Automatic Control
Department of Signals, Sensors & Systems

Nonlinear Control, 2E1262

Exam 14:00-19:00, Dec 16, 2003

Aid:
- Lecture notes from nonlinear course and textbook from basic course ("Reglerteknik" by Glad & Ljung or similar text approved by course responsible). Mathematical handbook (e.g., "Beta Mathematics Handbook" by Råde & Westergren). Other textbooks, handbooks, exercises, solutions, calculators etc. may **not** be used.

Observandum:

- Name and social security number (*personnummer*) on every page.
- Only one solution per page.
- A motivation must be attached to every answer.
- Specify number of handed in pages on cover.
- Each subproblem is marked with its maximum credit.

Preliminary Grading:

Grade 3: ≥ 20

Grade 4: ≥ 30

Grade 5: ≥ 40

Results: The results will be posted on the department's board on second floor, Osquldasväg 10.

Responsible: Karl Henrik Johansson, `kallej@s3.kth.se`

Please, remember to fill in the course evaluation linked on the course homepage.

Good Luck!

1. Consider the following nonlinear system

$$\dot{x}_1 = -3x_1 + x_1^3 - x_2 + u$$
$$\dot{x}_2 = x_1 - ax_2$$

(a) [3p] Determine the local stability properties of all equilibrium points to the nonlinear system if $u(t) \equiv 0$ and $a = 1$.

(b) [4p] Let $u(t) \equiv 0$ and $a = 0$. Prove that every solution starting close to the origin will approach the origin. You may use the function

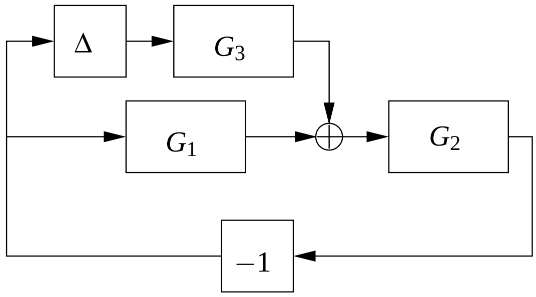
$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

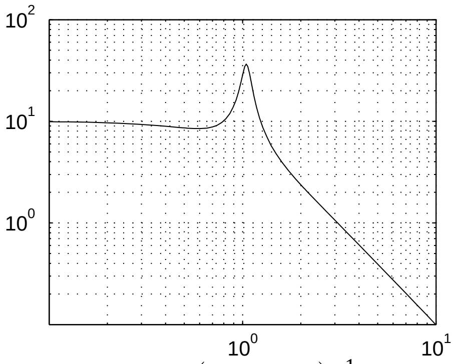
and LaSalle's invariant set theorem.

(c) [3p] If $a = 1$, determine a nonlinear state feedback control $u = k(x)$ such that the origin is globally asymptotically stable.

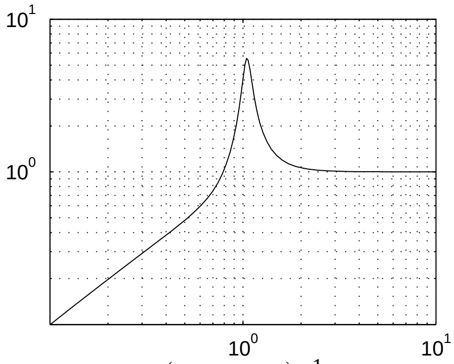
2.

(a) [3p] Consider the system depicted in the block diagram below. Here Δ denotes an unknown nonlinear system. Some relevant amplitude curves are also shown below. Use these to find a bound $\beta > 0$ such that the closed-loop system is stable for all Δ with gain $\gamma(\Delta) < \beta$ (with "gain" defined as in the lecture notes).

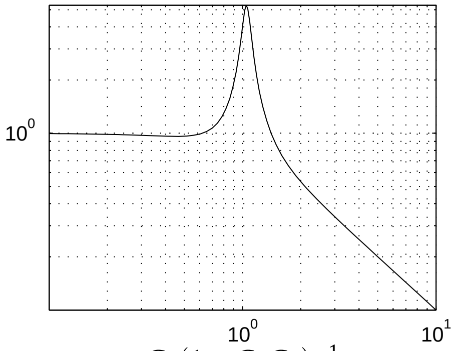




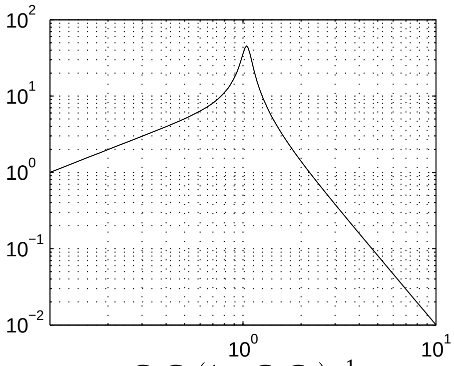
$G_2G_3(1+G_1G_2)^{-1}$



$(1+G_1G_2)^{-1}$



$G_2(1+G_1G_2)^{-1}$



$G_3G_1(1+G_1G_2)^{-1}$

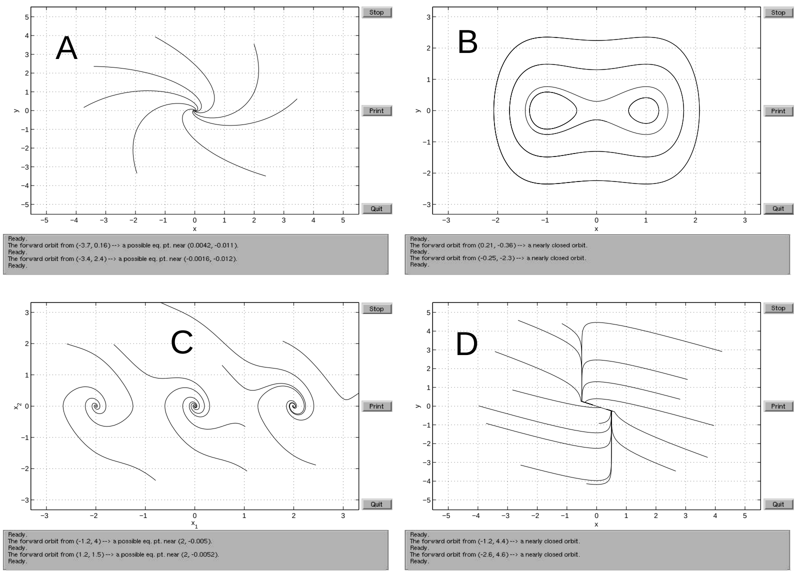
(b) [4p] Four nonlinear systems are shown below together with their phase portraits. Match systems and phase portraits. Motivate your answer for each pair.

$$(1) \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin(\pi x_1) - x_2 \end{aligned}$$

$$(2) \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3 \end{aligned}$$

$$(3) \quad \begin{aligned} \dot{x}_1 &= -2x_1 - \mathrm{sign}(x_1 + 2x_2) \\ \dot{x}_2 &= x_1 \end{aligned}$$

$$(4) \quad \begin{aligned} \dot{x}_1 &= -2x_1 + 2x_2 \\ \dot{x}_2 &= -2x_1 - 3x_2 \end{aligned}$$



(c) [3p] Investigate the stability of the origin for the system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + 2x_2 \\ \dot{x}_2 &= -x_1 - x_2^3, \end{aligned}$$

by using a quadratic Lyapunov function.

3.

(a) [4p] Show that the system

$$\dot{x} = -x^3 - x + u$$
$$y = x$$

with $x(0) = 0$ describes a passive mapping from u to y, that is, show that

$$\langle y, u \rangle_T = \int_0^T y(t)u(t)dt \geq 0$$

for all $T > 0$.

Hint: You may find the function $V(x) = x^2/2$ useful.

(b) [6p] A sliding mode controller is to be designed to stabilize the origin of the nonlinear system

$$\dot{x}_1 = -x_1^2 + x_2 + u$$
$$\dot{x}_2 = x_1 + x_2.$$

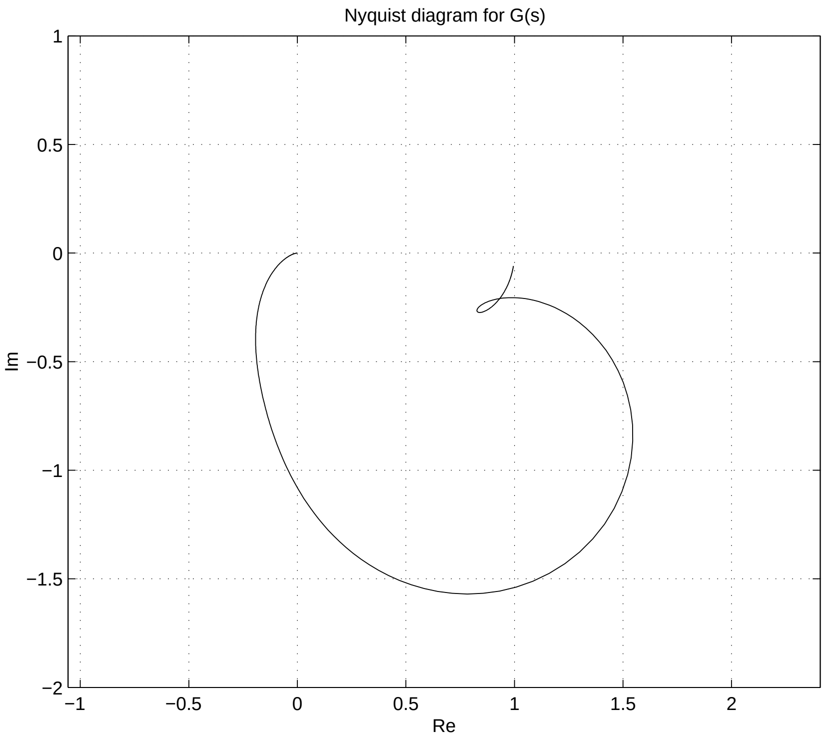
Choosing a suitable sliding surface $\{x : \sigma(x) = 0\}$ is a crucial part of the design. Consider three choices:

$$\sigma_1(x) = x_1 - x_2$$
$$\sigma_2(x) = x_1 + 4x_2$$
$$\sigma_3(x) = x_1^2 - x_2$$

Determine the equivalent control u_{eq} and the sliding mode dynamics for each of the suggested switching surfaces (if possible). Is the equivalent control well-defined for all sliding surfaces? Decide which one is the most suitable to use, and specify the complete control law for that one.

4.

(a) [6p] An exponentially stable linear system $G(s)$ is in negative feedback with a static nonlinearity $\psi(\cdot)$. The Nyquist diagram of $G(s)$ is shown below. What is the largest sector $\psi \in [0,\beta]$ for which *the circle criterion* guarantees stability for the closed loop? What is the largest sector $\psi \in [-k,k]$ for which *the small gain theorem* guarantees stability for the closed loop? (The notation $\psi \in [k_1,k_2]$ means that $(\psi(x) - k_1x)(\psi(x) - k_2x) \leq 0$.)



(b) [2p] Consider the scalar nonlinear control system

$$\dot{x} = x^2 u \tag{1}$$

where the $u = u(x)$ is a state-feedback control. Does there exist a linear control law $u(x) = kx$ that ensures that (1) is asymptotically stable? What about global asymptotic stability?

(c) [2p] Does there exist a quadratic control law $u(x) = kx^2$ that ensures that (1) is asymptotically stable? What about global asymptotic stability?

5. Consider the system

$$G(s) = \frac{1}{s+1}$$

with input u and output y. The system is controlled by the algorithm

$$u = \begin{cases} 1, & y < 0 \\ -1, & y > 0 \end{cases} \tag{2}$$

(a) [4p] Show that among all controls with $|u(t)| \leq 1$, the controller (2) is the one that drives the system to zero in the shortest amount of time.

Hint: You may solve this problem without applying Pontryagins Maximum Principle.

(b) [6p] Suppose the system is actually equal to

$$G(s) = \frac{1}{(s+1)(T_1 s+1)(T_2 s+1)}$$

with $T_1 = T_2 = 0.1$. If the controller (2) still is applied, there will be an oscillation. What is the amplitude and the frequency of the oscillation? Is the oscillation stable?

Hint: The calculations might be simpler if you consider $G^{-1}(i\omega)$.