Automatic Control
Department of Signals, Sensors & Systems

Nonlinear Control, 2E1262

Exam 14:00-19:00, Apr 16, 2004

Aid:
- Lecture notes from nonlinear course and textbook from basic course ("Reglerteknik" by Glad & Ljung or similar text approved by course responsible). Mathematical handbook (e.g., "Beta Mathematics Handbook" by Råde & Westergren). Other textbooks, handbooks, exercises, solutions, calculators etc. may **not** be used.

Observandum:

- Name and social security number (*personnummer*) on every page.
- Only one solution per page.
- A motivation must be attached to every answer.
- Specify number of handed in pages on cover.
- Each subproblem is marked with its maximum credit.

Preliminary Grading:

Grade 3: ≥ 23

Grade 4: ≥ 33

Grade 5: ≥ 43

Results: The results will be posted on the department's board on second floor, Osquldasväg 10.

Responsible: Karl Henrik Johansson, `kallej@s3.kth.se`

Good Luck!

1. Consider the following nonlinear system:

$$\dot{x}_1 = x_1 + x_2$$
$$\dot{x}_2 = \sin(x_1 - x_2) + u$$

(a) [2p] For $u = 0$ determine all equilibria and their stability properties.

(b) [1p] Show that the system is on strict feedback form.

(c) [5p] Design a controller based on back-stepping for the system.
Hint: Consider first the system

$$\dot{x}_1 = x_1 + x_2$$
$$\dot{x}_2 = u_1$$

(d) [2p] Prove that the controller you derived in (c) stabilizes the system.

2. Consider the second-order differential equation

$$\dot{x}_1 = -x_1^3 + x_2$$
$$\dot{x}_2 = x_1^6 - x_2^3$$

(a) [2p] Determine all equilibria and their stability properties.

(b) [5p] Show that the set

$$\Gamma = \{(x_1, x_2) \in \mathbb{R}^2 : \ 0 \le x_1 \le 1, x_2 \ge x_1^3, x_2 \le x_1^2\}$$

is invariant.

(c) [3p] Argue using Γ in (b) that the equilibrium in the origin is unstable.

3.

(a) [4p] Consider the following four systems:

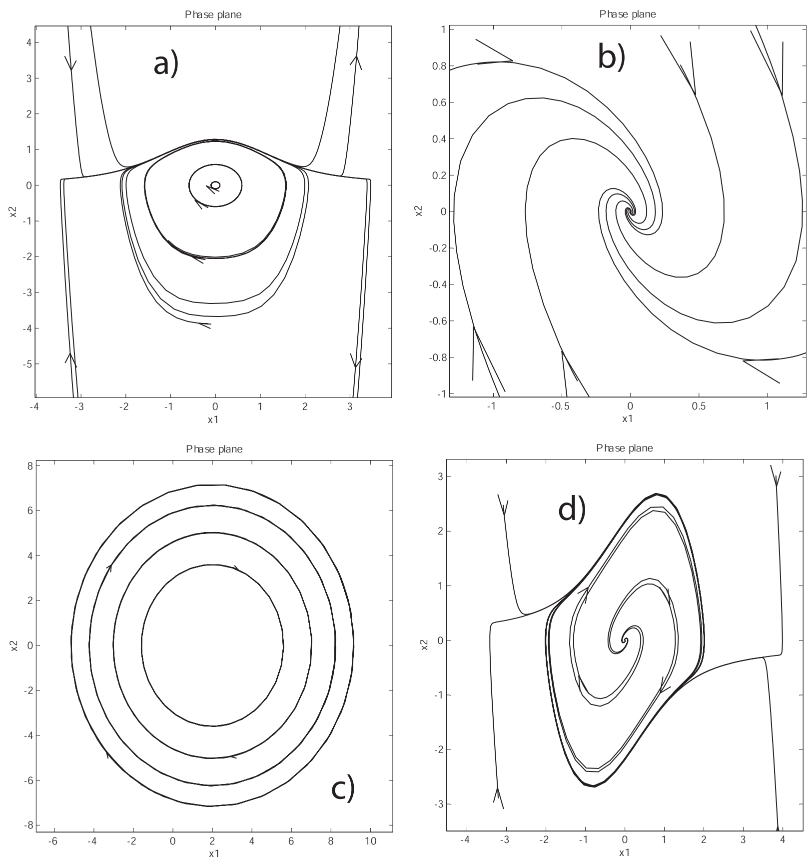
(i) $\dot{x}_1 = x_2$
$\dot{x}_2 = -x_2 - \sin(x_1)$

(ii) $\dot{x}_1 = x_2$
$\dot{x}_2 = (1 - x_1^2)x_2 - x_1$

(iii) $\dot{x}_1 = x_2$
$\dot{x}_2 = -x_1 + x_2 x_1^3/2$

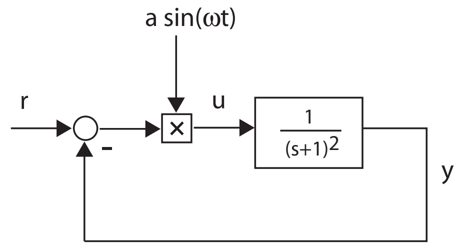
(iv) $\dot{x}_1 = x_2$
$\dot{x}_2 = -x_1 + 2$

Match each of the systems (i)–(iv) with a phase portrait (a)–(d) below. Each pair needs to be individually motivated.



(b) [2p] Three of the systems in (a) have an equilibrium in the origin and one has not. For the three systems that have an equilibrium in the origin, linearize the systems about this equilibrium point.

(c) [4p] A motor with time-varying dynamics is controlled with a proportional controller. The closed-loop system can be described by the system shown in the block-diagram below:



The control signal is thus given by

$$u(t) = [r(t) - y(t)]a \sin \omega t$$

Show that if $|a| < 1$, then the closed-loop system is BIBO stable.

4. Consider a static and odd nonlinearity f and let it define an input–output relation

$$y(t) = f(u(t))$$

(a) [3p] Show that the describing function N_f of such a nonlinearity is real valued.

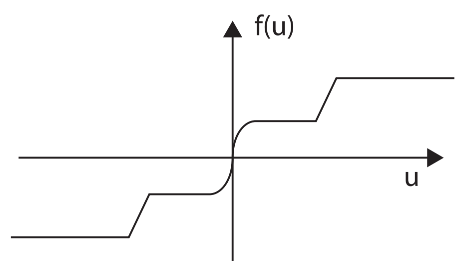
(b) [4p] Suppose $f(u) = u^5$. Show that the describing function for f is equal to

$$N_f(A) = \frac{5A^4}{8}$$

Hint:

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$
$$\int \cos^n x\,dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

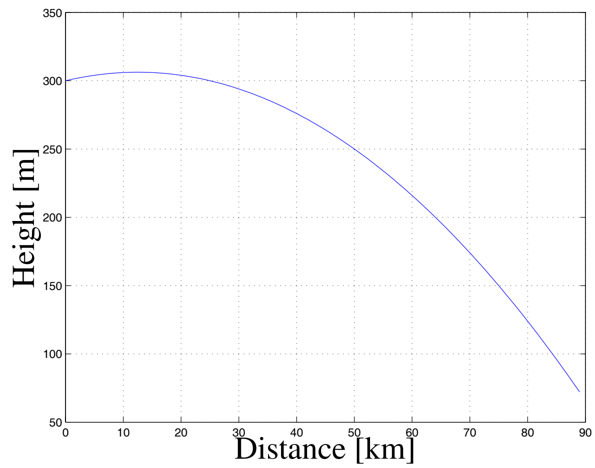
(c) [3p] Sketch qualitatively the describing function N_f of the nonlinearity f shown below. Motivate your graph.



5. You want to help a friend to win the ski race *Vasaloppet*. The problem is to minimize the time to complete the race, but only use a fixed amount of available fuel (including intake of blueberry soup). Suppose that the height curve of the race can be approximated by

$$g(x) = x - 0.04x^2 + 300$$

where x is the distance from start (in km) and $g(x)$ is the height (in m), as illustrated below.



Suppose the evolution of the velocity of the skier is given by

$$\dot{v} = -v - \frac{dg}{dx} + u$$

The distance from the start is the integrated velocity

$$\dot{x} = v$$

The spent energy for the whole race should equal

$$\int_0^{t_f} u^2(t)dt = 100$$

The performance criterion is to reach the goal $x(t_f) = 89$ km in as short time t_f as possible. Of course, the skier starts at rest with $x(0) = v(0) = 0$.

(a) [6p] Formulate the problem above as a time-optimal control problem for a third-order system with state $z = (z_1, z_2, z_3)$, where $z_1 = x$, $z_2 = v$, and $z_3 = \int_0^t u^2(s)dt$.

(b) [1p] Specify the Hamiltonian function for the problem in (a).

(c) [3p] Specify the adjoint equations (i.e., the equations for λ).

Note that you need not find a solution for the optimal control problem.