_1| < \sqrt{3}$ and $x_1 \neq 0$. Consider LaSalle's invariant set theorem (see lecture notes) with $E = \{x_1 = 0, x_2 \leq R\}$, for some $R > 0$. Then, the largest invariant set in E is $M = (0,0)$, because for $x(0)$ in E but not in M we have from $\dot{x}_1 = -x_2$ that $x(t)$ will leave E. The result now follows from LaSalle's invariant set theorem.

(c) With $u(x) = -x_1^3$, the nonlinearity in the system is canceled, so the closed-loop system is equal to

$$\dot{x}_1 = -3x_1 - x_2$$
$$\dot{x}_2 = x_1 - x_2.$$

This linear system is asymptotically stable (cf., $A(0,0)$ in (a)), thus the closed-loop system is globally asymptotically stable.

2. (a) Denote v the input of Δ and u the output. Then, we have

$$v = G_2G_3(1 + G_1G_2)^{-1}u =: Gu.$$

Small gain theorem guarantees closed-loop stability for all Δ such that

$$\gamma(\Delta) \sup_{\omega} |G(i\omega)| < 1.$$

Since $\sup_{\omega} |G(i\omega)| < 40$, we can choose $\beta = 1/40$.

(b) *System (1):* The equilibria points are at multiple locations, $x_0 = (n, 0)$ where $n = 0, 1, 2, \ldots$. If n is even, the system is asymptotic stable and if n is odd it is unstable. This corresponds to phase portrait C.

System (2): The equilibria are in the origin and in $(\pm 1, 0)$. If the system is linearized, the equilibrium in the origin can be shown to be unstable, but no conclusions can be drawn from the other two equilibria. However, from the vector field we can draw the conclusions that the system corresponds to phase portrait B.

System (3): The sign-term gives the system different character depending on the value of $x_1 + 2x_2$. All trajectories converge to this sliding set, therefore the corresponding phase portrait is D.

System (4): This system has complex stable eigenvalues. The trajectories spiral therefore to the origin, which corresponds to A.

(c) Let $V(x) = ax_1^2 + bx_2^2$. Then,

$$\dot{V} = -2ax_1^4 + 4ax_1x_2 - 2bx_1x_2 - 2bx_2^4.$$

Choose, for example, $a = 1/2$ and $b = 1$. Then,

$$\dot{V} = -x_1^4 - 2x_2^4 < 0, \quad \forall (x_1, x_2) \neq 0,$$

so the origin is globally asymptotically stable.

3. (a) The time derivative of V is

$$\dot{V} = -x^2(x^2 + 1) + yu \leq yu$$

Integrating the left-hand and the right-hand sides gives

$$V(x(T)) - V(x(0)) \leq \int_0^T y(t)u(t)dt$$

Since $V(x(T)) = x^2(T)/2 \geq 0$ and $V(x(0)) = 0$, it follows that the system is passive.

(b) For $\sigma_1 = x_1 - x_2 = 0$, we get the equivalent control $u_{\text{eq}} = x_1 + x_1^2$ because then

$$\dot{\sigma}_1 = \dot{x}_1 - \dot{x}_2 = -x_1^2 + x_2 + u_{\text{eq}} - x_1 - x_2 = 0$$

Inserted in the original system, this gives

$$\dot{x}_1 = -x_1^2 + x_2 + u_{\text{eq}} = x_1 + x_2 = 2x_1$$
$$\dot{x}_2 = x_1 + x_2 = 2x_2$$

Along the sliding surface $\{\sigma_1(x) = 0\}$, we have $x_1 = x_2$. The system is obviously unstable along the sliding surface.
For $\sigma_2 = x_1 + 4x_2 = 0$, we get $u_{\text{eq}} = -4x_1 + x_1^2 - 5x_2$ because then

$$\dot{\sigma}_2 = \dot{x}_1 + 4\dot{x}_2 = -x_1^2 + x_2 + u_{\text{eq}} + 4x_1 + 4x_2 = 0$$

Inserted in the system, this gives

$$\dot{x}_1 = -x_1^2 + x_2 + u_{\text{eq}} = -4x_1 - 4x_2 = -3x_1$$
$$\dot{x}_2 = x_1 + x_2 = -4x_2 + x_2 = -3x_2$$

Along the sliding surface $\sigma_2(x) = 0$, we have $x_1 = -4x_2$. Thus the system is stable along the sliding surface.

For $\sigma_3 = x_1^2 - x_2 = 0$, we get (similarly as above)

$$u_{\text{eq}} = \frac{1}{2x_1}(x_1 + x_2) + x_1^2 - x_2,$$

which is not well-defined for $x_1 = 0$.

Since σ_2 yields the only stable surface, we choose that one. The sliding mode controller can be written as (see lecture notes)

$$\begin{aligned} u &= -\frac{p^T f(x)}{p^T g(x)} - \frac{\mu}{p^T g(x)} \operatorname{sgn} \sigma_2(x) \\ &= x_1^2 - 4x_1 - 5x_2 - \mu \operatorname{sgn}(x_1 + 4x_2), \end{aligned}$$

where μ is a arbitrary positive constant that determines the rate of convergence to the sliding set.

4. (a) According to the circle criterion, the closed-loop system will be stable for a nonlinearity in the sector $[0, \beta]$ if the Nyquist curve stays to the right of the vertical line $-1/\beta$. From the Nyquist curve we see that we can take $\beta \approx 1/0.25 = 4$.

   The maximum gain of the linear system is equal to the largest distance from the origin of the Nyquist curve, which is about 2. The small gain theorem thus allows the sector to have $k \approx 1/2$.

   (b) $u = kx$ gives $\dot{x} = kx^3$. If $k < 0$ then the Lyapunov function $V(x) = x^2/2$ can be used to prove that $\dot{x} = kx^3$ is globally asymptotically stable.

   (c) $u = kx^2$ gives $\dot{x} = kx^4$. If $k > 0$ then $x \to \infty$ for $x(0) > 0$ and if $k < 0$ then $x \to -\infty$ for $x(0) < 0$; hence the system is unstable. For $k = 0$, we have the equation $\dot{x} = 0$ which is not asymptotically stable.

5. (a) In state-space form, the system $1/(s+1)$ is given by

$$\dot{x} = -x + u$$
$$y = x,$$

with $x(0) = x_0$. First suppose $x_0 > 0$. Since the system is of first order, the state x is driven to the origin in minimum time by the control $u^*(t) = \arg\min_{|u|\le 1} \dot{x}(t) = \arg\min_{|u|\le 1} -x(t) + u = -1$. Similarly, if $x_0 < 0$, then $u^*(t) = \arg\max_{|u|\le 1} \dot{x}(t) = 1$. Hence, the proposed control algorithm $u^* = -\operatorname{sgn} y$ is time optimal. (Note that the control never switches in the interval $(0, t_f)$.)

(b) The control in (a) corresponds to a relay in negative feedback. The describing function for the relay is equal to $N(A) = 4/(\pi A)$. As usual, we are looking for solutions to $G(i\omega)N(A) = -1$. Since

$$G^{-1}(i\omega) = \frac{100 - 21\omega^2}{100} + i\omega\frac{120 - \omega^2}{100}$$

intersects the negative real axis (that is, intersects $-N(A)$) for $\omega = \sqrt{120}$ with $G^{-1}(i\sqrt{120}) = -121/5$, we get

$$\frac{4}{\pi A} = \frac{121}{5}.$$

Hence, there will be an oscillation with frequency $\omega = \sqrt{120}$ and amplitude $A = 20/(121\pi)$. It follows from the same discussion as in the lecture notes on describing function that the oscillation is stable.