**Solutions to Exam in 2E1262 Nonlinear Control, Apr 16, 2004**

1. (a) The equilibria are given by $\bar{x} = (\bar{x}_1, \bar{x}_2) = k\pi/2(1,-1)$, $k \in \mathbb{Z}$. The linearized system about $\bar{x}$ is given by $\dot{z} = Az$ with

$$A = \begin{pmatrix} 1 & 1 \\ (-1)^k & (-1)^{k+1} \end{pmatrix}$$

which is unstable for every $k$, since

$$\det(sI - A) = (s-1)(s-(-1)^{k+1}) + (-1)^{k+1}$$

is an unstable polynomial.

(b) The system is on strict feedback form because it can be written as

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1,x_2) + g_2(x_1,x_2)u \end{aligned}$$

see Lecture 10.

(c) Using the notation of Lecture 10, we can choose $\phi_1(x_1) = -2x_1$ and $V_1(x_1) = x_1^2/2$, and thus

$$u_1 = \frac{d\phi_1}{dx_1}(x_1 + x_2) - \frac{dV_1}{dx_1} - (x_2 - \phi_1) = -5x_1 - 3x_2$$

Then, we choose

$$u = u_1 - \sin(x_1 - x_2) = -5x_1 - 3x_2 - \sin(x_1 - x_2)$$

(d) Consider Lyapunov function candidate

$$V(x) = V_2(x) = \frac{x_1^2}{2} + \frac{(2x_1 + x_2)^2}{2}$$

as suggested by the back-stepping lemma. Then, with the control as in (c), we have

$$\dot{V} = \frac{dV}{dx} f(x,u) = -15(x_1 + x_2/2)^2 - x_2^2/4 < 0, \quad \forall x \neq 0$$

so since $V$ is positive definite, the system is asymptotically stable.

2. (a) The equilibria are $(0,0)$ and $(1,1)$ with linearized systems given by

$$\dot{z} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z, \qquad \dot{z} = \begin{pmatrix} -3 & 1 \\ 6 & -3 \end{pmatrix} z$$

respectively. Hence, the origin is not locally stable, while $(1,1)$ is locally stable.

(b) Consider the banana shaped set $\Gamma$ (draw a picture). For each initial point $x(0)$ on the left boundary of $\Gamma$, we have $x_2(0) = x_1^2(0)$, and thus

$$\begin{aligned} \dot{x}_1(0) &= -x_1^3(0) + x_2(0) = -x_1^3(0) + x_1^2(0) > 0 \\ \dot{x}_2(0) &= x_1^6(0) - x_2^3(0) = 0 \end{aligned}$$

so the trajectory is directed inwards $\Gamma$. For each initial point $x(0)$ on the right boundary of $\Gamma$, we have $x_2(0) = x_1^3(0)$, and thus

$$\begin{aligned}\dot{x}_1(0) &= -x_1^3(0) + x_2(0) = 0 \\ \dot{x}_2(0) &= x_1^6(0) - x_2^3(0) = x_1^6(0) - x_1^9(0) > 0\end{aligned}$$

so the trajectory is again directed inwards $\Gamma$. Hence, $\Gamma$ is invariant.

(c) Draw a trajectory illustrating how a trajectory starting in $x(0) \in \Gamma$ close to the origin tends to the point $(1,1)$.

3. (a) (i) corresponds to (b) because the origin is a stable focus for (i). (ii) corresponds to (d) because (ii) has an unstable equilibrium in the origin. (iii) corresponds to (a) because the linearization of (iii) has a marginally stable equilibrium in the origin (linearized system with eigenvalues in $\pm i$). (iv) corresponds to (c) because (iv) has no equilibrium in the origin.

(b) For (i)–(iii), we have the linearized systems as

$$\dot{z} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} z, \quad \dot{z} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} z, \quad \dot{z} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z$$

(c) The closed-loop system consists of a system with gain less than or equal to $a$ and a linear system with gain equal to one. Small Gain Theorem hence gives the result.

4. (a) The describing function is given by $N_f(A) = (b_1 + ia_1)/A$, so we need to show that $a_1 = 0$. Recall that

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} y(\phi)\cos\phi\, d\phi = \frac{1}{\pi}\int_{-\pi}^{\pi} y(\phi)\cos\phi\, d\phi$$

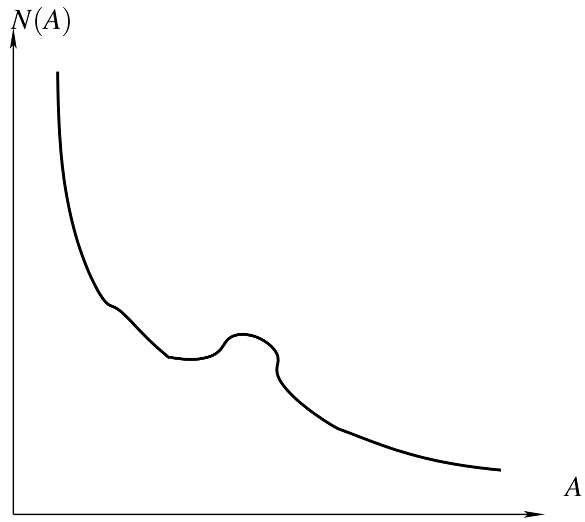
where $y(\phi) = f(A\sin\phi)$ is the output when the input is $u(\phi) = A\sin\phi$. Since $f$ and sin are odd functions, we have

$$\begin{aligned}a_1 &= \frac{1}{\pi}\int_{-\pi}^{0} y(\phi)\cos\phi\, d\phi + \frac{1}{\pi}\int_0^{\pi} y(\phi)\cos\phi\, d\phi \\ &= -\frac{1}{\pi}\int_0^{\pi} y(\phi)\cos\phi\, d\phi + \frac{1}{\pi}\int_0^{\pi} y(\phi)\cos\phi\, d\phi = 0\end{aligned}$$

(b) The describing function is given by $N_f(A) = (b_1 + ia_1)/A$ where $a_1 = 0$, see (a), and

$$\begin{aligned}b_1 &= \frac{1}{\pi}\int_0^{2\pi} y(\phi)\sin\phi\, d\phi = \frac{1}{\pi}\int_0^{2\pi} \sin^6\phi\, d\phi \\ &= \cdots = \frac{5A^5\pi}{8}\end{aligned}$$

(c) The describing function represents an amplitude-depending gain $N(A)$. A rough sketch is shown below:




5. (a) The optimal control problem (on generalized form) is given by $\min_u \int_0^{t_f} L\,dt = \min_u t_f$ with

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = -z_2 - \frac{dg}{dx}(z_1) + u$$
$$\dot{z}_3 = u^2$$

and $\psi(z(t_f)) = 0$, where $\psi_1(z) = z_1 - 89$ and $\psi_2(z) = z_3 - 100$. Here $z(0) = 0$.

(b) The Hamiltonian is given by

$$H = n_0 L + \lambda^T f = n_0 + \lambda_1 z_2 + \lambda_2(-z_2 - g'(z_1) + u) + \lambda_3 u^2$$

(c) The adjoint equations are given by

$$\dot{\lambda}(t) = -\frac{\partial H^T}{\partial z}(z^*(t), u^*(t), \lambda(t), n_0)$$
$$\lambda^T(t_f) = n_0 \frac{\partial \phi}{\partial z}(t_f, z^*(t_f)) + \mu^T \frac{\partial \psi}{\partial z}(t_f, z^*(t_f))$$

where $\phi = 0$, $\mu = (\mu_1, \mu_2)$ and

$$\frac{\partial \psi}{\partial z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence,

$$\dot{\lambda}_1 = \lambda_2 g''(z_1)$$
$$\dot{\lambda}_2 = \lambda_2 - \lambda_1$$
$$\dot{\lambda}_3 = 0$$

with

$$\lambda_1(t_f) = \mu_1$$
$$\lambda_2(t_f) = 0$$
$$\lambda_3(t_f) = \mu_2$$